ROTATIONS, ROTATION PATHS, AND QUANTUM SPIN

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1. ABSTRACT

This paper describes the construction of the universal covering group $Spin(n)$, $n > 2$, as a group of homotopy classes of paths starting from the identity in $O(n)$ to a reader unfamiliar with the theory of Lie groups, and outlines an explicit connection between rotation paths in $O(3)$, the physical meaning of $SO(3)$, and quantum mechanics.

2. PRELIMINARIES

I assume some familiarity with some aspects of the classical groups $O(n,\mathbb{R})$ and $SO(n,\mathbb{R})$, particularly their aspects as matrix groups. I also assume familiarity with basic topological concepts such as paths and homotopy.

Through this paper I make occasional references to the "physical world." It is not clear, even to physicists, what the physical world actually is. For simplicity's sake, I assume the physical world is $\mathbb{R}^3$. This is the subspace of all spatial dimensions in the special relativistic description of space-time (as opposed to the time dimension or other hypothesized dimensions), and the study of $\mathbb{R}^3$ suffices to explain almost all physical phenomena that are not on an extremely large or extremely small (sub-quantum) scale, and where relativistic effects are not present.

The notion of what it means to rotate something in $\mathbb{R}^3$, or any other metric space, can be rigorously defined. Physical rotation is *not* the same as applying an automorphism in the rotation group. If a physical object is rotated, it does not instantly take on its new configuration, but rather passes through a continuous series of intermediate configurations. Furthermore, there exist some objects that are not invariant under a rotation of 2π. To give a mathematically intuitive description of physical rotation, I introduce the notion of a rotation path, which is a path in $O(3)$, and introduce some formalism on rigid bodies and rotations applied continuously to them. I conclude by outlining a link between rotation paths and quantum physics.

I have sometimes seen the notion of a rotation path used to explain the existence of the group Spin(3), but I am unsure if they have been explicitly linked to physics as they are in this paper.

3. ROTATIONS IN SPACE

To talk about paths in $O(3)$, it must first be established that it has a topology.

Definition 3.1. A *Lie group* is a group that is also a smooth manifold, and in which group operations are a smooth map.

Lie groups have important properties, both as topological spaces and as groups.

Definition 3.2. The *general linear group* $GL_n(\mathbb{R})$, abbreviated as $GL(n)$, is the group of all automorphisms on the real finite-dimensional vector space $\mathbb{R}^n$.

Since an invertible $n \times n$ matrix is an automorphism on the vector space of $n \times 1$ column vectors, $GL(n)$ is isomorphic to the group of $n \times n$ matrices with nonzero determinant.

Definition 3.3. The *orthogonal group* $O(n)$ is the set of all automorphisms of $\mathbb{R}^n$ that preserve the Euclidean metric.

Theorem 3.4. *$GL(n)$ is a Lie group that is not connected.*

Proof. The set $M_n(\mathbb{R})$ of all $n \times n$ matrices is a real vector space of dimension n^2. The standard Euclidean metric gives a smooth topology on $\mathbb{R}^{n^2}$, and this topology therefore extends naturally to $M_n(\mathbb{R})$. Matrix multiplication is a polynomial map from $\mathbb{R}^{n^2}$ to itself, and inversion is a continuous rational map whenever a matrix is invertible, and so $GL(n)$ is Lie. Finally, the determinant map

$$\det : GL(n) \to \mathbb{R}$$

is a homomorphism of groups and a continuous map from $GL(n)$ to $\mathbb{R}^\times$, a manifold with two connected components. Therefore $GL(n)$ itself consists of at least two connected components. □

Theorem 3.5. *$O(n)$ is a Lie group that is not connected.*

As before, consider $O(n)$ as a matrix group acting on column vectors. The only matrices that preserve the Euclidean metric are those with determinant ± 1, so the determinant map is again a continuous map to a (discrete) topological space. It then follows, as with $GL(n)$, that $O(n)$ is not connected.

In fact, $GL(n)$ and $O(n)$ have precisely two connected components, but this does not hold in generalized orthogonal groups when the Euclidean metric is replaced. For example, the space-time metric in special relativity causes the orthogonal group to have four components [1].

Definition 3.6. The *special orthogonal group*, $SO(n)$, is the identity component of $O(n)$.

$SO(3)$ is sometimes called the rotation group, and elements of $SO(3)$ are sometimes called rotations.

4. Rotation paths

Definition 4.1. A *body* in $\mathbb{R}^n$is a set of (distinct) points $\mathcal{B}$ together with a set of functions $\mathcal{C} \subset \{f : \mathcal{B} \to \mathbb{R}^n\}$. $\mathcal{C}$ is called the *configuration space* of $\mathcal{B}$. The elements of $\mathcal{C}$ are called *configurations* of $\mathcal{B}$.

The structure of a body is determined by its configuration space.

Definition 4.2. A *rigid body* $\mathcal{B}$ is a body such that, for any two $f, g \in \mathcal{C}$ and any two points $\mathbf{b}_1, \mathbf{b}_2 \in \mathcal{B}$, $|f(\mathbf{b}_2) - f(\mathbf{b}_1)| = |g(\mathbf{b}_2) - g(\mathbf{b}_1)|$.

A rigid body can be characterized physically as a set of points in space such that the distance between any two points is fixed. In basic three-dimensional classical mechanics; for example, all objects are assumed to be rigid bodies in $\mathbb{R}^3$, typically with configuration space $SO(3)$.

Definition 4.3. A *rigid body with fixed point* $\mathbf{b}_o$ is a rigid body $\mathcal{B}$ such that, for for any two $f, g \in \mathcal{C}$, $f(\mathbf{b}_0) = g(\mathbf{b}_0)$. An *initial configuration* is a fixed, possibly arbitrary, configuration$I \in \mathcal{C}$.

Sometimes one speaks of an automorphism f:$\mathbb{R}^n \to \mathbb{R}^n$ used as a configuration. Here it is meant that the configuration is f acting on some initial configuration$I$, giving the configuration $f \circ I : B \xrightarrow{I} \mathbb{R}^3 \xrightarrow{f} \mathbb{R}^3$. If the configuration space is a group of automorphisms on $\mathbb{R}^n$, then I is the identity automorphism. This abuse of notation will be used throughout this paper.

A wheel on a fixed axle is a rigid body with a fixed point. Also, the Euclidean space$\mathbb{R}^3$ together with its isometric automorphisms is a rigid body, with fixed point $\mathbf{0}$ and configuration space $O(3)$.

When is it that the configuration space of a rigid body is $O(3)$?

Theorem 4.4. *For a rigid body $\mathcal{B}$ in $\mathbb{R}^n$, with fixed point the origin, if the configuration $I(\mathcal{B})$ contains a basis, the configuration space of a rigid body with a fixed point is a subset of $O(n)$. The largest possible configuration space is precisely$O(n)$.*

This follows from the fact that $O(n)$ is the set of all isometric automorphisms of $\mathbb{R}^n$. To see every element in $O(n)$ is a unique configuration: Any isometric automorphism (in fact, any automorphism at all) of $\mathbb{R}^n$ is determined completely by its action on a basis, and thus determined completely by its action on $I(\mathcal{B})$. Therefore, there can be no two elements of $O(n)$ that give the same configuration of $\mathcal{B}$.

Configuring a rigid body fixing a point $\mathbf{b}_o$ is equivalent to choosing the origin $\mathbf{0} = \mathbf{b}_o$ and then applying an isometric configuration in $O(n)$. The same result can be achieved by translating $\mathbf{b}_o$ to $\mathbf{0}$, rotating, and then translating it back to $\mathbf{b}_o$.

Corollary 4.5. *For a rigid body with fixed point $\mathbf{b}_o$, the configuration space is a subset of the set $\{f : \mathcal{B} \to \mathbb{R}^n \mid f(\mathbf{b}) = \mathbf{b}_o + O(I(\mathbf{b}) - \mathbf{b}_o)\}$.*

The reader might notice that in these definitions, I am allowing as configurations the improper rotations in $O(n)$, that is, elements that are not in $SO(n)$, to be configurations of rigid bodies. However, I will show that only rotations in $SO(n)$ are, in a sense, physically valid configurations of rigid bodies.

Definition 4.6. A (proper) *rotation* in $\mathbb{R}^n$ is an element of $SO(n)$. $SO(3)$ (or more generally $SO(n)$) is sometimes called the *rotation group.*

This definition would seem to completely describe what it means to rotate a rigid body in space. $SO(n)$ contains all oriented isometric automorphisms of $\mathbb{R}^n$. Therefore, the spatial configuration of a rigid body with a fixed point in $\mathbb{R}^3$ can completely be described in terms of a fixed configuration rotated by $SO(3)$.

However, there are two problems with this. The first is that it does not adequately explain why non-oriented isometries of physical space, the so-called improper rotations that are elements of $O(3)$ but not $SO(3)$, are not allowed in physics. The second is that it is sometimes noted in math and science texts that a "rotation" of 2π is not homotopic to the identity, and this is sometimes cited as a somewhat unconvincing justification for the fact that the spin of quantum particles change sign under a rotation of 2π. However, it makes no sense to speak of homotopies of rotations since rotations are points, not paths.

This calls for a new definition. If one were to pick up, say, a pencil, and rotate it in space, the pencil's direction varies continuously from its initial configuration to its final configuration. The same is true of any other rotation. There appears to be no such thing as "instantly" rotating something; to rotate an object from one configuration to another, one must continuously pass it through a path of intermediate rotations.

Definition 4.7. A *rotation path* is a (continuous) path in $O(n)$ starting from the identity.

It immediately follows that a rotation path must be contained in $SO(n)$, the identity component of $O(n)$.

Lemma 4.8. *Choose a basis B in $\mathbb{R}^n$. A map $\gamma : [0,1] \to O(n)$ is a (continuous) rotation path if, and only if, for all $\mathbf{x} \in B$, the function $\gamma_{\mathbf{x}} : [0,1] \to \mathbb{R}^n$ defined by $\gamma_{\mathbf{x}}(t) \mapsto \gamma(t)(\mathbf{x})$ is a path.*

Proof. From the above isomorphism between automorphisms of real vector spaces and $n \times n$ matrices acting on column vectors, we see that the action of a matrix on a vector is a polynomial map. Fix any basis $B = (\mathbf{x}_1 \cdots \mathbf{x}_n)$ and let $\iota_B : O(n) \to M_n(\mathbb{R})$ be an isomorphism of vector spaces mapping elements in $O(n)$ to matrices in $M_n(\mathbb{R})$ with respect to the basis B. Consider the function $\gamma\prime = \iota_B \circ \gamma$, which is simply γ in matrix form relative to the basis B. Specifically, consider the i-th component of the action of $\gamma\prime$ on the column vector x_j. The action of $\gamma\prime$ on the vector x_j is continuous if and only if it is continuous for each component of x_j. each of which is determined completely by $\gamma\prime_{i,j}$. (Specifically, $\gamma\prime_{i,j}$ is the function defined by $\gamma\prime_{i,j}(t) = (\gamma\prime(t))_{i,j}$.) This is true if and only if $\gamma\prime$ must be a continuous path in $M_n(\mathbb{R})$, which is true if and only if γ is a continuous path in $GL(n)$ (and thus $O(n)$ by definition of continuous rotation path). □

The next theorem follows immediately:

Theorem 4.9. *For a rigid body $\mathcal{B}$ with fixed point the origin and initial configuration I, consider a function of the form $f : \mathcal{B} \times [0,1] \to \mathbb{R}^n$. If, for any set S in $\mathcal{B}$ such that $I(S)$ is a basis, the restriction of f to any $x \in I(S)$ is a continuous path in $\mathbb{R}^n$, then $f(\mathcal{B},t) = \gamma(t)(I(\mathcal{B}))$ where $\gamma : [0,1] \to O(n)$ is a rotation path.*

This gives the physical meaning of a rotation path.

It is easy to see that the effect of any rotation path on a rigid body can be characterized in terms of its endpoint in $SO(n)$. The purpose of introducing rotation paths thus kills two birds with one stone: It explains why improper rotations in $O(n)$ are not physically allowed, and describes all the ways to continuously rotate an object, or $\mathbb{R}^n$.

Rotation paths can be formed into a group in several ways. The most useful such formulation is concerned not with the paths themselves but rather their homotopy classes.

5. THE UNIVERSAL COVERING GROUP Spin(n)

Definition 5.1. For a topological space X, a (topological) *cover* is a topological space C, the *covering space*, together with a continuous map $\varphi : C \to X$, the *covering map*, such that for each $x \in X$ there exists an open neighborhood of x, $N(x)$, such that each component of $\varphi^{-1}(N(x))$ is mapped homeomorphically onto $N(x)$. A topological space is said to cover another if a covering map exists from the former to the latter. A cover is a *double cover* if each fiber over $x \in X$ has cardinality 2. For $n \in \mathbb{Z}^+$, a cover is an *n-cover* if each fiber has cardinality n.

The cardinality of every fiber over x is the same for all x. A cover is like a homeomorphism, where the condition that $\varphi^{-1}(x)$ be a continuous bijection with the weaker condition that $\varphi^{-1}(x)$ be the union of several continuous bijections in a neighborhood of x. All covering maps are locally homeomorphic.

Definition 5.2. For a topological space X, the *universal cover* $\varphi : U \to X$ is a cover such that, for any cover $g : D \to X$, there exists a cover $f : U \to D$ such that $g \circ f = \varphi$. The universal cover, if it exists, is unique up to homotpy.

Example 5.3. For any $n \in \mathbb{Z}^+$, the map $p_n : \mathbb{C}^\times \to \mathbb{C}^\times$ given by $p_n(z) = z^n$ is an n-fold cover. For the circle group $\mathbb{T}$, the map $\varphi : \mathbb{R} \to \mathbb{T}$, $\varphi(x) \mapsto \cos(x) + i\sin(x))$, is the universal cover.

The existence of the universal cover is given by construction.

Theorem 5.4. *For any compact manifold X, and a fixed point x_o, let U be the set of all homotopy classes $[\gamma]$, where $\gamma$is any path with $\gamma(0) = x_o$. Then U is the universal cover of X.*

We can construct a base on this topology as follows. For a homotopy class of paths $[\gamma] \in U$ and an open neighborhood $D \in X$ where $\gamma(1) \in D$, define $[\gamma, D]$ to be the set of all paths homotopic to any path formed by γ concatenated to some path contained entirely in D. Roughly, two paths $\alpha$and β are close to each other if $\alpha(1)$ is close to $\beta(1)$. This construction is rather clearly a base; a rigirous proof can be found in [3].[1]

The universal cover is unique up to homeomorphism and simply connected. An intuitive justification for this is given by the fact that, for any path in X , its endpoint can be smoothly moved backwards along the path until it reaches its starting point. Its uniqueness follows from the fact that a local homeomorphism of simply connected groups must be a global homeomorphism as well. Proofs can be found in [3].

Theorem 5.5. *If X is a connected Lie group, then so is its universal cover, U. U is then called the universal covering group, and can be constructed with the above construction with $x_o = e$, the identity. Multiplication in U is defined by $[\alpha][\beta] = [\alpha\beta]$ where $(\alpha\beta)(x) = \alpha(x)\beta(x)$.*

The proof that U is Lie rests on the fact that smooth operations on elements of X are also smooth on the points of paths in U.[2]

The universal covering group U is unique up to an isomorphism of Lie groups. (Kolk, p.62-68) gives an alternate construction of the universal covering group by means of quotients of path groups, together with a proof that this is isomorphic as a Lie group to the group I just described, and a proof of uniqueness.

Note that the above construction does not work for X not connected. However, there does exist a universal cover for most topological spaces–any path-connected, locally connected, semi-locally simply connected space has a universal cover. Though the universal cover is unique to homeomorphism, the universal covering group of a non-connected Lie group is generally not unique up to isomorphism.

The universal cover of $SO(n)$ gives, up to homotopy, all rotation paths in $O(n)$. Furthermore, it defines a group operation on these paths.

Definition 5.6. For $n > 2$, the *spin group* $\text{Spin}(n)$ is defined as the universal cover of $SO(n)$.

$\text{Spin}(n)$ is always a double cover when $n > 2$ [1]. The restriction $n > 2$ is necessary because, in 1 and 2 dimensions, $\text{Spin}(n)$ is defined as a double cover, but it is not the universal cover.

Finally, we consider the topology of $SO(3)$ and $\text{Spin}(3)$.

Definition 5.7. The real projective space, RP^n, is any space homeomorphic to the space formed by taking the usual n-sphere, $\{\mathbf{x} \in \mathbb{R}^{n+1} \mid |\mathbf{x}| = 1\}$, and identifying antipodal points.

[1]The cited text deals with Riemann surfaces, but its discussion of the universal cover holds for all manifolds, often by simply substituting “manifold” for “surface”.

Theorem 5.8. *$SO(3)$is diffeomorphic to the real projective space RP^3.*

Proof. Every rotation $x \in SO(3)$ has an eigenvector with eigenvalue 1 in $\mathbb{R}^3$. As x must preserve magnitude of vectors orientation of vector triples, it can therefore be completely described in terms of this eigenvector and the angle by which vectors orthogonal to it are rotated. Consider a closed 3-disc, D, with radius π. We then have a map $D \to SO(3)$, sending the vector $\mathbf{z}$ to a rotation with $\mathbf{z}$ as its fixed eigenvector and $|\mathbf{z}|$ its magnitude of rotation, and sending $\mathbf{0}$ to the identity. This map sends ∂D to $\mathbf{0}$ and identifies each vector with its antipode. All 3-dimensional smooth manifolds homeomorphic to each other are also diffeomorphic [6], so $SO(3) \simeq RP^3$. □

Theorem 5.9. *The universal covering group of $SO(3)$,* Spin(3)*, is a double cover diffeomorphic to S^3.*

Proof. This follows from the diffeomorphism $SO(3) \simeq RP^3$, and the fact that S^3 is a simply connected cover of RP^3. □

It is interesting to note that, as I write this (8/15/06), the New York Times is publishing an article announced that mathematicians claim to have completed and verified the proof of the Poincare conjecture, which suffices to prove that $\mathrm{Spin}(3) \simeq S^3$.

The construction of the universal cover gives $\mathrm{Spin}(n)$ as the group of all rotation paths, up to homotopy, of a rigid body with a point fixed. It follows that $\mathrm{Spin}(n)$ is the group of all continuous rotation paths, up to homotopy, of a rigid body with a point fixed. So we have a complete description, up to homotopy and up to a fixed point, of all the ways to rotate a rigid body, bundled up in the group $\mathrm{Spin}(n)$. Since $\mathbb{R}^n$ itself can be considered to be a rigid body, we also have all the ways, up to homotopy, to rotate $\mathbb{R}^n$ about the origin.

6. SPIN

I conclude by making explicit the connection between the spin group, constructed as a group of rotation paths, and quantum physics.

In 1927 Wolfgang Pauli formulated the theory of so-called spin $1/2$ particles in $\mathbb{R}^3$. Paul Dirac, in 1928, extended the theory to relativistic space-time.

In classical mechanics, a rotating object possess an angular momentum vector. If one imagines ones self looking at the vector head-on, the object spins counter-clockwise about this vector, and the magnitude of the vector is equal to the object's angular momentum. For example, if this page were spinning counterclockwise, its angular momentum vector would be pointed at you. Angular momentum is proportional to both the speed of the rotation and the rotational inertia of the object.. Rotations in $SO(3)$ act upon this vector just as they act upon any other object in physical space.

In quantum mechanics in $\mathbb{R}^3$, a spinning[2] object posses instead a spinor. The introduction of spinors, rather than vectors, to describe angular momentum, is motivated by the fact that one simply cannot have a particle literally spinning along some axis in quantum mechanics. First, the uncertainty principle demands that the angular momentum of a particle not have a definite direction. Second, quantum particles are believed to be point particles with no diameter.

I state a series of definitions and facts that explain the connection between rotation paths, spin and spinors. These facts should be familiar to those who have studied quantum mechanics. Those not so fortunate can verify this information in texts such as [5].

[2]Spinning is just an adjective. Quantum particles, as far as physicists know, do not rotate at all, and posess an angular momentum intrinsic to the particle itself.

Definition 6.1. A *spinor* is a complex 2×1 column vector, denoted $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$.

Spinors form a 2-dimensional complex vector space.

Fact 6.2. *Spinors form a representation of the group* $\mathrm{Spin}(3)$.

This is already an important connection between $\mathrm{Spin}(3)$ and physics, but not the connection we are aiming for.

Definition 6.3. The *anuglar momentum vector* of a particle is the vector such that, when the component of angular momentum along that vector is measured, the result is always $\hbar/2$, where $\hbar$ is the important physical constant Planck's constant.

It may not be clear what I mean by measuring a component of angular momentum. A classical particle rotating has three components of angular momentum, which are the components of the angular momentum vector in some orthonormal basis. The uncertainty principle demands that certain components of angular momentum of a quantum spinning particle not have a definite value. The anuglar momentum vector of a quantum particle is the component of angular momentum that always has a definite (positive) value.

Fact 6.4. *There exists a 2:1 continuous map sending normed spinors to angular momentum vectors.*

This map is typically described using the Pauli matricies, which can be used to generate $\mathrm{Spin}(3) \simeq SU(2)$. $SU(2)$ is the complex matrix group analogous to $SO(2)$, and is the group generated by all matricies X such that $\det(X) = 1$ and $X^{t*} = X^{-1}$, where X^{t*} is the transpose conjugate of X.

Fact 6.5. *The element in* $\mathrm{Spin}(3)$ *that is a rotation of* 2π *flips the sign of a spinor, but leaves the angular momentum vector invariant.*

It now immediately follows:

Theorem 6.6. *Action on spinors is described completely by* $\mathrm{Spin}(3)$*, the group of rotation paths in* $O(3)$*. Furthermore, each rotation path in* $\mathrm{Spin}(3)$ *has precisely the same effect on a particle's angular momentum vector as it does on the angular momentum vector of a classically spinning object.*

The spin group, $\mathrm{Spin}(3)$, considered as homotopy classes of rotation paths, elegantly describes mathematically the behavior of spin $1/2$ particles under rotation.

References

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