Geodesics: REU 2006

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Abstract

One of my main aims for the summer was to work into a subject with which I had no prior experience. This paper bears the fruits of that labor. I chose geodesics as a project because it's a particular aspect of geometry that I find interesting, and I believe that the theorem, while perhaps mainly useful as a tool for proving more complex theorems, is interesting itself. The aim of the paper is to introduce the reader to a few definitions, and move straight into the proof.

1 Definitions

Affine Connection 1 *An affine connection* ∇ *on a differentiable manifold M is a mapping*

$$\nabla : X(M) \times X(M) \longrightarrow X(M)$$

which is denoted by $(X,Y) \longrightarrow \nabla_X Y$ *and which satisfies the following properties:*

1. $\nabla_{fX+gY} Z = f\nabla_X Z + g\nabla_Y Z$
2. $\nabla_X (Y + Z) = \nabla_X Y + \nabla_X Z$
3. $\nabla_X (fY) = f\nabla_X Y + X(f)Y$

in which $X,Y,Z \in X(M)$ *and* $f, g \in D(M)$ *where* $X(M)$ *is the set of all vector fields of class* C^∞ *on* M *and* $D(M)$ *is the ring of real-valued functions of class* C^∞ *defined on* M*.*

Geodesic 1 *A parametrized curve* $\gamma : I \longrightarrow M$ *is a* geodesic at $t_0 \in$ I *if* $\frac{D}{dt}(\frac{d\gamma}{dt}) = 0$ *at the point* t_0*; if* γ *is a geodesic at* t*, for all* t $\in$ *I, we say that* γ *is a* geodesic*. If [a,b]* $\subset$ *I and* $\gamma : I \longrightarrow$ *is a geodesic, the restriction of* γ *to [a,b] is called a* geodesic segment joining $\gamma(a)$ *to* $\gamma(b)$*.*

Tangent Bundle 1 *A* tangent bundle TM on a manifold M *is the set of pairs* $(q, v), q \in M, v \in T_q M$*. If (U,*$\boldsymbol{x}$*) is a system of coordinates on* M*, then any vector in* $T_q M, q \in \boldsymbol{x}(U)$*, can be written as* $\sum_{i=1}^{n} y_i(\frac{\partial}{\partial x_i})$

Coderivative 1 We'll define *coderivative* by a proposition:
Let M *be a differentiable manifold with an affine connection* ∇*. There exists a unique correspondence which associates to a vector field* V *along the differentiable curve* $c : I \longrightarrow M$ *another vector field* $\frac{DV}{dt}$ *along c, called the covariant derivative of* V *along c, such that:*

1. $\frac{D}{dt}(V + W) = \frac{DV}{dt} + \frac{DW}{dt}$*, where* W *is a vector field along* c *and* f *is a differentiable function on* I*.*
2. $\frac{D}{dt}(fV) = \frac{df}{dt}V + f\frac{DV}{dt}$
3. *If* V *is induced by a vector field* $Y \in X(M)$*, i.e. V(t)=Y(c(t)), then* $\frac{DV}{dt} = \nabla_{\frac{dc}{dt}} Y$*.*

Proof. Let us suppose initially that there exists a correspondence satisfying items 1, 2, and 3 of the definition. Let $\mathbf{x} : U \subset \mathbb{R}^n \longrightarrow M$ be a system of coordinates with c(I) $\cap x(U) \neq \emptyset$ and let $(x_1(t), \ldots, x_n(t))$ be the local expression of c(t), t $\in I$. Let $X_i = \frac{\partial}{\partial x_i}$. Then we can express the field V locally as $V = \sum_j \upsilon^j X_j, j = 1, \ldots, n$ where $\upsilon^j = \upsilon^j(t) and X_j = X_j c(t)$. By properties 1 and 2, we have

$$\frac{DV}{dt} = \sum_j \frac{d\upsilon^j}{dt} X_j + \sum_j \upsilon^j \frac{DX_j}{dt}.$$

By property 3 and by the first property of the definition of affine connection,

$$\frac{DX_j}{dt} = \nabla_{\frac{dc}{dt}} X_J = \nabla_{\sum \frac{dx_i}{dt} X_i} X_j = \sum_i \frac{dx_i}{dt} \nabla_{X_i} X_j, \;\; i, j = 1, \ldots, n.$$

Therefore,

$$\frac{DV}{dt} = \sum_j \frac{d\upsilon^j}{dt} X_j + \sum_{i,j} \frac{dx_i}{dt} \upsilon^j \nabla_{X_i} X_j.$$

The above expression shows that if there is a correspondence that satisfies the conditions given, then such a correspondence is unique. To show existence, define $\frac{DV}{dt}$ in $x(U)$ as above. If $y(W)$ is another coordinate neighborhood, with $y(W) \cap x(U) \neq \emptyset$ and we define $\frac{DV}{dt}$ in $y(W)$ as above, then the definitions agree in $y(W) \cap x(U)$, by the uniqueness of $\frac{DV}{dt}$ in $x(U)$. It follows from this fact that the definition can be extended over all of M, and this concludes the proof. □

2 Theorems

Proposition 1 *Given $p \in M$, there exist an open set $V \subset M, p \in V$, numbers $\delta > 0$ and $\varepsilon_1 > 0$ and a C^∞ mapping*

$$\gamma : (-\delta, \delta) \times U \longrightarrow M, U = \{(q, \upsilon); q \in V, \upsilon \in T_q M, |\upsilon| < \varepsilon_1\}$$

such that the curve $t \longrightarrow \gamma(t, q, \upsilon), t \in (-\delta, \delta)$ is the unique geodesic of M which, at the instant t=0, passes through q with velocity υ, for each $q \in V$ and for each $\upsilon \in T_q M with |\upsilon| < \varepsilon_1$.

This proposition asserts that if $|\upsilon| < \varepsilon_1$, then the geodesic exists in some interval and is unique. Its proof follows from the following lemma.

Lemma: Homogeneity of a geodesic 1 *If the geodesic $\gamma(t, q, \upsilon)$ is defined on the interval $(-\delta, \delta)$, then the geodesic $\gamma(t, q, a\upsilon), a \in \mathbb{R}, a > 0$, is defined on the interval $(-\frac{\delta}{a}, \frac{\delta}{a})$ and*

$$\gamma(t, q, a\upsilon) = \gamma(at, q, \upsilon).$$

Proof. Let $h : (-\frac{\delta}{a}, \frac{\delta}{a}) \longrightarrow M$ be a curve given by h(t) $= \gamma(at, q, \upsilon)$. Then h(0) $= q$ and $\frac{dh}{dt}(0) = a\upsilon$. In addition, since $h'(t) = a\gamma'(at, q, \upsilon)$,

$$\frac{D}{dt}(\frac{dh}{dt}) = \nabla_{h'(t)} h'(t) = a^2 \nabla_{\gamma'(at,q,\upsilon)} \gamma'(at, q, \upsilon) = 0,$$

where, for the first equality, we extend *h'(t)* to a neighborhood of *h(t)* in M. Therefore, h is a geodesic passing through q with velocity $a\upsilon$ at the instant $t = 0$. By uniqueness,

$$h(t) = \gamma(at, q, \upsilon) = \gamma(t, q, a\upsilon).\square$$

Proposition 1 with the above lemma allows us to extend the definition of a geodesic from that of an interval to a uniformly large neighborhood of p.

Proposition 2 *Given $p \in M$, there exist a neighborhood V of p in M, a number $\varepsilon > 0$ and a $\mathcal{C}^\infty$ mapping $\gamma : (-2,2) \times \mathcal{U} \longrightarrow M, \mathcal{U} = \{(q,w) \in TM; q \in V, w \in T_qM, |w| < \varepsilon\}$ such that $t \longrightarrow \gamma(t,q,w), t \in (-2,2)$, is the unique geodesic of M which, at the instant t=0, passes through q with velocity w, for every $q \in V$ and for every $w \in T_qM$, with $|w| < \varepsilon$.*

Proof. The geodesic $\gamma(t,q,v)$ in Proposition 1 is defined for $|t| < \delta$ and for $|v| < \varepsilon_1$. From the lemma of homogeneity, $\gamma(t,q,\frac{\delta v}{2})$ is defined for $|t| < 2$. Taking $\varepsilon < \frac{\delta \varepsilon_1}{2}$, we obtain that the geodesic $\gamma(t,q,w)$ is defined for $|t| < 2$ and $|w| < \varepsilon$.□