CONSTRUCTING FROBENIUS ALGEBRAS

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ABSTRACT. We discuss the relationship between algebras, coalgebras, and Frobenius algebras. We describe a method of constructing Frobenius algebras, given certain finite-dimensional algebras, and we demonstrate the method with several concrete examples.

1. Introduction

This paper was written for the 2007 summer math REU at the University of Chicago. It describes algebraic structures called Frobenius algebras and explains some of their basic properties. To make the paper accessible to as many readers as possible, we have included definitions of all the most important concepts. We only assume that the reader is familiar with basic linear algebra.

Our discussion begins with the definition of an algebra, a coalgebra, and a Frobenius algebra. Then we show how to explicitly construct a coalgebra given a finite-dimensional algebra. We show that the resulting structure is closely related to a Frobenius algebra. The paper concludes with several examples.

2. Algebras, Coalgebras, and Frobenius Algebras

If V and W are vector spaces over the same field, then the *tensor product* $V \otimes W$ is the vector space spanned by the symbols $v \otimes w$ with $v \in V$, $w \in W$, subject to the relations $v_1 \otimes w + v_2 \otimes w = (v_1 + v_2) \otimes w$, $v \otimes w_1 + v \otimes w_2 = v \otimes (w_1 + w_2)$, and $(av) \otimes w = v \otimes (aw) = a(v \otimes w)$ for all v, v_1, $v_2 \in V$, w, w_1, $w_2 \in W$, and all scalars a. If $f : V \to V'$ and $g :\to W'$ are linear transformations, then there is a linear map $f \otimes g : V \otimes W \to V' \otimes W'$ defined on generators by $(f \otimes g)(v \otimes w) = f(v) \otimes g(w)$.

Definition 1. An *algebra* A over a field K is a vector space over K together with a K-linear vector multiplication $\mu : A \otimes A \to A$, $x \otimes y \mapsto xy$ and a K-linear unit map $\eta : K \to A$ such that the following three diagrams commute.

$$\begin{array}{ccc} A \otimes A \otimes A & \xrightarrow{\mathrm{id}_A \otimes \mu} & A \otimes A \\ {\scriptstyle \mu \otimes \mathrm{id}_A}\downarrow & & \downarrow{\scriptstyle \mu} \\ A \otimes A & \xrightarrow[\mu]{} & A \end{array}$$

$$\begin{array}{ccc} K \otimes A & \xrightarrow{\eta \otimes \mathrm{id}_A} & A \otimes A \\ & \searrow & \downarrow{\scriptstyle \mu} \\ & & A \end{array} \qquad \begin{array}{ccc} A \otimes A & \xleftarrow{\mathrm{id}_A \otimes \eta} & A \otimes K \\ {\scriptstyle \mu}\downarrow & \swarrow & \\ A & & \end{array}$$

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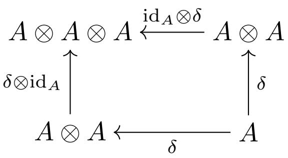
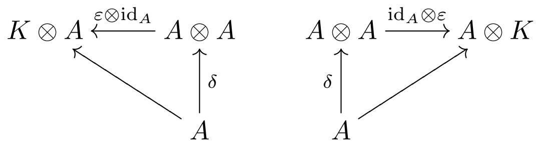
Here the diagonal maps are given by scalar multiplication: $c \otimes a \mapsto ca$ and $a \otimes c \mapsto ca$. The rectangular diagram expresses the fact that multiplication is associative, and the two triangular diagrams express the *unit condition.*

If we reverse all arrows in the above diagrams, we obtain the axioms of a coalgebra.

Definition 2. A *coalgebra* A over a field K is a vector space over K together with two K-linear maps $\delta : A \to A \otimes A$ and $\varepsilon : A \to K$ such that the following three diagrams commute.





Here the diagonal maps are given by $a \mapsto 1 \otimes a$ and $a \mapsto a \otimes 1$. The map δ is called *comultiplication*, and the map ε is called the *counit.* The rectangular diagram expresses a property called *coassociativity*, and the two triangular diagrams express the *counit condition.*

Definition 3. A *Frobenius algebra* is a finite-dimensional algebra A over a field K together with a map $\sigma : A \times A \to K$ which satisfies

$$\begin{aligned}
&\sigma(xy, z) = \sigma(x, yz) \\
&\sigma(x_1 + x_2, y) = \sigma(x_1, y) + \sigma(x_2, y) \\
&\sigma(x, y_1 + y_2) = \sigma(x, y_1) + \sigma(x, y_2) \\
&\sigma(ax, y) = \sigma(x, ay) = a\sigma(x, y)
\end{aligned}$$

for all x, y, z, x_1, x_2, y_1, $y_2 \in A$, $a \in K$, and $\sigma(x, y) = 0$ for all x only if $y = 0$. This last condition is called *nondegeneracy*, and the map σ is called the *Frobenius form* of the algebra.

3. Dual Spaces and Dual Maps

In this section we show how to explicitly construct a coalgebra given a finite-dimensional algebra. If we then compose the multiplication map with the counit map, we obtain a map which satisfies all of the axioms of a Frobenius form, except possibly nondegeneracy.

Definition 4. If V is a vector space over a field K, then the *dual space* denoted V^* is the set of K-linear maps $V \to K$.

Theorem 1. *If V is a finite-dimensional vector space over a field K, then V^* is a vector space over K.*

Proof. Clearly V^* is closed under pointwise addition. Since K is a field, the addition operation is associative and commutative. The constant function 0 is an additive identity for V^*, and every element $v^* \in V^*$ has an additive inverse $-v^*$. Hence V^* is an abelian group with respect to addition.

Let v^*, v_1^*, and v_2^* be maps in V^*. Since these maps take values in K, we have

$$\begin{aligned} a(v_1^*(x) + v_2^*(x)) &= av_1^*(x) + av_2^*(x) \\ (a+b)v^*(x) &= av^*(x) + bv^*(x) \\ a(bv^*(x)) &= (ab)v^*(x) \\ 1v^*(x) &= v^*(x) \end{aligned}$$

for all $x \in V$, $a, b \in K$, so V^* satisfies all of the axioms of a vector space. □

Theorem 2. *If V has a basis $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$, then the map $\boldsymbol{e}_i^*$ defined by*

$$(*) \qquad \boldsymbol{e}_i^*\left(\sum_{j=1}^{n} c_j \boldsymbol{e}_j\right) = c_i$$

is linear. Moreover, the $\boldsymbol{e}_i^*$ form a basis for V^*.

Proof. We have

$$\begin{aligned} \mathbf{e}_i^*\left(\sum_{j=1}^{n} c_j \mathbf{e}_j + \sum_{j=1}^{n} d_j \mathbf{e}_j\right) &= \mathbf{e}_i^*\left(\sum_{j=1}^{n} (c_j + d_j)\mathbf{e}_j\right) \\ &= c_i + d_i = \mathbf{e}_i^*\left(\sum_{j=1}^{n} c_j \mathbf{e}_j\right) + \mathbf{e}_i^*\left(\sum_{j=1}^{n} d_j \mathbf{e}_j\right) \end{aligned}$$

and

$$\mathbf{e}_i^*\left(a\sum_{j=1}^{n} c_j \mathbf{e}_j\right) = \mathbf{e}_i^*\left(\sum_{j=1}^{n} ac_j \mathbf{e}_j\right) = ac_i = a\mathbf{e}_i^*\left(\sum_{j=1}^{n} c_j \mathbf{e}_j\right),$$

so the map is linear. Now if $\alpha_1 \mathbf{e}_1^*(x) + \cdots + \alpha_n \mathbf{e}_n^*(x) = 0$ for all x, then we must have $\alpha_1, \ldots, \alpha_n = 0$, for if $x = \mathbf{e}_i$ then $\alpha_1 \mathbf{e}_1^*(x) + \cdots + \alpha_n \mathbf{e}_n^*(x) = \alpha_i$. This shows that the $\mathbf{e}_i^*$ are linearly independent. The set $\mathbf{e}_1^*, \ldots, \mathbf{e}_n^*$ is clearly a spanning set for V^*, so it is a basis for V^*. □

Theorem 3. *The map*

$$\begin{aligned} \varphi_V : V &\to V^* \\ \boldsymbol{e}_i &\mapsto e_i^* \end{aligned}$$

is a vector space isomorphism.

Proof. Since each map in V^* can be represented as a *unique* linear combination of the basis maps $\mathbf{e}_1^*, \ldots, \mathbf{e}_n^*$, the map φ_V is injective. Since the basis maps $\mathbf{e}_1^*, \ldots, \mathbf{e}_n^*$ span V^*, the map φ_V is also surjective. This map φ_V is linear by definition, so it is a vector space isomorphism. □

Theorem 4. *If V and W are finite-dimensional vector spaces, then there is an isomorphism $(V \otimes W)^* \cong V^* \otimes W^*$.*

Proof. Since V and W are finite-dimensional, Theorem 3 says that the maps $\varphi_V : V \to V^*$ and $\varphi_W : W \to W^*$ are isomorphisms. It follows that the tensor product

$$\varphi_V \otimes \varphi_W : V \otimes W \to V^* \otimes W^*$$

is an isomorphism. Theorem 3 also implies that $(V \otimes W)^* \cong V \otimes W$. Thus

$$(V \otimes W)^* \cong V \otimes W \xrightarrow{\cong} V^* \otimes W^*,$$

and by composing isomorphisms, we get $(V \otimes W)^* \cong V^* \otimes W^*$. $\square$

Definition 5. Suppose that V and W are vector spaces over K. If $\alpha : V \to W$ is a K-linear map, then we write

$$\begin{aligned} \alpha^* : W^* &\to V^* \\ f &\mapsto f \circ \alpha. \end{aligned}$$

and call α^* the *dual map*. Note that α sends V into W while α^* sends W^* into V^*. In this sense the dual map reverses arrows.

Theorem 5. *Suppose that V, W, and X are vector spaces over K. If $\alpha : V \to W$ is a K-linear map, then α^* is a K-linear map. Moreover, if $\alpha : V \to W$ and $\beta : W \to X$ are K-linear maps, then $(\beta \circ \alpha)^* = \alpha^* \circ \beta^*$.*

Proof. If α is a K-linear map, then

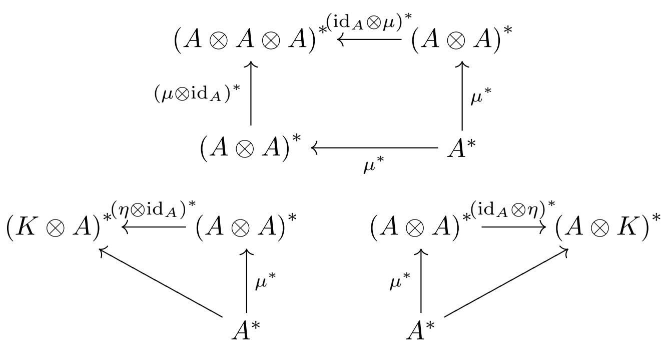
$$\alpha^*(f_1 + f_2) = (f_1 + f_2) \circ \alpha = f_1 \circ \alpha + f_2 \circ \alpha = \alpha^*(f_1) + \alpha^*(f_2)$$

and

$$\alpha^*(af) = af \circ \alpha = a\alpha^*(f)$$

for all f, f_1, $f_2 \in W^*$, $a \in K$, so α^* is a K-linear map. For any map $f : X \to K$, we also have $(\beta \circ \alpha)^*(f) = f \circ (\beta \circ \alpha) = (f \circ \beta) \circ \alpha = \alpha^*(f \circ \beta) = \alpha^*(\beta^*(f))$. $\square$

We are now in a position to construct a coalgebra from an algebra. Let A be a finite-dimensional algebra over a field K. If we replace each vector space in Definition 1 with its corresponding dual space, and if we replace the maps with their corresponding dual maps, then we obtain the following three commutative diagrams.



If we apply Theorems 4 and 5, we obtain a coalgebra A^* with comultiplication μ^* and counit η^*. Fix an isomorphism $\varphi : A \xrightarrow{\cong} A^*$ on basis elements by $\varphi(e_i) = e_i^*$, where the first basis element is the unit element $\mathbf{1} = \eta(1)$. This map transforms the coalgebra A^* into a coalgebra A with counit $\varepsilon = \eta^* \circ \varphi : A \to K$.

The following theorem shows how this construction relates to Frobenius algebras.

Theorem 6. *The map $\varepsilon \circ \mu$ has all of the properties of a Frobenius form, except possibly nondegeneracy.*

Proof. By the definition of ε and μ we have

$$(\varepsilon \circ \mu)(xy, z) = \varepsilon(xyz) = (\varepsilon \circ \mu)(x, yz)$$

for all x, y, $z \in A$. Since ε is linear, we also have

$$\begin{aligned}(\varepsilon \circ \mu)(x_1 + x_2, y) &= \varepsilon((x_1 + x_2)y) = \varepsilon(x_1 y + x_2 y) \\ &= \varepsilon(x_1 y) + \varepsilon(x_2 y) = (\varepsilon \circ \mu)(x_1, y) + (\varepsilon \circ \mu)(x_2, y)\end{aligned}$$

and

$$(\varepsilon \circ \mu)(ax, y) = \varepsilon(axy) = a\varepsilon(xy) = a(\varepsilon \circ \mu)(x, y),$$

and similarly, $(\varepsilon \circ \mu)(x, y_1 + y_2) = (\varepsilon \circ \mu)(x, y_1) + (\varepsilon \circ \mu)(x, y_2)$ and $(\varepsilon \circ \mu)(x, ay) = a(\varepsilon \circ \mu)(x, y)$ for all x, y, x_1, x_2, y_1, $y_2 \in A$, $a \in K$. This shows that $\varepsilon \circ \mu$ has all of the properties of a Frobenius form, except possibly nondegeneracy. □

Conversely, one can show that every Frobenius algebra has both algebra and coalgebra structure. This property of Frobenius algebras allows us to define "topological quantum field theories", which are important in topology and physics. For more information on topological quantum field theories, see [1].

4. Examples

Example 1. One can easily check that the field of complex numbers together with the inclusion map $\eta : \mathbb{R} \hookrightarrow \mathbb{C}$ and ordinary multiplication $\mu : \mathbb{C} \otimes \mathbb{C} \to \mathbb{C}$ is a finite-dimensional algebra. Choose the canonical basis $\{\mathbf{e}_1 = 1, \mathbf{e}_2 = i\}$ for $\mathbb{C}$. Then equation $(*)$ determines the basis maps for $\mathbb{C}^*$. Since the unit map η is simply the inclusion, we have

$$\begin{aligned}\eta^*(\mathbf{e}_1^*) &= \mathbf{e}_1^* \circ \eta = \mathrm{id}_{\mathbb{R}} \\ \eta^*(\mathbf{e}_2^*) &= \mathbf{e}_2^* \circ \eta = 0.\end{aligned}$$

This proves that $\varepsilon = \Re$ where $\Re$ is the "real part function" defined by $\Re(x+iy) = x$ for real x and y. The map $\sigma(x, y) = \Re(xy)$ is nondegenerate, so it is a Frobenius form for the algebra.

Example 2. Suppose that $G = \{t_0, \ldots, t_n\}$ is a finite group written multiplicatively with identity element t_0. One can show that the set $K[G]$ of formal linear combinations $\sum c_i t_i$ $(c_i \in K)$ together with the unit map

$$\begin{aligned}\eta : K &\to K[G] \\ a &\mapsto at_0,\end{aligned}$$

and multiplication μ given by multiplication in G, is a finite-dimensional algebra. Since $\{t_0, \ldots, t_n\}$ is a basis for $K[G]$, equation $(*)$ gives a basis $\{t_0^*, \ldots, t_n^*\}$ for $K[G]^*$. Then we have

$$\begin{aligned}\eta^*(t_0^*) &= t_0^* \circ \eta = \mathrm{id}_K \\ \eta^*(t_i^*) &= t_i^* \circ \eta = 0 \ (i \neq 0).\end{aligned}$$

By the definition of ε, we have

$$\begin{aligned}\varepsilon : K[G] &\to K \\ t_0 &\mapsto 1 \\ t_i &\mapsto 0 \ (i \neq 0).\end{aligned}$$

Finally, since the composite map $\sigma = \varepsilon \circ \mu$ is nondegenerate, it is a Frobenius form for the algebra.

Example 3. It is again easy to check that the set $\mathrm{Mat}_n(K)$ of $n \times n$ matrices over a field K, together with the unit map

$$\begin{aligned} \eta : K &\to \mathrm{Mat}_n(K) \\ a &\mapsto \begin{pmatrix} a & 0 & \dots & 0 \\ 0 & a & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a \end{pmatrix} \end{aligned}$$

and ordinary matrix multiplication $\mu : \mathrm{Mat}_n(K) \otimes \mathrm{Mat}_n(K) \to \mathrm{Mat}_n(K)$, is a finite-dimensional algebra. Choose the canonical basis $\{\mathbf{e}_{1,1}, \dots, \mathbf{e}_{n,n}\}$ for $\mathrm{Mat}_n(K)$ in which each matrix $\mathbf{e}_{i,j}$ contains a 1 in the (i,j) position and 0's everywhere else. Then equation $(*)$ determines a basis $\{\mathbf{e}^*_{1,1}, \dots, \mathbf{e}^*_{n,n}\}$ for $\mathrm{Mat}_n(K)^*$. Since $\eta^*(\mathbf{e}^*_{ij}) = \mathbf{e}^*_{ij} \circ \eta$, it follows that

$$\eta^*(\mathbf{e}^*_{ij}) = \begin{cases} \mathrm{id}_K, & i = j \\ 0 & i \neq j. \end{cases}$$

By forming linear combinations, we see that the counit is the *matrix trace* obtained by summing all of the coordinates along the main diagonal of a matrix:

$$\varepsilon(X) = \mathrm{tr}(X) = \sum_i (X)_{ii}.$$

Since the map $\sigma(X,Y) = \mathrm{tr}(XY)$ is nondegenerate, it has all of the properties of a Frobenius form.

References

[1] Kock, Joachim. *Frobenius Algebras and 2D Topological Quantum Field Theories*. Cambridge: Cambridge University Press, 2003.