ABSTRACTS: 2007 SUMMER VIGRE REU

All concepts mentioned in the abstracts will be carefully defined

APPRENTICE PROGRAM

Apprentices attend courses offered in the full program during week 1. They are also welcome to attend such courses in later weeks. Several of the courses in the full program are designed to be accessible to apprentice participants.

Weeks 2–4: Laci Babai

NUMBER THEORY AND LINEAR ALGEBRA

Fun topics and surprising interconnections that are not usually discussed in algebra classes will be highlighted while introducing these basic fields. More details will be posted later. Attendance at week 1 of the "Discrete Mathematics" program is recommended.

FULL PROGRAM

1. Laci Babai. Discrete mathematics. Weeks 1–8.

2a. Gregory Lawler. Random walks and Brownian motion. Weeks 1–2.

2b. Robert Fefferman. When does a Fourier series converge? Weeks 3–4.

2c. Thomas Fiore. Mathematics and music. Weeks 5–6.

3a. Benson Farb. Mapping class groups. Weeks 1–4.

3b. Uri Bader. Groups of two by two matrices. Weeks 5–6.

3c. Matthew Kerr. Toric geometry. Weeks 7–8.

4a. Peter May. Surfaces, categories, algebras, and 2D TQFT'S. Weeks 1–4.

4b. Vigleik Angelveik. Products on topological spaces. Weeks 5–8.

1. DISCRETE MATHEMATICS

Weeks 1 – 8: Laci Babai

The course will be divided into two separate modules with very little overlap. Details will be posted shortly.

Weeks 1 – 4. NUMBERS, GROUPS, AND GRAPHS

Weeks 5 – 8. FINITE AND TRANSFINITE COMBINATORICS:
The world of Paul Erdős

2a. RANDOM WALKS AND BROWNIAN MOTION

Weeks 1 – 2: Gregory Lawler

An introduction to random processes including some areas which are active research areas today. Topics will come from the following:

— Random walkers on an integer grid. Does one return infinitely often? Stirling's formula. Central limit theorem.
— Self-avoiding walks
— What is a Markov chain?
— Using Markov chains as a simulation tool
— Brownian motion as a limit of random walk.
— Properties of Brownian motion (e.g., nondifferentiability)
— Relationship between random walks / Brownian motion and some partial difference/differential equations such as the heat equation.

I will not assume that students have had a course in probability, but such knowledge will be useful.

Most of my lectures will come from my books:
G. Lawler, Lectures on Contemporary Probability (AMS)
G. Lawler, Introduction to Stochastic Processes (CRC) (Chapter 8)

2b. WHEN DOES A FOURIER SERIES CONVERGE?

Weeks 3 – 4: Robert Fefferman

This will be a set of elementary lectures based on the idea of determining when a Fourier series converges. Centuries ago the French mathematician Fourier wrote down a "proof" that every periodic function could be written as an infinite sum of sine and cosine functions. The proof was non-rigorous to say the least, and it is a very interesting, important, and surprising investigation that one becomes involved in when one asks when and in which precise senses Fourier was right (and when he was wrong). We will carry this out with a minimum of background assumed (pretty much just a very good grasp of Calculus).

2c. MUSIC AND MATHEMATICS

Weeks 5 – 6: Thomas Fiore

Have you ever wondered if the relationship between mathematics and music extends to the composition and interpretation of musical works? If you have, then this course is for you. If you already know the answer because you participated last year, then this course is still for you because we will have new material.

A central concern of music theory is to find a good way of hearing a piece of music and to communicate that way of hearing. Music theorists often draw upon mathematics to create conceptual categories towards this end. In recent years, basic tools from group theory, combinatorics, and topology have entered the realm of musical analysis.

We will discuss two currents in modern mathematical music theory: David Lewin's Transformational Theory and Guerino Mazzola's Topos of Triads. Lewin's theory asks: which transformations are idiomatic for a work of music? For example, any fugue contains transpositions and inversions of the subject, and recognizing this pattern makes a fugue more enjoyable for both listener and performer. Another instance of a transformation assigns to a major chord its relative minor, something that we hear on the radio every day. Lewin applied the theory of groups and group actions to great effect in his analyses, and we will recount how in various musical examples. Group theory is used both as a language and as a vehicle for musical insight.

Mazzola's Topos of Triads, on the other hand, builds on Grothendieck's notion of topos. Topos theory has found application in algebraic geometry, logic, and category theory. It is a great surprise that such a sophisticated notion has anything to do with music at all! We will attempt to understand this pioneering work in elementary terms.

Familiarity with any notions mentioned above is not necessary for this module. Nor is the ability to read music a prerequisite, since we aim to see and hear mathematics in action.

3a. MAPPING CLASS GROUPS

Weeks 1 – 4: Benson Farb

The mapping class group of a surface is the group of self-homeomorphisms modulo homotopy (=continuous deformation). The study of these groups lies at the intersection of geometric/combinatorial group theory, low-dimensional topology, complex analysis, and algebraic geometry.

This course will be an introduction to the basics of the theory. The course will be taught at a number of levels at once, and will hopefully be interesting from the freshmen through graduate student level. A number of accessible open questions will be presented.

Prerequisites: The basics of point-set topology and group theory. Knowledge of fundamental group would be useful, but is not essential.

Selected topics: surface topology, Alexander trick, Dehn twists, the classical modular group, braid groups, the complex of curves, the finite generation theorem, the lantern and chain relations, Hurwitz's formula and the 84(g-1) theorem, the Dehn-Nielsen-Baer theorem, the symplectic representation. We will also try to indicate relationships with 3-manifold theory and symplectic geometry.

3b. GROUPS OF TWO BY TWO MATRICES

Weeks 5 – 6: Uri Bader

Some objects in mathematics are so fundamental that they are, surprisingly or not, to be found all over the place. We will discuss one such: The group of invertible two by two matrices, along with some of its subgroups. Our journey will take us from geometry through arithmetic to dynamics and back to geometry, viewing along the way concepts such as

— Hyperbolic geometry,
— Conformal dynamics,
— Galois groups,
— The space of lattices in R^2,
— Continuous fractions, Surfaces and their covers

and many more, all in a rather elementary way.

3c. TORIC GEOMETRY

Weeks 7 – 8: Matthew Kerr

This will be a hands-on introduction to several concepts in complex algebraic geometry, including rational functions, divisors, resolutions of singularities, and (as an application at the end) families of elliptic curves and elliptic fibrations. We will also say something about mirror symmetry in physics. We will work almost entirely in terms of examples.

An algebraic variety X is the solution set of some algebraic equations. It is usually looked at dually in terms of (i) the underlying complex manifold, and (ii) the commutative ring consisting (for example) of regular algebraic functions on X.

In the special case where X is a toric variety, we also think of it as a fiber bundle with multidimensional torus fibers (think circle, surface of donut, etc.) over a polytope. The combinatorics of the polytope provide a visual means to understand (i), (ii), and their interplay.

4a. SURFACES, CATEGORIES, ALGEBRAS, AND 2D TQFT'S

Weeks 1–4: Peter May

In an influential philosophical paper, Sir Michael Atiyah axiomatized "Topological Quantum Field Theories". While motivated by physics, these are really quite elementary mathematical structures. However, although they are easy to define, they are very hard to construct rigorously in dimensions greater than 2. However, in dimension 2, even their construction is elementary.

The idea in dimension 2 is to think about closed strings (perhaps many of them) moving in time and tracing out surfaces with boundaries. We think of such a surface as giving a transition from a state space of strings at time 0 to a state space of strings at time 1.

One can classify all such surfaces with boundaries up to a suitable notion of equivalence (homeomorphism or diffeomorphism, in dimension 2 it makes no difference), and one can put algebraic structure on the set of equivalence classes.

As we shall explain, this construction gives us a category. Categories give a precise way to define mathematical structures specified by objects and maps. Functors give a precise way to compare two such mathematical structures. A 2D TQFT is a functor from our topologically defined category of surfaces to a purely algebraic category, usually taken to be the category of vector spaces over a field, such as the field of real or complex numbers. The algebraic structure on the source category corresponds to algebraic structure on the target category. When one works this all out, one reaches the conclusion that 2D TQFT's are equivalent to certain kinds of algebras, called commutative Frobenius algebras.

We will develop from scratch everything that is needed to make this completely rigorous. It is doable fun to actually work out concrete examples of commutative Frobenius algebras. Participants might well find some new ones.

The essential point is the beautiful and surprising interaction between various branches of mathematics, and the language that allows mathematicians to turn comparisons by analogy into theorems of comparison.

There is a helpful and accessible little book on this topic:

Joachim Kock. Frobenius algebras and 2D topological quantum field theories.

4b. PRODUCTS ON TOPOLOGICAL SPACES

Weeks 5 – 8: Vigleik Angeltveit

It is clear that we can always add n-tuples of real numbers, but when can we multiply them? If $n = 2$ we can, by thinking of a vector (a, b) in $\mathbb{R}^2$ as the complex number $a + bi$. And the complex numbers have many good properties. In particular, $\mathbb{C}$ is associative, meaning that $(zv)w = z(vw)$, and commutative, meaning that $zw = wz$.

If $n = 4$ there is also such a multiplication. We think of a vector (a, b, c, d) in $\mathbb{R}^4$ as the quaternion $a + bi + cj + dk$. Just as $i^2 = -1$ in $\mathbb{C}$, there are relations between i, j and k in the quaternions $\mathbb{H}$, namely $i^2 = j^2 = k^2 = ijk = -1$. But $\mathbb{H}$ is not commutative. For example, $ij = -ji$. It is still true that $\mathbb{H}$ is associative, so $\mathbb{H}$ has some of the good properties $\mathbb{C}$ has.

If $n = 8$ we can do the same thing, using what is called the octonions $\mathbb{O}$. In the octonions, $(zv)w$ is not always equal to $z(vw)$ either, so the octonions are not as nice as $\mathbb{C}$ and $\mathbb{H}$.

These are the only dimensions where we can equip $\mathbb{R}^n$ with a multiplication such that if $a \neq 0$ and $b \neq 0$ then $ab \neq 0$, and by restricting to unit vectors these examples give us a multiplication on the spheres S^1, S^3 and S^7 inside $\mathbb{R}^2$, $\mathbb{R}^4$ and $\mathbb{R}^8$. In this course I will explain what these things are, and some of the things we can do with them. Our focus will be the interplay between the topological properties and the algebra of spaces with a continuous multiplication.