All Convex Polyhedra are Homeotopic to the Sphere

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Definitions and Assumptions

Define the following characteristics to be:
V(K) = the number of vertices within a graph K
E(K) = the number of edges within a graph K
F(K) = the number of faces a graph K separates the plane or surface into
Definition: A graph G is a *tree* if G is connected and has no simple cycles, i.e. a cycle with no repeated vertices besides the starting and ending vertex.

Assumptions:
Jordan Curve Theorem: Any simple closed curve separates the space it's contained within into two distinct parts

1 Lemma: Euler's Theorem in 2 Dimensions

Definition: The Euler characteristic for any graph K in 2 dimensions is defined by:

$$\chi(K) = V(K) - E(K)$$

Claim 1: $\forall$ connected graphs K χ(K)$\leq$1
Claim 2: χ(K)=1 $\Leftrightarrow$ K is a *tree*
Proof of Claim 1: Define G_k to be a connected graph with k edges. For G_0 there are two possible graphs:
The graph consisting of a single vertex satisfying:

$$\chi(G_0) = 1 - 0 = 1$$

And the Grapch consisting of no vertices satisfying:

$$\chi(G_0) = 0 - 0 = 0$$

Thus for k=0, $\chi(G_k)\leq$1. Now we will induct on the number of edges in order to prove Claim 1.
Assume that $\chi(G_n)\leq$1 with V(G_n)=l and E(G_n)=n. In order to create any

graph G_{n+1} there are only two possible methods for adding an additional edge: (1) Add an edge between v_i and v_j with $v_i, v_j \in G_n$. Thus:

$$V(G_{n+1}) = V(G_n) = l$$

$$E(G_{n+1}) = E(G_n) + 1 = n + 1$$

$$\chi(G_{n+1}) = V(G_{n+1}) - E(G_{n+1}) = l - (n+1) = (l-n) - 1 = \chi(G_n) - 1 < \chi(G_n)$$

(2) Add an edge between v_i and v_j with $v_i \in G_n$ and $v_j \notin G_n$. Thus:

$$V(G_{n+1}) = V(G_n) + 1 = l + 1$$

$$E(G_{n+1}) = E(G_n) + 1 = n + 1$$

$$\chi(G_{n+1}) = V(G_{n+1}) - E(G_{n+1}) = (l+1) - (n+1) = (l-n) + 1 - 1 = \chi(G_n)$$

Hence ∀k, $\chi(G_k) \leq 1$//
Lemma: Adding an edge to a connected graph G using method (1) creates a simple cycle.
Proof: G is connected ⇒ ∃ a path from v_i to v_j s.t. no vertex is repeated. If a new edge is added connecting v_i to v_j, this edge will add allow the path to extend from v_i to v_i without crossing any other vertices twice.//
Proof of claim 2: We shall assume that G_0 with 0 vertices is not a tree, though it does make intuitive sense that the nonexistent vertices could have infinite cycles between nothingness. Thus our base case shall be G_0 with V(G_0)=1, and hence as shown above $\chi(G_0)$=1.
Assume G_k is a tree.⇒ G_k is generated from G_0 solely by method(2) by the Lemma. Thus:

$$\chi(G_k) = \chi(G_{k-1}) = ... = \chi(G_1) = \chi(G_0) = 1$$

Since method (1) preserves the Euler Characteristic.

Now assume $\chi(G_k)$=1. We shall also assume that G_k is generated by at least one edge of type (1).

$$\Rightarrow \chi(G_k) = \chi(G_{k-1}) = ...\chi(G_{j+1} < \chi(G_j) = ... = \chi(G') = \chi(G) = 1$$

$$\Rightarrow \chi(G_k) < 1$$

A contradiction of our initial assumption, thus all edges of G_k must be added by method (2)⇒ G_k is a tree.//

2 Proof of Euler's Theorem in 3 Dimensions

Definition: For any surface or solid K in 3 Dimensions the euler characteristic χ of K is denoted:

$$\chi(K) = V(K) - E(K) + F(K)$$

Definition: Let a net on a convex surface be defined as a graph of connected vertices and edges separating the surface into faces.

Claim: $\forall$ nets P on a convex surface χ(N)=2

Proof: Let P_k be a net with k edges, and P_0 be a net on a surface consisting of a single vertex be the smallest possible net on a surface. Thus:

$$\chi(P_0) = 1 - 0 + 1 = 2$$

Thus $\chi(P_k)$=2 for k=0. We will now procede by induction on the edges. Assume P_n is a net with V(P_n)=l, E(P_n)=n, and F(P_n)=m and that $\chi(P_n)$=2. As in the previous section, in order to create P_{n+1} from P_n there are two methods for adding an additional edge, with the same consequences for the vertices and edges as above, with the addition that for method (1) creates a simple closed curve, and thus separates 1 of the existing faces into 2 distinct faces, though method (2) creates no additional faces. Thus the following is obtained by each method:
Method (1):

$$V(P_{n+1}) = V(P_n) = l$$
$$E(P_{n+1}) = E(P_n) + 1 = n + 1$$
$$F(P_{n+1}) = F(P_n) + 1 = m + 1$$

$$\chi(P_{n+1}) = V(P_{n+1}) - E(P_{n+1}) + F(P_{n+1}) = l - (n+1) + (n+1) = (l - n + m) - 1 + 1 = \chi(P_n)$$

Method (2):

$$V(P_{n+1}) = V(P_n) + 1 = l + 1$$
$$E(G_{n+1}) = E(G_n) + 1 = n + 1$$
$$F(P_{n+1}) = F(P_n) = m$$

$$\chi(P_{n+1}) = V(P_{n+1}) - E(P_{n+1}) + F(P_{n+1}) = (l+1) - (n+1) + m = (l - n + m) + 1 - 1 = \chi(P_n)$$

Using our inductive assumption we find that $\chi(P_{n+1})$=$\chi(P_n)$=2.
Thus χ(P)=2 for all nets on convex surfaces, and hence χ(P)=2 for all polyhedra, a subset of those nets.//

3 Euler Characteristic an Homeotopy to the Sphere

For this section we will be showing the equivalency of 3 statements:
(1)χ(K)=2
(2)Any embedded loop in K separates K into two distinct parts
(3)K is homeotopic to the sphere
Proof: **(3)$\Rightarrow$(2)**: First by the Jordan Curve Theorem we can say already that (3)$\Rightarrow$(2) since the sphere has one surface, any closed curve would have to separate it.

Now we will show that (2)⇒(1)⇒(3) to complete the proof. In doing this we will use the lemma stating that the euler characteristic of any tree is equal to 1.
To do this we will consider a closed combinatorial surface K, essentially a triangulation of the surface, or a net composed solely of triangles.
(2)⇒(1): First create a maximal tree T, in K. Since the tree is maximal ⇒

$$V(T) = V(K)$$

Then create a dual graph Γ with each face of K being a vertext of Γ, and adding an edge between every pair of vertices $v_i, v_j \in \Gamma$ s.t. the faces $f_i, f_j \in K$ corresponding to v_i, v_j satisfy:

$$f_i \cap f_j \notin T$$

Thus Γ will have the property

$$F(K) = V(\Gamma)$$

and the two graphs together will satisfy:

$$E(K) = E(T) + E(\Gamma)$$

Thus combining these results we obtain:

$$\chi(K) = V(K) - E(K) + F(K) = V(T) - [E(T) + E(\Gamma)] + V(\Gamma) =$$
$$[V(T) - E(T)] + [V(\Gamma) - E(\Gamma)] = \chi(T) + \chi(\Gamma)$$

By the Lemma we know that χ(T)=1 since T is a tree, so to show that χ(K)=2, we simply need to show that χ(Γ)=1, or that Γ is a tree.
Suppose Γ is a tree
⇒ Γ has an embedded loop
⇒ Γ separates K, and hence that T isn't connected, but T is connected, a contradiction. Thus Γ must be a tree, and χ(K)=2 **(1)⇒(3)**: Now K itself is composed of a neighborhood around T and a neighborhood around Γ, which we have just shown are both trees, and the neighborhood around a tree is homeomorphic to a disk. If we take the neighborhood around T and Γ and combine them together we obtain an area homeomorphic to a sphere.

4 All Convex Polyhedra are Homeomorphic to the sphere

Proof: In the previous two sections we found that:
1)χ(K)=2 ∀ convex polyhedra
2)χ(K)=2⇔ All embedded loops in K separate⇔ K is homeotopic to the sphere.

Thus we can now say that all convex polyhedra are homeomorphic to the sphere//