AN ANALYTIC PROOF OF THE ROGERS-RAMANUJAN IDENTITIES

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Abstract. The Rogers-Ramanujan Identities were discovered independently by Leonard James Rogers and Srinivasha Ramanujan; the 1st identity was found by Rogers in 1894 and by Ramanujan in 1913. These identities were stepping stones towards building a general theory of Rogers-Ramanujan continued fractions and certain elliptic modular equations appearing in Ramanujan's lost notebook. G.H. Hardy believed in the inexistence of a simple proof of the results. This remains the case even as of today.

Contents

1. Preliminaries

Definition 1.1. Let n be a nonnegative integer. A *partition* of n is a representation of n as an unordered sum of nonnegative integers. The summands are the *parts* of the partition.

Examples. There are seven partitions of six, namely 5, $4+1$, $3+1+1$, $3+2$, $2+2+1$, $2+1+1+1$, and $1+1+1+1+1$. The partition $2+2+1$ of 5 has three parts. $5+3+1$ is a partition of 9 into odd parts, while $2+2+2+3$ is a partition of 9 into an odd number of even parts.

The following theorem two theorems will be the most useful and most important in understanding and proving the Rogers-Ramanujan identities.

Theorem 1.2. *Let $p(n|A)$ denote the number of partitions of n taken from a set A of nonnegative integers. If $|q| < 1$, then*

$$\sum_{n=0}^{\infty} p(n|A)q^n = \prod_{n\in A} \frac{1}{1-q^n}.$$

Proof. This proof is an imitation of the one given in ([2], 3-5). Write $A = \{a_1, a_2, ...\}$. Then

Date: August 17, 2007.

$$
\begin{aligned}
\prod_{n\in A}\frac{1}{1-q^n} &= \prod_{n\in A}(1+q^n+q^{2n}+...) \\
&= (1+q^{a_1}+q^{2a_1}+...)(1+q^{a_2}+q^{2a_2}+...) \\
&= \sum_{a_1\geq 0}\sum_{a_2\geq 0}...q^{a_1h_1+a_2h_2+...} \quad .
\end{aligned} \tag{1.3}
$$

Observe that the exponent on q is the partition $a_1h_1 + a_2h_2 + ...$ Therefore the appearance of q^n in the expansion is once for every partition into parts belonging to A. Without loss of generality, suppose $0 < q < 1$. Notice that

$$
\begin{aligned}
\sum_{j=0}^{M} p(j|A)q^j &\leq \prod_{i=1}^{n}(1-q^{a_i})^{-1} \\
&\leq \prod_{i=1}^{\infty}(1-q^{a_1})^{-1} < \infty
\end{aligned} \tag{1.4}
$$

and

$$
\begin{aligned}
\sum_{j=0}^{\infty} p(j|A)q^j &\geq \prod_{i=1}^{n}(1-q^{a_i})^{-1} \\
&\longrightarrow \prod_{i=1}^{\infty}(1-q^{n})^{-1}
\end{aligned} \tag{1.5}
$$

Therefore

$$
\sum_{j=0}^{\infty} p(j|A)q^j = \prod_{i=1}^{\infty}(1-q^{a_i})^{-1} = \prod_{n\in A}\frac{1}{1-q^n}. \tag{1.6}
$$

□

Theorem 1.7 (Jacobi's Triple Product Identity)**.** *For* $|q| < 1$ *and* $x \neq 0$*,*

$$
\sum_{n=-\infty}^{\infty} x^n q^{\frac{n(n+1)}{2}} = \prod_{n=1}^{\infty}(1-q^n)(1+xq^n)(1+x^{-1}q^{n-1}).
$$

Proof. Define

$$
F(x) = \prod_{n=1}^{\infty}(1+xq^n)(1+x^{-1}q^{n-1}).
$$

F admits a Laurent series expansion about 0, say

$$
F(x) = \sum_{n=-\infty}^{\infty} a_n(q)x^n.
$$

By definition of F, F satisfies the functional equation $F(xq) = x^{-1}q^{-1}F(x)$. If we compare coefficients of x^n on both sides of the functional equation, we get

$$
q^n a_n(q) = q^{-1}a_{n+1}(q)
$$

Iterating this recursion formula yields

$$a_n(q) = q^{\frac{n(n+1)}{2}} a_0(q).$$

If $p(n)$ denotes the number of partitions of n, then by Theorem 1.2, we have

$$a_0(q) = \sum_{n=0}^{\infty} p(n) q^n = \prod_{n=1}^{\infty} \frac{1}{1-q^n}.$$

Hence

$$\begin{aligned} \prod_{n=1}^{\infty} (1+xq^n)(1+x^{-1}q^{n-1}) &= F(x) \\ &= a_0(q) \sum_{n=-\infty}^{\infty} x^n q^{\frac{n(n+1)}{2}} \\ &= \prod_{n=1}^{\infty} \frac{1}{1-q^n} \sum_{n=-\infty}^{\infty} x^n q^{\frac{n(n+1)}{2}}. \end{aligned} \tag{1.8}$$

□

The proof of the Rogers-Ramanujan Identities will be much smoother and look less intimidating if we introduce some compact notation.

Definition 1.9. Let a and Q be real numbers, and let n be a nonnegative integer. We define $(a;q)_0 = 1$ and for $n \geq 1$,

$$(a;Q)_n = (1-a)(1-aQ)\dots(1-aQ^{n-1}) = \prod_{j=1}^{n-1} (1-aQ^j).$$

Also we define $(a;Q)_\infty = lim_{n\to\infty}(a;Q)_n$.

With this compact notation, Jacobi's Triple Product Identity can be written as

$$\sum_{n=-\infty}^{\infty} x^n y^{\frac{n(n+1)}{2}} = (y;y)_\infty(-x;y)_\infty(-(xy)^{-1};x)_\infty. \tag{1.10}$$

Corollary 1.11 (Euler's Pentagonal Number Theorem)**.** *If $|q| < 1$, then*

$$\sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{n(3n-1)}{2}} = (q;q)_\infty. \tag{1.12}$$

Remark 1.13. Note that the numbers $1, 5, 12, 22, 35, ..., n(3n-1)/2, ...$ are the *pentagonal numbers*. Recall the series-product identity

$$\sum_{n=0}^{\infty} p(n)\, q^n = \prod_{n=1}^{\infty} \frac{1}{1-q^n} = \frac{1}{(q,q)_\infty}, \tag{1.14}$$

where $p(n)$ denotes the number of partitions of n. By Euler's Pentagonal Number Theorem, we obtain a remarkable transformation formula

$$\sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{n(3n-1)}{2}} = \frac{1}{\sum_{n=0}^{\infty} p(n)\, q^n}. \tag{1.15}$$

Proof. (Corollary 1.7). In Equation (1.6), set $x = -q^{-2}$ and $y = q^3$ to obtain

$$\begin{aligned} \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{n(3n-1)}{2}} &= (q^3; q^3)_\infty (q^2; q^3)_\infty (q; q^3)_\infty \\ &= (q; q)_\infty \end{aligned} \tag{1.16}$$

since each symbol $(;)_\infty$ respectively contains a distinct congruence class modulo 3 (so then the product will have the exponents of q range through all the positive integers). ☐

This proof illustrates the usefulness and chiqueness of Jacobi's Triple Product Identity.

2. Rogers-Ramanujan Identities

The following is the combinatorial version of the Rogers-Ramanujan Identity:

Theorem 2.1. *The number of partitions of n whose parts differ by at least two is equinumerous with the number of partitions on n whose parts are congruent to* 1 *or* 4 *mod 5. The number of partitions of n where the difference between parts is at least two and 1 is excluded as a part is equinumerous with the number of partitions of n whose parts are congruent to* 2 *or* 3 *mod 5.*

In the first part of the theorem, notice that if $x_1 + x_2 + \ldots + x_k$ *is such a partition of n, then* $1 \leq x_1 \leq x_2 - 2$, $x_2 \leq x_3 - 2$, $\ldots$, $x_{k-1} \leq x_k - 2$. *Thus there are unique numbers* y_1, y_2,$\ldots$,y_k, *such that*

$$\begin{aligned} x_1 &= 1 + y_1 \\ x_2 &= 3 + y_2 \\ x_3 &= 5 + y_3 \\ &\vdots \\ x_k &= (2k-1) + y_k \end{aligned} \tag{2.2}$$

Given a partition of n into positive k 2-distinct parts (difference between parts is at least two), there corresponds therefore a partition of $n - (1+3+5+...+(2k-1)) = n - k^2$ *into at most k positive parts. Let* $A(n)$ *denote the number of partitions of n into 2-distinct parts. Then its generating function is given by*

$$
\begin{aligned}
\sum_{n\geq 0} A(n)\, q^n \quad &= \sum_{0\leq y_1\leq y_2\leq \ldots\leq y_k} q^{(1+y_1)+(3+y_2)+\ldots((2k-1)+y_k)} \\
&= q^{k^2} \sum_{0\leq y_1\leq y_2\leq \ldots\leq y_k} q^{y_1+y_2+\ldots+y_k} \\
&= q^{k^2} \sum_{0\leq y_1\leq y_2\leq \ldots\leq y_{k-1}} q^{y_1+y_2+\ldots+y_{k-1}} \frac{q^{y_{k-1}}}{1-q} \\
&= \frac{q^{k^2}}{1-q} \sum_{0\leq y_1\leq y_2\leq \ldots\leq y_{k-2}} q^{y_1+y_2+\ldots+y_{k-2}} \frac{q^{2y_{k-2}}}{1-q^2} \\
&= \frac{q^{k^2}}{(1-q)(1-q^2)} \sum_{0\leq y_1\leq \ldots\leq y_{k-3}} q^{y_1+y_2+\ldots+y_{k-3}} \frac{q^{3y_{k-3}}}{1-q^3} \\
&\vdots \\
&= \frac{q^{k^2}}{(1-q)(1-q^2)\ldots(1-q^k)} \\
&= \frac{q^{k^2}}{(q;q)_k}
\end{aligned}
\tag{2.3}
$$

Following the above argument, we have, in Theorem 2, for every partition of n into 2-distinct parts excluding 1 as a part, there corresponds a partition of $n-(2+4+6+\ldots+2k) = n-(k^2+k)$ *into at most k parts. Therefore if* $B(n)$ *denotes the number of such partitions, then similarly, we would obtain the generating function*

$$
\sum_{n\geq 0} B(n)\, q^n \quad = \frac{q^{k^2+k}}{(q;q)_k}
\tag{2.4}
$$

By Theorem 1.3, it follows that this combinatorial version is equivalent to the following theorem:

Theorem 2.5. *(Rogers-Ramanujan Identities, Analytic Version)*

(1) [1st Rogers-Ramanujan Identity]

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty}.$$

(2) [2nd Rogers-Ramanujan Identity]

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_n} = \frac{1}{(q^2;q^5)_\infty (q^3;q^5)_\infty}.$$

Proof. Define

$$
Q_{k,i}(q) = \frac{1}{(q;q)_\infty} \sum_{n=0}^{\infty} (-1)^n q^{(2k+1)\frac{n(n+1)}{2} - in} (1 - q^{(2n+1)i}).
\tag{2.6}
$$

Then

$$\begin{aligned} Q_{k,i}(q) &= \frac{1}{(q;q)_\infty}\sum_{n=0}^{\infty}(-1)^n q^{(2k+1)\frac{n(n+1)}{2}-in} \\ &\quad + \frac{1}{(q;q)_\infty}\sum_{n=-\infty}^{-1}(-1)^n q^{(2k+1)\frac{(-n-1)(-n)}{2}-i(-n-1)+i(2(-n-1)+1)} \\ &= \frac{1}{(q;q)_\infty}\sum_{n=-\infty}^{\infty}(-1)^n q^{(2k+1)\frac{n(n+1)}{2}-in}. \end{aligned} \tag{2.7}$$

Recall Jacobi's Triple Product Identity in Section 1:

$$\sum_{n=-\infty}^{\infty} x^{\frac{n(n+1)}{2}} y^n = (y;y)_\infty (x;y)_\infty ((xy)^{-1};x)_\infty.$$

Taking $x = q^{-i}$ and $y = -q^{2k+1}$, it follows that

$$\begin{aligned} Q_{k,i}(q) &= \frac{1}{(q;q)_\infty}\left\{(q^{2k+1};q^{2k+1})_\infty (q^{2k+1-i};q^{2k+1})_\infty (q^i;q^{2k+1})_\infty\right\} \\ &= \prod_{n\neq 0,\pm i\ (mod\ 2k+1)} \frac{1}{1-q^n} \end{aligned} \tag{2.8}$$

Taking $k = 2$ and $i = 1$, (1) follows. Taking $k = 2$ and $i = 2$, (2) follows. □

References

[1] G.E. Andrews. *Integer Partitions.* Cambridge University Press, Cambridge, 2004

[2] G.E. Andrews. *The Theory of Partitions.* Cambridge University Press, Cambridge, 1984