ANALYTIC PROPERTIES OF BROWNIAN MOTION

ELAN BECHOR

Abstract. This paper largely deals with some analytic properties of Brownian motion and those of its discrete counterpart, random walks. I prove that the random walker returns to the origin infinitely with probability one if and only if his coin is not biased. I then then show how to construct a continuous Brownian motion over the reals. While demonstrating this existence is in itself non-trivial, I show that it has a number of pathological properties: continuous everywhere but nowhere differientiable and unbounded variation in any non-empty closed interval.

Contents

1. Definitions and useful lemmas of basic probability theory

Definition 1.1. A real-valued random variable is a measurable function over sample space Ω.

Definition 1.2. The expected value of a discrete variable X is

$$\mathbb{E}[X] = \sum_{z} zP(X = z)$$

where P is the probability measure.

Definition 1.3. An event is a subset in the probability space to which the probability measure assigns (in this case) a real number between 0 and 1.

Definition 1.4. The variance of a random variable X is defined as $Var[x] = \mathbb{E}[(X - \mathbb{E}[X])^2]$. It is easy to show that variance is an additive function.

Definition 1.5. A variable with a normal distribution (and variance t) is one whose probability denisity function is $\frac{e^{-x^2/2t}}{\sqrt{2\pi}}$. That is, $P(a \leq X \leq b) = \int_a^b \frac{e^{-x^2/2t}}{\sqrt{2\pi}}\, dx$.

Definition 1.6. Given some probability space, a stochastic process is a set of random variables $\{\mathcal{W}_t : t \in T\}$ indexed by a time set T.

Definition 1.7. The dyadics are the set D $= \bigcup_{n=1}^{\infty} \{\frac{k}{2^n} : k \in \mathbb{N}\}$.

The following are two useful lemmas that will be used to prove whether certain events occur infinitely often.

Date: 19 August 2007.

Lemma 1.8. *First Borel-Cantelli Lemma: Let If $A_1, A_2, \ldots$ are a countable sequence of events, and $\sum_{n=1}^{\infty} P(A_n) < \infty$, then $P(A_n \text{ i. o.}) = 0$*

Proof. The events A_n i. o is equivalent to $E = \bigcap_{k=1}^{\infty} \bigcup_{i=k}^{\infty} A_i$.. That is, it is clear that $x \in E$ if and only if it occurs in infinitely many A_i. Let T be the sequence of tail unions, i.e $T_n = \bigcup_{i=n}^{\infty} A_i$. Clearly, $E \subset T_n$ for all n, and by the axioms of probability theory, $P(E) \leq P(T_n)$. We know that $P(T_n) \leq \sum_{i=n}^{\infty} P(A_i)$ by Boole's inequality – in other words, collective probability is maximized when all the events are independent. But by hypothesis, we know that this sum approaches 0 by the vanishing condition of summable sequences. $\square$

Lemma 1.9. *Second Borel-Cantelli Lemma: If $A_1, A_2, \ldots$ are independent events and $\sum_{n=1}^{\infty} P(A_n)$ diverges, then $P(A_n \text{ i. o.}) = 1$*

Proof. In order to prove this, we show that the complement of the event $E = \bigcap_{k=1}^{\infty} \bigcup_{i=k}^{\infty} A_i$. has measure 0. To begin with, the complement of an intersection of union of sets is the union of the intersection of their complements, i.e. $E^c = \bigcup_{k=1}^{\infty} \bigcap_{i=k}^{\infty} A_i^c$. But for each k in this union,

$$P(\cap_{i=n}^{\infty} A_i^c) = \prod_{i=n}^{\infty} (1 - P(A_i))$$

This is true in virtue of the events being independent. To show that this right hand side is equal to 0, we utilize the fact that $1 - a \leq e^{-a}$, a consequence of the former being a 1st order Taylor approximation in an alternating series and thus the remainder is positive. Hence,

$$\prod_{i=n}^{\infty} (1 - P(A_i)) \leq e^{-\sum_{i=n}^{\infty} P(A_i)} = 0$$

since $\sum_{i=k}^{\infty} P(A_i)$ diverges by hypothesis. $\square$

2. Random Walks

Definition 2.1. A random walk beginning at x is a sum of n random variables,

$$S_n = x + X_1 + \ldots + X_n$$

where $P(X_i = 1) = P(X_i = -1) = 1/2$ for all i.

In general, this is described as a walker moving either to the left or to the right by using a fair coin flip for each decision. It is trivial to see that $E[S_n] = 0$ for all n, since the probability of moving left or right is equal. More difficult to calculate is the average distance after n steps. Average distance can be defined in many ways, but the two that this paper will calculate are root mean squared distance ($\sqrt{\mathbb{E}[S_n^2]}$) and $\mathbb{E}[|S_n|]$, i.e. average absolute distance from the origin.

Proposition 2.2. *The expected root mean square after n coin flips is $\sqrt{n}$ for all n.*

Proof.

$$\begin{aligned}\mathbb{E}[S_n^2] &= \mathbb{E}[(\sum_{j=1}^{n} X_j)^2] \\ &= \mathbb{E}[\sum_{j=1}^{n} \sum_{k=1}^{n} X_j X_k]\end{aligned}$$

We then regroup the sum into terms that have identical indices and terms that do not. The n terms in the former sum to n, since both $(-1)(-1) = 1$ and $(1)(1) = 1$

$$= n + \sum_{j \neq k} \mathbb{E}[X_j X_k]$$

But this latter term is equal to zero, since $P(X_j X_k = 1) = P(X_j X_k = -1) = 1/2$. So we have established that $\mathbb{E}[S_n^2] = n$ or that $\sqrt{\mathbb{E}[S_n^2]} = \sqrt{n}$. □

$\mathbb{E}[|S_n|]$ is unfortunately much more difficult to calculate, and also impossible to calculate from first principles, so we will omit some steps of the proof and use some unproven identites.

Proposition 2.3. $\mathbb{E}[|S_n|] \sim \sqrt{\frac{2n}{\pi}}$

Proof. We must treat separately the cases when n is odd and when n is even, and in this paper only the even case will be explored. See references for a more complete proof. We begin by looking at the probability distribution $P(S_n = k)$ for $k = -n, \ldots, n$. Since n is even, S_n must be even, and it is not too hard to see that $P(S_n = k)$ will look like Pascal's triangle multiplied by a factor of 2^{-n} and interspersed zeroes at all odd positions. That is, since the difference between the number of heads and tails is k,

$$P(S_n = k) = 2^{-n} \binom{n}{\frac{k+n}{2}} = \frac{n!}{(\frac{n+k}{2})!(\frac{n-k}{2})!}$$

And hence the average absolute distance from the origin is:

$$\mathbb{E}[|S_n|] = 2^{-n} \sum_{k=-n,-n+2,\ldots}^{n} |k| \frac{n!}{(\frac{n+k}{2})!(\frac{n-k}{2})!}$$

Since n is even, letting $n = 2b$, and modifying the formula so that the index runs through $-b, -b+1, \ldots, b$, we have

$$= 2^{-n} \sum_{k=-b,-b+1,\ldots}^{b} |2k| \frac{n!}{(\frac{2b+2k}{2})!(\frac{2n-2k}{2})!} = \frac{n!}{2^{-n+2}} \sum_{k=1}^{b} \frac{k}{(b+k)!(b-k)!}$$

The right hand side of the equation comes from symmetry and the fact that the sum at 0 contributes nothing. We are then at an impasse without using exotic gamma identities, namely,

$$\sum_{k=1}^{b} \frac{k}{(b+k)!(b-k)!} = \frac{1}{\Gamma(b)\Gamma(1+b)} \text{ and } \Gamma(\frac{1}{2}+b) = \frac{\prod_{j=1}^{n} 2j-1}{2^n} \sqrt{\pi}$$

We can go through a process of simplifications and finally arrive at

$$= \frac{2}{\sqrt{\pi}} \frac{\Gamma(1+b)}{\Gamma(\frac{1}{2}+b)}$$

Using the final fact that $\frac{\Gamma(1+\frac{n}{2})}{\Gamma(\frac{1}{2}+\frac{n}{2})} \sim \sqrt{\frac{n}{2}}$ we arrive at the desired conclusion. □

Now that we have answered how far, on average, the random walker goes from the origin, it is a natural question to ask how often the walker *returns* to the origin. As it turns out, if the coin is exactly fair, the walker will return to the origin infinitely often. In general, one proves this using an estimate of $P(S_n = 0)$ derived from Stirling's formula, but I will use a different approach with the Euler-Wallis product instead.

$$\frac{2}{\pi} = \prod_{n=1}^{\infty} \frac{(2n-1)(2n+1)}{(2n)(2n)} \tag{2.4}$$

Proposition 2.5. *A random walker will return to the origin infinitely often*

Proof. It should be clear enough that if a random walker begins at the origin,

$$p_n = P(S_{2n} = 0) = 2^{-2n}\binom{2n}{n}$$

since the cases when the walker returns to the origin are when the number of heads flipped equal the number of tails. We then build a recursive formula for p_n. Since $p_{n+1} = \binom{2n+2}{n+1}2^{-2n-2}$,

$$\frac{p_{n+1}}{p_n} = \frac{\binom{2n+2}{n+1}}{4\binom{2n}{n}} = \frac{(2n+2)(2n+1)}{4(n+1)(n+1)} = \frac{2n+1}{2n+2}$$

Hence we can form the recursive product

$$p_n = \prod_{k=1}^{n} \frac{2n-1}{2n}$$

At this point, we must note that

$$\frac{1}{2n+1}\prod_{n=1}^{k} \frac{(2n-1)(2n+1)}{(2n)(2n)} = \prod_{n=1}^{k}(\frac{2n-1}{2n})^2$$

To finish the proof, we make use of one unproven lemma: since $\prod_{n=1}^{\infty}(\frac{2n-1}{2n})$ converges absolutely, $\prod_{n=1}^{\infty}(\frac{2n-1}{2n})^2 = (\prod_{n=1}^{\infty}\frac{2n-1}{2n})^2$. Thus,

$$\prod_{n=1}^{\infty}(\frac{2n-1}{2n}) = \sqrt{\lim_{n\to\infty}\frac{1}{2n+1}\prod_{n=1}^{k}\frac{(2n-1)(2n+1)}{(2n)(2n)}} = \frac{1}{\sqrt{n\pi}}$$

Since $\sum \frac{1}{\sqrt{n}}$ diverges, this means that the expected number of returns, $\sum P(S_{2n} = 0)$, diverges. We now assume for the purpose of contradiction that the number of returns is finite with non-zero probability. This is true if and only if the walker returns to the origin with probability less than 1. Supposing this to be true, we can attempt to find a probability distribution for R, the number of returns to the origin. Since $P(R = n+1) = qP(R = n)$, where q is the probability the walker returns to the origin after starting at 0, $\mathbb{E}[R] = \sum_{n=1}^{\infty} nq^n(1-q) = \frac{1}{1-q}$ which contradicts our earlier work that showed the expected number of returns to the origin was infinite. Thus with probability one the walker will return to the origin infinitely often. $\square$

Next, I will use the First Borel-Cantelli Lemma to show that a *biased* random walker returns to the origin only a finite number of times with probability 1.

Proposition 2.6. *If a random walker moves to the left (without loss of generality) with probability* p *and to the right with probability* $(1-p)$, *it will return to the origin only finitely many times with probability 1.*

Proof. Just as with the standard random walker, this is proved by generating a recursive formula for $p_n = P(S_{2n} = 0)$ and then using its infinite product to find some asymptotic estimate. Noting that $p_n = 2nn(1-p)^n pn$, we then find the recursive formula:

$$\frac{p_{n+1}}{p_n} = \frac{(2n+2)(2n+1)}{(n+1)(n+1)}(1-p)p = \frac{2n+1}{2n+2}[4(1-p)(p)]$$

Letting $q = [4(1-p)(p)]$ and noting that $q < 1$ if and only if $q \neq \frac{1}{2}$, we can see from our work above that

$$p_n \sim \prod_{k=1}^{n} \frac{1}{\sqrt{\pi n}}$$

By the comparison test, $\sum p_n$ converges, and so by the First Borel-Cantelli Lemma, the walker returns only finitely often with probability 1. □

3. Brownian Motion

Definition 3.1. A Brownian motion is a stochastic process such that the following three properties hold

- For each t, the random variable W_t has $\mathcal{N}(0,t)$, i.e. it's normally distributed with variance t and mean 0.
- For each $0 \leq s < t$ in the indexed time set, the random variable $W_t - W_s$ has $\mathcal{N}(0, t-s)$.
- With probability one the it is continuous everywhere.

Theorem 3.2. *There exists a sample path $t \to W_t$ whose domain on $[0,1]$ and consequently all of $\mathbb{R}$.*

Proof. The major difficulty lay in proving the existence of a Brownian motion on an uncountable set. Therefore, to make things easier, we begin first by defining W_t inductively on for each t in the $\mathcal{D} \cap [0,1]$; from there, we prove that this countable set satisfies the properties of a Brownian motion (the hard one is continuity). Then we extend this set analytically to all of $\mathbb{R}$. Start with a countable set of $\mathcal{N}(0,1)$ variables Z_q defined on the dyadics and then define W on the integers with the recurrence relation

$$W_{k+1} - W_k = Z_k$$

. The nth inductive step fills in part of the remaining gap: it defines W_q for $q \in \mathcal{D}_n + 1 \cap [0,1]$. Thus, in the inductive step, we are *given* $W_{\frac{k}{2^n}}$ and $W_{\frac{(k+1)}{frm-e^n}}$ for $k \in \mathbb{N}$ and must define $W_{(2k+1)/2^{n+1}}$. The latter is defined as:

$$W_{(2k+1)/2^{n+1}} - W_{k/2^n} = \frac{1}{\sqrt(2)}[W_{(k+1)/2^n} - W_{k/2^n}] + 2^{-n/2-1/2} Z_{(2k+1)/2^{n+1}}$$

The factor of $\frac{1}{\sqrt{2}}$ in the first part of the right hand side ensures that the variance will be appropriate (2^{-n-1}) and the latter half ensures that the interval is independent. Now we must establish that this set of random variables (defined only on the dyadics at this point) is continuous, and in fact it is easiest to show a stronger claim: it is uniformly continuous.

Claim 3.3. The above construction is uniformly continuous on the dyadics, i.e. for every $\epsilon > 0 \exists \delta$ such that if $|t-s| < \delta$ then $|W_t - W_s| < \epsilon$

Proof. It is sufficient to prove that the sequence defined as

$$Z_n = sup\{|W_s - W_t| : |s-t| \leq 2^{-n}, s, t, \in \mathcal{D} \cap [0,1]\}$$

converges to 0. This is very easy to see, since given any ϵ we will be guaranteed the ability to find some point k in the sequence past which all terms are less than ϵ. Then 2^{-k-1} will suffice for our δ. Let us further define

$$M_n = \max_{1,2,\ldots,2^n} \sup\{|W_q - W_{\frac{(k-1)}{2^n}}| : q \in \mathcal{D} \cap [0,1]\}$$

Now, it's clear that

$$|W_q - W_{\frac{(k-1)}{2^n}}| : q \in \mathcal{D} \cap [(k-1)2^n, \frac{k}{2^n}]\} \subset \{|W_s - W_t| : |s-t| \leq 2^{-n}, s, t, \in \mathcal{D} \cap [(k-1)2^n, \frac{k}{2^n}]\}$$

and thus $M_n \leq Z_n$, but we can also put an upper bound on Z_n. If $Z_n \geq M_n$, it is because the supremum occurs between two points, say $a < b$, in different slices of the unit interval, say $\frac{(k-1)}{2^n}$ and $\frac{k}{2^n}$. So by the triangle inequality, we can establish that

$$|W_a - W_b| \leq |W_a - W_{\frac{(k-1)}{2^n}}| + |W_{\frac{(k-1)}{2^n}} - W_{\frac{k}{2^n}}| + |W_{\frac{k}{2^n}} - W_b| \leq 3M_n$$

Hence it is clearly sufficient to prove that $M_n \to 0$. Now, since each $W_{\frac{k}{2^n}} - W_{\frac{(k-1)}{2^n}}$ is independent, by Boole's inequality for any $\epsilon > 0$

$$P(M_n \geq \epsilon) \leq 2^n P(\sup\{|W_q| : q \in \mathcal{D} \cap [0, \frac{1}{2^n}]\} \geq \epsilon) = 2^n P(\sup\{|W_q| : q \in \mathcal{D} \cap [0,1]\} \geq 2^{\frac{n}{2}} \epsilon_n)$$

the latter equation coming from the properties of the normal distribution. The following fact will not be proven here, but the proof itself is not very complicated:

$$P(\sup\{|W_q| : q \in \mathcal{D} \cap [0,1]\} \geq a) \leq \frac{4e^{-\frac{a^2}{2}}}{a} \tag{3.4}$$

Defining $\epsilon_n = \frac{1}{n}$, we can use the above two relations to show that

$$\sum_{n=1}^{\infty} 2^n P(\sup\{|W_q| : q \in \mathcal{D} \cap [0, \frac{1}{2^n}] \ \geq \frac{1}{n}) \leq \sum_{n=1}^{\infty} \frac{2^{\frac{n}{2}}}{e^{\frac{2^{n-1}}{n^2}}}$$

which clearly converges. Hence, by the First Borel-Cantelli Lemma, M_n will with probability 0 surpass ϵ_n infinitely often; i.e., ϵ_n will almost surely surpass M_n, and thus M_n will almost surely converge to 0. Our previous work shows that this demonstrates that $t \to W_t$ (the sample path) is uniformly continuous. Now we must simply extend the sample path, now defined only on the dyadics, to the entire unity interval.

Lemma 3.5. *$\mathcal{D}$ is a dense subset of $\mathcal{R}$; i.e. for every $t \in \mathcal{R}$, there is a converging sequence $t_n \to t$ where $t_n \in \mathcal{D}$*

Proof. It is sufficient to prove that between any two real numbers there exists some dyadic, and without loss of generality assume that these two real numbers (not equal to any dyadics) $a < b$ are on the unit interval. It is clear that for every $\epsilon > 0 \exists 2^k$ such that $2^k \epsilon > 1$. From this, it is clear that for every a and b, $\exists k$ such that $2^k a - 2^k b > 1$. Since by hypothesis neither $2^k a$ nor $2^k b$ are integers, $\exists m \in \mathbb{N}$ such that $a \leq \frac{m}{2^k} \leq b$. □

We can instantly recognize the sequence as Cauchy, since a sequence of geometrically approaching dyadics can be chosen. Now, let $\{t \to W_{t_n}\}$ be a sequence of converging sample paths. The limit, $t \to W_t$, is clearly continuous, since $\forall \epsilon > 0 \exists \delta$ such that if $0 < |t-s| < \delta$, we can choose an appropriate n such that $|W_{t_n} - W_t| \leq \frac{\epsilon}{3}$, $|W_{s_n} - W_s| \leq \frac{\epsilon}{3}$, and $|W_{s_n} - W_{t_n}| \leq \frac{\epsilon}{3}$. Then, by the triangle inequality, $|W_t - W_s| \leq \epsilon$ and we have successfully shown that that a sample path can be created on the unit interval – with no loss of generality. □

Theorem 3.6. *While Brownian motion is continuous everywhere, it has a number of pathological properties. Surprisingly, with probability 1 a Brownian motion is differentiable nowhere.*

Lemma 3.7. *In order to show that a function is differentiable at some point, it must also be 1-Hölder continuous at some point t, i.e. that* $\exists \epsilon, C > 0$ *such that* $\forall x, y \in [t-\epsilon, t+\epsilon]$ *such that* $|W_x - W_y| \leq C|x-y|$.

Proof. If a function is differentiable at some point, it must also be continuous in some neighborhood $[t-\epsilon, t+\epsilon]$ around that point. By the mean value theorem, if a function f is differentiable at $f(t)$, then given any two points $(x < y)$ inside the continuous neighborhood, $\exists C = f'(c)$ such that $C = \frac{f(x)-f(y)}{x-y}$; i.e. $|f(x) - f(y)| = C|x-y|$. □

With that in mind, let $M(k,n)$ be the maximum of the triplet

$$\{|W_{\frac{k}{n}} - W_{\frac{k-1}{n}}|, |W_{\frac{k+1}{n}} - W_{\frac{k}{n}}|, |W_{\frac{k+2}{n}} - W_{\frac{k+1}{n}}|\}$$

and let M_n be the minimum of $M(1,n), \ldots, M(n,n)$. If there is a differentiable point $t \in [0,1]$, there must be some constant C_s which is

$$\sup\{C : \inf\{C : \text{C works as a constant for Hölder continuity for} \epsilon > 0\}\}$$

. Then by Hölder continuity, $\forall \epsilon > 0 \exists n_o$ such that for every $n \geq n_o, M_n \leq \frac{C_s}{n}$. Now, disjoint intervals of a Brownian motion are by definition independent, so by Boole's inequality,

$$P(M_n \leq \frac{C}{n}) \leq (P(|W_{\frac{1}{n}}| \leq \frac{C}{n}))^3$$

This is equal to the cube of the probability distribution integral from $-\frac{C}{n}$ to $\frac{C}{n}$, and hence

$$\leq (\frac{C}{n})^3$$

We know from basic convergence theorems that this will converge to some finite some. Hence, by the First Borel-Cantelli Lemma, for any constant C,

$$P(M_n \leq \frac{C}{n} \text{ i. o.}) = 0$$

In other words, with probability one Brownian motion is nowhere differentiable. □

This explains some of the other pathological properties of Brownian motion. For example, any given interval of a Brownian motion has unbounded variation. Roughly speaking, this means Brownian motion has infinite "up or down motion" in any given interval.

Corollary 3.8. *Given any interval of a Brownian motion and any finite constant C and its variation* $V = \lim_{n\to\infty} \sum_{k=0}^{2^n-1} |W_{\frac{k}{2^n}} - W_{\frac{k-1}{2^n}}|$, $P(V \leq C) = 0$.

Proof. The variation limit is monotonously increasing. But if even V_0 were were bounded on some interval, it would be 1-Hölder continuous there as well, a contradiction. ☐

References

[1] Borel-Cantelli Lemma. PlanetMath.org. 2007.
[2] Evans, Lawrence. An Introduction to Stochastic Differential Equations. UC Berkeley. 2005.
[3] Lawler, Gregory. Random Walks and Brownian Motion. Preprint. 2007.
[4] Weisstein, Eric W. "Random Walk–1-Dimensional." From MathWorld–A Wolfram Web Resource. http://mathworld.wolfram.com/RandomWalk1-Dimensional.html. 2007.