# CONCEPTUALIZING MUSIC THROUGH MATHEMATICS AND THE GENERALIZED INTERVAL SYSTEM

JOHN STERNBERG

## 1. Introduction

Anyone who has listened to a piece of music and tried to anticipate the next few measures understands that music is full of recognizable patterns. For centuries, theorists have attempted to analyze these patterns in order to better appreciate musical works. Only in the last century, however, has music theory been significantly extended into the world of mathematics.

One of the primary figures behind this development, David Lewin connected music theory with abstract algebra by creating the Generalized Interval System, or GIS. The GIS forms an excellent basis for analysis because it incorporates the notions of pitch classes and pitch-class intervals which are so fundamental to music theory. At the same time, the GIS provides mathematic structure by rendering the musical notions of transposition and inversion as mathematical group actions on the musical space associated with a GIS. The following sections will formally define the Generalized Interval System, and then provide examples of both the musical and mathematical importance of this construct.

## 2. Generalized Interval Systems

**Definition 2.1.** A *generalized interval system* (GIS) is an ordered triple $(S, G, int)$, where the set $S$ is the *musical space* of the GIS, the group $G = (G, *)$ is the group of *intervals*, and a function $int : S \times S \longrightarrow G$, called the *interval function*, satisfying the following two conditions:

- For all $r, s, t \in S$, $int(r, s) * int(s, t) = int(r, t)$,
- For all $s \in S$ and $g \in G$, there exists a unique $t \in S$ which lies the interval $g$ from $s$. In other words, there exists a unique $t \in S$ such that $int(s, t) = g$ [2].

**Example 2.2.** The most common example of a GIS seen in music represents the chromatic scale starting on C. In this GIS, the set $\mathbb{Z}_{12}$ represents the pitch classes C=0, C♯=1, B=2, etc. It should be noted that a pitch class consists of all pitches with a given letter name. This means that the pitches A at 440Hz and A at 880Hz are both in the same pitch class. We define the operation $+_{12}$ such that if $r, s \in \mathbb{Z}_{12}$ then $r +_{12} s = r + s \bmod 12$. For simplicity, $+_{12}$ shall henceforward be denoted simply $+$. We further define an interval function by $int(x, y) = (-x) + y$ for all $x, y \in \mathbb{Z}_{12}$. With these definitions, the triple $(\mathbb{Z}_{12}, (\mathbb{Z}_{12}, +), int)$ clearly represents a GIS. In fact, this particular interval system provides the starting point for all of atonal analysis because it represents the basic scale of which all other western scales are subsets, and incorporates the standard notion of pitch-class intervals.

## 3. Group Actions

An important characteristic of a GIS is that it is equivalent to a simply transitive group action. This is an important mathematical fact, and to prove it shall require some definitions:

**Definition 3.1.** If $G$ is a group and $X$ is a set, then a *(left) group action* of $G$ on $X$ is a binary function $G \times X \longrightarrow X$ (where the image of $g \in G$ and $x \in X$ is written $g \cdot x$), which satisfies the following two axioms:

- $(g * h) \cdot x = g \cdot (h \cdot x)$ for all $g, h \in G$ and $x \in X$.
- $e \cdot x = x$ for every $x \in X$.

**Definition 3.2.** A *group homomorphism* is a map $f : G \longrightarrow H$ between two groups $G$ and $H$ such that $f(g_1 * g_2) = f(g_1) * f(g_2)$ for all $g_1, g_2 \in G$

*Remark* 3.3. If $f$ is a group homomorphism from $G$ into $H$, then:

$$f(e_G) = f(e_G * e_G) = f(e_G) * f(e_G) \Rightarrow e_H = f(e_G).$$

**Lemma 3.4.** *A group action of group $G$ on set $S$ is the same as a group homomorphism $G \longrightarrow Sym(S)$ where $Sym(S)$ is the symmetric group on $S$.*

*Proof.* ($\Rightarrow$) Given a group action of $G$ on $S$, we define a function $\rho : G \longrightarrow Sym(S)$ by $\rho : g \mapsto F_g$ where $F_g(s) = g \cdot s$.

- $F_g \in SYM(S)$ for all $g \in G$.
  - If there exist $g \in G$ and $r, s \in S$ for which $F_g(r) = F_g(s)$, then:

$$\begin{aligned} g \cdot r &= g \cdot s \\ g^{-1} \cdot (g \cdot r) &= g^{-1} \cdot (g \cdot s) \\ (g^{-1} * g) \cdot r &= (g^{-1} * g) \cdot s \\ e \cdot r &= e \cdot s \\ r &= s. \end{aligned}$$

    This means that $\rho$ is injective.
  - Suppose that there exist $s \in S$ and $g \in G$ such that $g \cdot r \neq s$ for all $r \in S$. Consider that $g^{-1} \cdot s = t$ for some $t \in S$.

$$g \cdot (g^{-1} \cdot s) = (g * g^{-1}) \cdot s = e \cdot s = s.$$

    This contradicts the assumption that $g \cdot r \neq s$ for all $r$. Therefore $\rho$ is surjective.

  We thus have that $F_g$ is a bijection from $S$ into $S$ for all $g \in G$. This means that $F_g$ is a permutation of $S$ and therefore an element of $SYM(S)$.
- Given $g, h \in G$ and $s \in S$,

$$(\rho(g * h))(s) = F_{g*h}(s) = (g * h) \cdot s = g \cdot (h \cdot s) = (F_g \circ F_h)(s) = (\rho(g) \circ \rho(h))(s).$$

  So $\rho$ is a group homomorphism.

($\Leftarrow$) If $\rho : G \longrightarrow Sym(S)$ is a group homomorphism, then define $g \cdot s = (\rho(g))(s)$.

– For all $g, h \in G, s \in S$,

$$\begin{aligned}(g * h) \cdot s &= (\rho(g * h))(s) \\ &= (\rho(g) * \rho(h))(s) \\ &= (\rho(g) \circ \rho(h))(s) \\ &= (\rho(g))((\rho(h))(s)) \\ &= (\rho(g))(h \cdot s) \\ &= g \cdot (h \cdot s).\end{aligned}$$

– for all $s \in S$,

$$e_G \cdot s = (\rho(e_G))(s) = (e_{Sym(S)})(s) = id(s) = s.$$

□

**Definition 3.5.** A group action is *transitive* if for every $x, y \in X$, there exists some $g \in G$ such that $g \cdot x = y$.

**Definition 3.6.** A group action is *free* or *simple* if for all $g, h \in G$ with $g \neq h$ and all $x \in X$, $g \cdot x \neq h \cdot x$. Or, equivalently, if $g \cdot x = h \cdot x$ for some $x \in X$ and $g, h \in G$, then $g = h$.

**Definition 3.7.** A group action is *simply transitive* if it is both transitive and free. This means that given $x, y \in X$, there exists a unique $g \in G$ such that $g \cdot x = y$.

**Example 3.8.** Any group acts upon itself on the left.

*Proof.* Given a group, $(G, *)$, consider the map $G \times G \longrightarrow G$ defined by $(g, x) \mapsto g * x$:

- The first axiom of a group action follows from the associativity of the group operation:

$$g \cdot (h \cdot x)) = g * (h * x) = (g * h) * x = (g \cdot h) \cdot x.$$

- The second axiom follows from the definition of the identity:

$$e \cdot x = e * x = x.$$

□

**Example 3.9.** The left action of a group onto itself is simply transitive.

*Proof.* We again have the map $G \times G \longrightarrow G$ such that $(g, x) \mapsto g * x$.

- Given $x \in G$, let $g = (y * x^{-1})$. We know $g \in G$ because $G$ is a group, and:

$$g \cdot x = g * x = (y * x^{-1}) * x = y * (x^{-1} * x) = y * e = y.$$

  Therefore the action is transitive.
- Suppose there exist $g, h \in G$ such that $g \cdot x = h \cdot x$. In this case, we have:

$$\begin{aligned}g * x = g \cdot x &= h \cdot x = h * x \\ g * x * x^{-1} &= h * x * x^{-1} \\ g &= h.\end{aligned}$$

Therefore, $f$ is both free and transitive, or simply transitive. □

**Definition 3.10.** Given GIS $(S, G, int)$ and $g \in G$, we define *transposition by $g$* as the map $T_g : S \longrightarrow S$ such that $int(s, T_g(s)) = g$ for all $s \in S$.

**Example 3.11.** Recall the GIS $(\mathbb{Z}_{12}, \mathbb{Z}_{12}, int)$ from example 2.2. For all $n, m, p \in \mathbb{Z}_{12}$,

$$(T_m(n) = p) \Leftrightarrow (int(n,p) = m) \Leftrightarrow (-n + p = m) \Leftrightarrow (n + m = p).$$

Therefore, in $(\mathbb{Z}_{12}, \mathbb{Z}_{12}, int)$, $T_m(n) = n + m$.

**Lemma 3.12.** *$T_{h*g} = T_g \circ T_h$.*

*Proof.* Given GIS $(S, G, int)$, $T_{h*g}(s)$ is defined by $int(s, T_{h*g}(s)) = h * g$ for all $s \in S$. By definition of $T_g$ and $T_h$, $int(s, T_h(s)) = h$ and $int(T_h(s), T_g(T_h(s))) = g$ for all $s \in S$. Finally, by definition of $int$,

$$int(s, T_h(s)) * int(T_h(s), T_g(T_h(s))) = int(s, T_g(T_h(s))) = h * g.$$

Therefore, $T_g \circ T_h = T_{h*g}$. □

**Definition 3.13.** A group isomorphism is a map $F : G \longrightarrow H$ between two groups $G$ and $H$ defined such that:

- $f$ is a group homomorphism, and
- $f$ is bijective.

**Definition 3.14.** A group *anti-homomorphism* is a map $f : G \longrightarrow H$ between two groups $G$ and $H$ defined such that $f(g_1) * f(g_2) = f(g_2 * g_1)$.

**Definition 3.15.** A group *anti-isomorphism* is a bijective group anti-homomorphism.

**Theorem 3.16.** *The transposition group $TRANS = \{T_g : g \in G\}$ is anti-isomorphic to $G$.*

*Proof.* Consider the map $f : G \longrightarrow TRANS$ defined by $f(g) = T_g$.

- For all $g, h \in G$,

$$f(g * h) = T_{g*h} = T_h \circ T_g = f(h) * f(g).$$

  Thus, $f$ is a group anti-homomorphism.
- Consider $\sigma : TRANS \longrightarrow G$ defined by $\sigma(T_g) = g$. We know that $\sigma$ is well-defined because $T_g$ is generated uniquely by the interval function. For all $g \in G$ we have,

$$(\sigma \circ f)(g) = \sigma(f(g)) = \sigma(T_g) = g,$$

  and

$$(f \circ \sigma)(T_g) = f(\sigma(T_g)) = f(g) = T_g.$$

  This means that $\sigma$ is both a left and right inverse of $f$, so $f$ is a bijection and an isomorphism.

□

**Theorem 3.17.** *Every GIS $(S, G, int)$ gives rise to a simply transitive group action on $S$.*

*Proof.* Given a GIS $(S, (G, *), int)$, define a group action $TRANS \times S \longrightarrow S$ by $(T_g, s) \mapsto T_g(s)$ for all $g \in G$ and $s \in S$.

1. This defines a group action.
   - Given $g, h \in G$ and $s \in S$,

$$(g \cdot (h \cdot s) = T_g(T_h(s)) = (T_g \circ T_h)(s) = (T_g * T_h)(s) = (T_g * T_h) \cdot s.$$

– For all $s \in S$:

$$e \cdot s = T_e(s) \Rightarrow int(s, T_e(s)) = e.$$

Choose arbitrary $t \in S$ and let $int(t, s) = \alpha$. In this case, $T_\alpha(t) = s$.

$$int(t, T_e(s)) = int(t, s) * int(s, T_e(s)) = \alpha * e = \alpha.$$

Therefore, $T_\alpha(t) = T_e(s)$. But remember that $s = T_\alpha(s)$ because $\alpha$ was chosen that way. Therefore, $e \cdot s = T_e(s) = s$.

2. This is simply transitive.
   – Transitivity of $f$ follows from the fact that the interval function is defined for all elements of $S$. Given $s, t \in S$ there must exist some $g \in G$ such that $int(s, t) = g$. By definition, $g \cdot s = T_g(s) = t$.
   – To show that $f$ is free, suppose that $g \cdot s = h \cdot s$ for some $s \in S$ and $g, h \in G$.

$$g \cdot s = T_g(s) = T_h(s) = h \cdot s$$

This implies that:

$$int(s, T_g(s)) = int(s, T_h(s))$$

But intervals are unique by definition, so $g = h$.

We now have that TRANS gives rise to a simply transitive group action on S, so, by Theorem 3.16, G does so as well because it is anti-isomorphic to TRANS. □

**Theorem 3.18.** *Every simply transitive group action of a group $G$ on a set $S$ gives rise to a GIS.*

## 4. Hindemith

Turning now to an excerpt of *Ludus Tonalis* by composer Paul Hindemith, we shall examine the GIS constructed by the transposed and inverted forms of a repeated theme. But first, we must define inversion.

**Definition 4.1.** We define *inversion about* $n$ for some $n \in \mathbb{Z}_{12}$ as the function $I_n : \mathbb{Z}_{12} \longrightarrow \mathbb{Z}_{12}$ such that $I_n(x) = -x + n$.

**Definition 4.2.** A *pitch-class segment* or *pcseg* is an ordered subset of $\mathbb{Z}_{12}$, denoted $\langle x_1, x_2, \ldots, x_n \rangle$, which represents a series of musical pitch classes.

**Example 4.3.** At the end of the second measure of Hindemith's "Fuga Tertia in F" from *Ludus Tonalis* one finds the notes ⟨E♭,D,C♯,A⟩ in the treble clef (Note: this and other pcsegs referenced in examples can be found circled on the attached copy of "Fuga Tertia."). These notes form a pcseg, which is denoted $\langle 3, 2, 1, 9 \rangle$ in the standard $\mathbb{Z}_{12}$ notation described previously.

**Definition 4.4.** The *dihedral group of order* $2n$ (where $n \in \mathbb{N}$) is defined as $\langle \sigma, \tau |\ \sigma^n = 1,\ \tau^2 = 1,\ \tau\sigma\tau = \sigma^{-1} \rangle$.

**Lemma 4.5.** *The dihedral group of order 24 is isomorphic to the group of all transpositions $T_n$ and inversions $I_n$ where $n \in \mathbb{Z}_{12}$ and the group operation is function composition. The dihedral group shall therefore be denoted $T/I$ henceforward.*

*Proof.* Consider that $T/I = \langle T_1, I_0 \rangle$. Given GIS (S,G,int), then for all $s \in S$:

$$T_1^{12}(s) = (T_1 \circ T_1 \circ T_1 \circ T_1 \circ T_1 \circ T_1 \circ T_1 \circ T_1 \circ T_1 \circ T_1 \circ T_1 \circ T_1)(s) = s + 12 = s.$$

$$I_0^2(s) = (I_0 \circ I_0)(s) = -(-s) = s.$$

$$(I_0 \circ T_1 \circ I_0)(s) = -(-s+1) = s - 1 = T_1^{-1}(s).$$

□

*Remark* 4.6. Both $T_n$ and $I_n$ can be extended so that they map pitch-class segments to pitch-class segments by defining $T_n(\langle x_1, x_2, \ldots, x_m \rangle) = \langle T_n(x_1), T_n(x_2), \ldots, T_n(x_m) \rangle$ and $I_n(\langle x_1, x_2, \ldots, x_m \rangle) = \langle I_n(x_1), I_n(x_2), \ldots, I_n(x_m) \rangle$.

**Theorem 4.7.** *Given a pcseg $\langle x_1, x_2, \ldots, x_m \rangle$ which contains any interval not equal to 0 or 6, then $T/I$ acts simply transitively on the set $S$ of transposed and inverted forms of $\langle x_1, x_2, \ldots, x_m \rangle$.*

*Proof.* We define a group homomorphism $\rho : T/I \longrightarrow Sym(S)$ by $\rho(T_n)(\langle x_1, x_2, \ldots, x_m \rangle) = \langle T_n(x_1), T_n(x_2), \ldots, T_n(x_m) \rangle$ and $\rho(I_n)(\langle x_1, x_2, \ldots, x_m \rangle) = \langle I_n(x_1), I_n(x_2), \ldots, I_n(x_m) \rangle$. $T_n$ and $I_n$ are permutations of $S$ for all $n \in \mathbb{Z}_{12}$. Therefore, the dihedral group is a subgroup of $Sym(S)$ by Lemma 4.5. As a result, the embedding $id : T/I \longrightarrow Sym(S)$ is a group homomorphism because for all $g, h \in T/I$:

$$id(g * h) = id(g \circ h) = g \circ h = id(g) \circ id(h) = id(g) * id(h).$$

By Lemma 3.4, this is equivalent to a group action of $T/I$ on $S$.

This group action is transitive because $S$ was defined as the collection of transposed and inverted forms of a given pcseg, and $T/I$ is a group, so given $r, s \in S$, one may simply compose appropriate elements and inverse elements of $T/I$ to achieve $g \in T/I$ such that $g \cdot r = s$.

This group action is also free because $g \cdot s$ is obtained by adding some element of $\mathbb{Z}_{12}$ to $s$ or by adding $-s$ to some element of $\mathbb{Z}_{12}$. Therefore, the properties of addition tell us that only with the same element of $\mathbb{Z}_{12}$ will $g \cdot s = h \cdot s$. □

**Example 4.8.** Considering the same pcseg introduced in Example 4.3, its set of transposed and inverted forms is as follows:

| Transposed Forms | Inverted Forms |
|---|---|
| $\langle 3,2,1,9 \rangle$ | $\langle 9,10,11,3 \rangle$ |
| $\langle 4,3,2,10 \rangle$ | $\langle 10,11,12,4 \rangle$ |
| $\langle 5,4,3,11 \rangle$ | $\langle 11,0,1,5 \rangle$ |
| $\langle 6,5,4,0 \rangle$ | $\langle 0,1,2,6 \rangle$ |
| $\langle 7,6,5,1 \rangle$ | $\langle 1,2,3,7 \rangle$ |
| $\langle 8,7,6,2 \rangle$ | $\langle 2,3,4,8 \rangle$ |
| $\langle 9,8,7,3 \rangle$ | $\langle 3,4,5,9 \rangle$ |
| $\langle 10,9,8,4 \rangle$ | $\langle 4,5,6,10 \rangle$ |
| $\langle 11,10,9,5 \rangle$ | $\langle 5,6,7,11 \rangle$ |
| $\langle 0,11,10,6 \rangle$ | $\langle 6,7,8,0 \rangle$ |
| $\langle 1,0,11,7 \rangle$ | $\langle 7,8,9,1 \rangle$ |
| $\langle 2,1,0,8 \rangle$ | $\langle 8,9,10,2 \rangle$ |

**Definition 4.9.** Given $S$, the collection of transposed and inverted forms of a pcseg $\langle x_1, x_2, \ldots, x_n\rangle$, let $K : S \longrightarrow S$ be the function that maps a pcseg to its inverted form with the first two pitch classes reversed (i.e. $\langle x_2, x_1, \ldots, y_{n-1}, y_n\rangle$). This is known as a type of *contextual inversion*.

**Theorem 4.10.** *The contextual inversion, $K$, commutes with all $T/I$ operations* [1].

**Example 4.11.** Turning back to "Fuga Tertia in F," we find, in the treble clef of measure 25, the series of pitches $\langle F\sharp, G, A\flat, C\rangle$. In $\mathbb{Z}_{12}$ notation, this pcseg is denoted $\langle 6,7,8,0\rangle$, and the intervals within the pcseg are two minor seconds and a major third. These intervals appear again in the next two measures, where the pcsegs $\langle 9,10,11,3\rangle$ and $\langle 0,1,2,6\rangle$ appear. One can immediately see that these pcsegs are a minor third apart from one another. This means that repeated application of $T_3$ allows one to obtain the second and third pcsegs from the first:

$$\langle 6,7,8,0\rangle \xrightarrow{T_3} \langle 9,10,11,3\rangle \xrightarrow{T_3} \langle 0,1,2,6\rangle.$$

In addition to this example of transposition, one notices that beginning in the bass clef of measure 25 is the pcseg $\langle 7,6,5,1\rangle$. This pcseg happens to be $K(\langle 6,7,8,0\rangle)$, as one can see from the following table of transposed and inverted forms of $\langle 6,7,8,0\rangle$.

| Transposed Forms | Inverted Forms |
|---|---|
| $\langle 6,7,8,0\rangle$ | $\langle 6,5,4,0\rangle$ |
| $\langle 7,8,9,1\rangle$ | $\langle 7,6,5,1\rangle$ |
| $\langle 8,9,10,2\rangle$ | $\langle 8,7,6,2\rangle$ |
| $\langle 9,10,11,3\rangle$ | $\langle 9,8,7,3\rangle$ |
| $\langle 10,11,0,4\rangle$ | $\langle 10,9,8,4\rangle$ |
| $\langle 11,0,1,5\rangle$ | $\langle 11,10,9,5\rangle$ |
| $\langle 0,1,2,6\rangle$ | $\langle 0,11,10,6\rangle$ |
| $\langle 1,2,3,7\rangle$ | $\langle 1,0,11,7\rangle$ |
| $\langle 2,3,4,8\rangle$ | $\langle 2,1,0,8\rangle$ |
| $\langle 3,4,5,9\rangle$ | $\langle 3,2,1,9\rangle$ |
| $\langle 4,5,6,10\rangle$ | $\langle 4,3,2,10\rangle$ |
| $\langle 5,6,7,11\rangle$ | $\langle 5,4,3,11\rangle$ |

Hindemith also transposes this chord twice by a minor third; however, in this case he transposes down. This is equivalent to $T_{-3} = T_9$, and we get:

$$\langle 6,5,4,0\rangle \xrightarrow{T_9} \langle 3,2,1,9\rangle \xrightarrow{T_9} \langle 0,11,10,6\rangle\,.$$

In this example, Hindemith applies $K$ and then $T_9$ to $\langle 6,7,8,0\rangle$, but Theorem 4.10 tells use that, as the following diagram shows, these two functions can be applied in either order.

$$\begin{tikzcd}
\langle 6,7,8,0\rangle \arrow[r, "K"] \arrow[d, "T_9"'] & \langle 7,6,5,1\rangle \arrow[d, "T_9"] \\
\langle 3,4,5,9\rangle \arrow[r, "K"'] & \langle 4,3,2,10\rangle
\end{tikzcd}$$

**Definition 4.12.** Given $S$, the set of transposed and inverted forms of a pcseg $\langle x_1, x_2, \ldots, x_m\rangle$, define $K' : S \longrightarrow S$ such that $K'$ maps a pcseg to its inverted form with the last two pitch classes switched (i.e. $\langle y_1, y_2, \ldots, y_{m-2}, x_m, x_{m-1}\rangle$). This is an alternate type of *contextual inversion*.

**Theorem 4.13.** *The function $K'$ commutes with all $T/I$ operations* [1].

**Example 4.14.** Hindemith uses this second type of contextual inversion later on in "Fuga Tertia," when, in measure 32, the pcseg $\langle 7, 11, 0, 1\rangle$ appears. As in the previous example, Hindemith transposes the pcseg by a minor third to obtain the notes for the next two measures.

$$\langle 7,11,0,1\rangle \xrightarrow{T_3} \langle 10,2,3,4\rangle \xrightarrow{T_3} \langle 1,5,6,7\rangle$$

Hindemith also uses a contextual inversion to obtain a pcseg for use in the treble clef. In this case, the inversion is not $K$ but $K'$, which generates $\langle 6, 2, 1, 0\rangle$, as evidenced by the following table.

| Transposed Forms | Inverted Forms |
|---|---|
| $\langle 7,11,0,1\rangle$ | $\langle 5,1,0,11\rangle$ |
| $\langle 8,0,1,2\rangle$ | $\langle 6,2,1,0\rangle$ |
| $\langle 9,1,2,3\rangle$ | $\langle 7,3,2,1\rangle$ |
| $\langle 10,2,3,4\rangle$ | $\langle 8,4,3,2\rangle$ |
| $\langle 11,3,4,5\rangle$ | $\langle 9,5,4,3\rangle$ |
| $\langle 0,4,5,6\rangle$ | $\langle 10,6,5,4\rangle$ |
| $\langle 1,5,6,7\rangle$ | $\langle 11,7,6,5\rangle$ |
| $\langle 2,6,7,8\rangle$ | $\langle 0,8,7,6\rangle$ |
| $\langle 3,7,8,9\rangle$ | $\langle 1,9,8,7\rangle$ |
| $\langle 4,8,9,10\rangle$ | $\langle 2,10,9,8\rangle$ |
| $\langle 5,9,10,11\rangle$ | $\langle 3,11,10,9\rangle$ |
| $\langle 6,10,11,0\rangle$ | $\langle 4,0,11,10\rangle$ |

Again, this contextually inverted form is transposed twice by a major sixth over the course of measures 34 and 35.

$$\langle 6,2,1,0\rangle \xrightarrow{T_9} \langle 3,11,10,9\rangle \xrightarrow{T_9} \langle 0,8,7,6\rangle$$

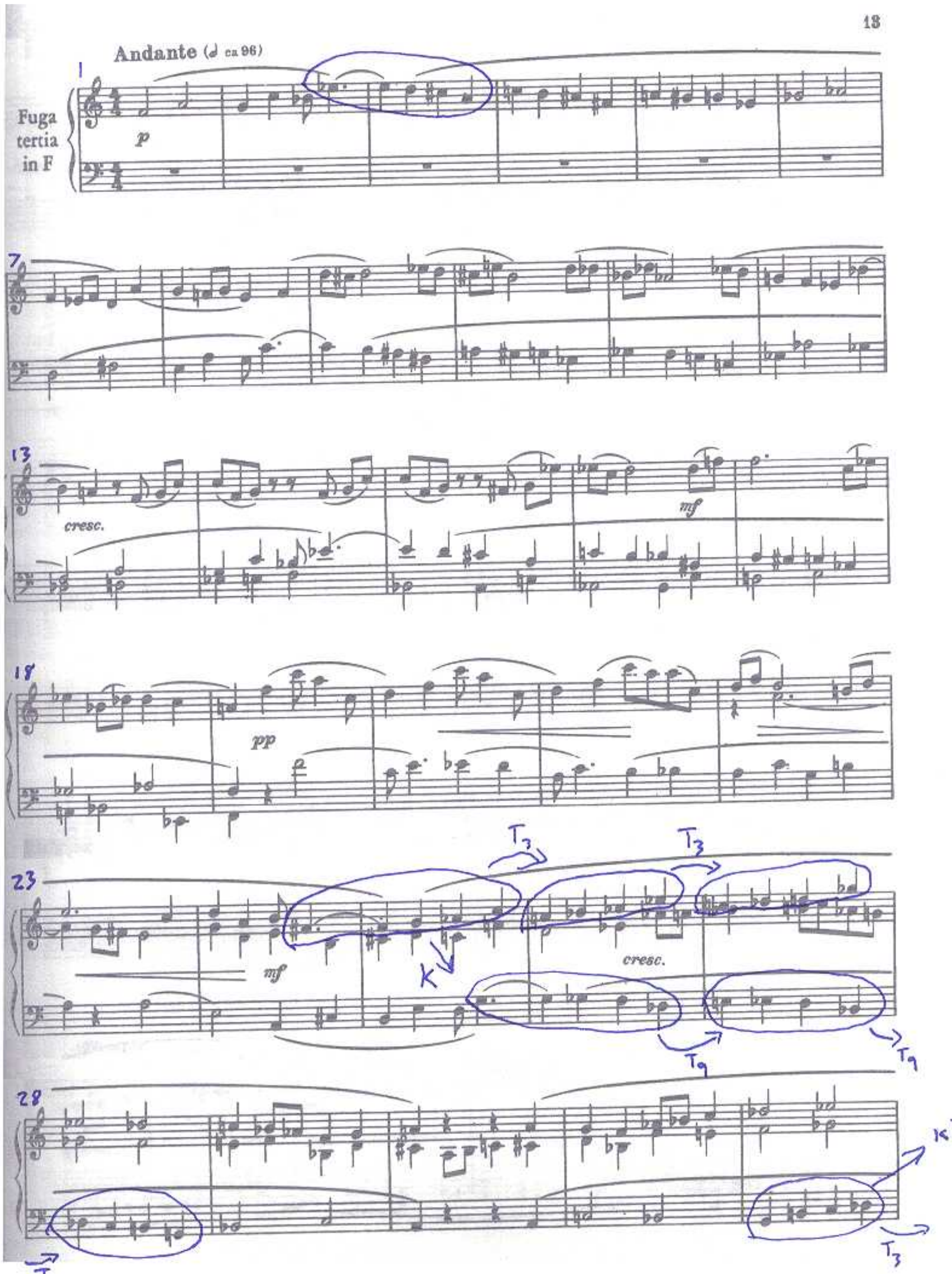
. In this case, as with $K$, we see that $K'$ commutes with $T_9$.

$$
\begin{tikzcd}
\langle 7,11,0,1\rangle \arrow[r, "T_9"] \arrow[d, "K'"'] & \langle 4,8,9,10\rangle \arrow[d, "K'"] \\
\langle 6,2,1,0\rangle \arrow[r, "T_9"'] & \langle 3,11,10,9\rangle
\end{tikzcd}
$$

It is important to note that while Examples 4.11 and 4.14 may be mathematically quite similar, by starting with a pcseg in the treble clef for Example 4.11 and in the bass clef for Example 4.14, Hindemith creates two very distinct sounds. In the former case, the pitches begin close together and move apart, whereas in the second example the pitches begin far away and move together. This difference creates two quite distinct musical effects.

## 5. Conclusion

Having thus explored the nature of the GIS, its equivalency to a group homomorphism, and the importance of the $T/I$ group in analyzing contextual inversion, one begins to understand the mathematical nature of music and the importance of the GIS. The implications of the GIS go far beyond those touched on here, however, and the interested scholar will find much to be learned in the work of Lewin and his contemporaries.


13
Andante (♩ ca 96)
Fuga tertia in F
p
cresc.
mf
pp
cresc.
T3
T9
K
K'


## 6. Acknowledgements

The following individuals and organizations have contributed significantly to the production of this paper:

- Thomas Fiore
- Masoud Kamgarpour
- The VIGRE program
- The University of Chicago
- The National Science Foundation

## References

[1] Thomas M. Fiore and Ramon Satyendra. Generalized contextual groups. *Music Theory Online*, 11(3), 2005, http://mto.societymusictheory.org.

[2] David Lewin. *Generalized Musical Intervals and Transformations*. Yale University Press, New Haven, 1987.