Categories and Natural Transformations

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1 Introduction

The motivation for studying Category Theory is to formalise the underlying similarities between a broad range of mathematical ideas and use these generalities to gain insights into these more specific structures. Because of its formalised generality, we can use categories to understand traditionally vague concepts like “natural” and “canonical” more precisely. This paper will provide the basic concept of categories, introduce the language needed to study categories, and study some examples of familiar mathematical objects which can be categorized. Particular emphasis will go to the concept of natural transformations, an important concept for relating different categories.

2 Categories

2.1 Definition

A **category** $\mathscr{C}$ is a collection of objects, denoted $Ob(\mathscr{C})$, together with a set of morphisms, denoted $\mathscr{C}(X,Y)$, for each pair of objects $X, Y \in Ob(\mathscr{C})$. These morphisms must satisfy the following axioms:

1. For each $X, Y, Z \in Ob(\mathscr{C})$, there is a composition function

$$\circ : \mathscr{C}(Y,Z) \times \mathscr{C}(X,Y) \to \mathscr{C}(X,Z)$$

   We can denote this morphism as simply $g \circ f$

2. For each $X \in Ob(\mathscr{C})$, there exists a distinguished element $1_X \in \mathscr{C}(X,X)$ such that for any $Y, Z \in Ob(\mathscr{C})$ and $f \in \mathscr{C}(Y,X)$, $g \in \mathscr{C}(X,Z)$ we have

$$1_X \circ f = f \text{ and } g \circ 1_X = g$$

3. Composition is associative: $h \circ (g \circ f) = (h \circ g) \circ f$

Remark: We often write $X \in \mathscr{C}$ instead of $X \in Ob(\mathscr{C})$
Remark: We say a category $\mathscr{C}$ is *small* if the collection $Ob(\mathscr{C})$ forms a set.

2.2 Examples

We discover categories in many areas of mathematics. The most obvious examples follow:

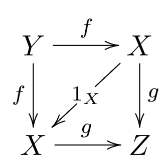
1. **Sets:** The objects are simply sets, and the morphisms are functions between sets.
2. **Groups:** The objects are groups, and the morphisms are homomorphisms.
3. **Abelian Groups:** The objects are abelian groups, and the morphisms are homomorphisms.
4. **Topological Spaces:** In this non-algebraic example, the objects are topological spaces, and the morphisms are continuous maps.
5. **Monoid:** This example is less intuitive. Consider a category with one object, $Ob(\mathscr{C}) = \star$. The set of morphisms $\mathscr{C}(\star, \star)$ form a structure which has an identity morphism along with an associative composition function. Such a category, then, is exactly a monoid in which the elements are the morphisms of the object in this category.
6. **Group:** We again consider a category with one object. If we require, in addition to the normal categorical axioms, that each morphism have an inverse morphism, we obtain a single group, the elements of which are the morphisms from this one object to itself.

And so on. Most algebraic structures with the obvious designations given to objects and morphisms form categories, although it is not always entirely obvious how to write the extra structure they exhibit in categorical terms. Note that only examples five and six are small categories; examples one through four are large (since, for example, there is no set of all sets).

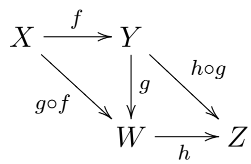
2.3 Commutative Diagrams

Sometimes it is more useful to state the axioms of categories in terms of commutative diagrams instead of equivalent functions. The axioms, stated thus, follow:

1. Identity Axiom: For each $X \in Ob(\mathscr{C})$, there exists a distinguished element $1_X \in \mathscr{C}(X, X)$ such that for any $Y, Z \in Ob(\mathscr{C})$ and $f \in \mathscr{C}(Y, X)$, $g \in \mathscr{C}(X, Z)$ the following diagram commutes:



2. Associativity Axiom: For all $X, Y, W, Z \in \mathscr{C}$, and $f : X \to Y, g : Y \to W$, and $h : W \to Z$, the following diagram commutes:



2.4 Isomorphisms

A morphism $f : X \to Y$ is an **isomorphism** if there exists a morphism $g : Y \to X$ such that $f \circ g = Id_Y$ and $g \circ f = Id_X$.

3 Functors

As we would expect, in order to make this study a fruitful one, some extra structure and sorting are required. In this section we will define and describe maps between categories, which give us grounds on which to finally define "naturality" and formalise the notion of "sameness."

3.1 Definition

Let $\mathscr{C}$ and $\mathscr{D}$ be categories. A **(covariant) functor** $F : \mathscr{C} \to \mathscr{D}$ is a morphism that makes the following associations: to each $C \in Ob(\mathscr{C})$, it associates an object $F(C) \in \mathscr{D}$, and to each morphism $f \in \mathscr{C}(C, C')$ it associates a morphism $F(f) \in \mathscr{D}(F(C), F(C'))$ such that:

1. $F(1_C) = 1_{F(C)}$ for all $C \in Ob(C)$
2. $F(g \circ f) = F(g) \circ F(f)$ for all $f : C \to C'$, $g : C' \to C''$

3.2 Examples

1. **Constant Functor** This functor maps every object $C \in Ob(\mathscr{C})$ to one object $D \in Ob(\mathscr{D})$, and every morphism in $\mathscr{C}$ to the identity morphism on D.
2. **Forgetful Functor** Generally, the forgetful functor is one which goes from a category with more structure to one with less, "forgetting" that extra structure. For example, the forgetful functor which takes groups to sets takes the elements of groups to their underlying sets, and the homomorphisms between the groups to generic functions, stripping away the group structure and leaving only the set underneath. This functor is defined for almost all of the examples we mentioned: We could take abelian groups to groups, or groups to monoids.

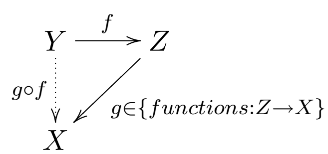
3. **Free Functor** This is the opposite notion of a forgetful functor. Free functors impose extra structure on categories with less. For example, a free functor could take the category of sets to the category of groups by taking each set to the free group generated by that set, and the functions to homomorphisms between these free groups.

3.3 Contra-variant functors

The difference between a contra-variant functor and a (covariant) functor is the morphism that F associates $f \in \mathscr{C}(C, C')$. The same axioms hold, except that each $f \in \mathscr{C}(C, C')$ gets sent to a morphism, $F(f) \in \mathscr{D}(F(C'), F(C))$. That is, a covariant functor preserves the direction of our arrows, while a contravariant functor reverses them.

3.4 Example

Let $X \in Sets$. Consider the functor $F(-, X) : Sets \to Sets$, where "$-$" denotes any set, defined on objects by $F : Y \mapsto \{functions : Y \to X\}$. Now for morphisms, suppose we have $F(Y, X)$ and $F(Z, X)$. The question is how to define this relationship, given a function $f : Y \to Z$. We want to know, given this function, is it more natural to define $F(Y, X) \to F(Z, X)$ or $F(Y, X) \leftarrow F(Z, X)$? Looking at the following diagram which states this relationship more explicitly, the answer becomes clear:



That is, given a map $f : Y \to Z$, and a map $g : Z \to X$, we can induce a map $g \circ f : Y \to X$. The correct way to define the relationship between $F(Y, X)$ and $F(Z, X)$, then, is

$$F(Y, X) \longleftarrow F(Z, X)$$

$$g \circ f \longleftarrow\!\shortmid\ g,$$

forming a perfectly acceptable contra-variant functor.

3.5 Elementary Properties

We can derive some properties immediately from the axioms. For example:

1. F transforms each commutative diagram in $\mathscr{C}$ to a commutative diagram in $\mathscr{D}$. That all of the objects in the diagram in $\mathscr{C}$ get sent to objects in an equivalent diagram in $\mathscr{D}$ is obvious from the definition of a functor. Similarly, for every arrow $C \to C'$ in a commutative diagram in $\mathscr{C}$, the functor F will associate a morphism

$F(f) \in \mathscr{D}(F(C), F(C'))$. Any property requiring the identity morphism in $\mathscr{C}$ will be associated with the identity morphism in $\mathscr{D}$, and any composition in $\mathscr{C}$ goes to an equivilant composition in $\mathscr{D}$. Thus every combination of maps between objects in $\mathscr{C}$ is preserved in the equivalent diagram in $\mathscr{D}$, making that diagram in $\mathscr{D}$ commute.

4 Natural Transformations

A morphism between categories is a functor; a map between functors (both of which must have the same input and output categories) is a natural transformation.

4.1 Definition

Let F and G be functors from $\mathscr{C}$ to $\mathscr{D}$. A **natural transformation** $\eta : F \to G$ is a collection of maps $\eta_C : F(C) \to G(C)$, one for each $C \in \mathscr{C}$, such that for any $C, C' \in \mathscr{C}$ and any $f \in \mathscr{C}(C, C')$, the following diagram commutes:

$$\begin{array}{ccc} F(C) & \xrightarrow{F(f)} & F(C') \\ {\scriptstyle \eta_C}\downarrow & & \downarrow{\scriptstyle \eta_{C'}} \\ G(C) & \xrightarrow{G(f)} & G(C') \end{array}$$

That is, whether we first take the functor F on an object C and then take η into $G(C')$, or whether we first take η into G and then use the functor G to go into $G(C')$, is irrelevant; we will obtain the same morphism from $F(C)$.

Remark: Unsurprisingly, a natural transformation is a natural isomorphism when each η_C is an isomorphism.

4.2 Example

Consider the statement "every group is naturally isomorphic to its opposite group." (The opposite group G^{op} of G has the same underlying set, and the operations reversed: $a * b \in G \mapsto b *^{op} a \in G^{op}$) This can clearly be defined in terms of a functor from groups to groups, such that $f^{op} = f$ for any group homomorphism $f : G \to H$. In order for this to define a functor, we need $f^{op} : G^{op} \to H^{op}$ to be a homomorphism. To see that this is in fact the case, we observe that, for all $a, b \in G$ and $f : G \to H$:

$$f^{op}(a *^{op} b) = f(b * a) = f^{op}(a) *^{op} f^{op}(b)$$

This statement also shows us clearly what we need to do to verify naturality: we must show that the identity functor between groups is naturally isomorphic to the opposite functor, defined above. Thus we must give an isomorphism $\eta_G : G \to G^{op}$ such that the relevant diagram defining naturality commutes.

Proof: Let $\eta_G(a) = a^{-1}$ for all $a \in G$. We have $(ab)^{-1} = b^{-1}a^{-1}$ and $(a^{-1})^{-1} = a$, showing that our η_G is its own inverse. Now set $f : G \to H$, a group homomorphism. We can write the following diagram, keeping in mind that $f^{op} = f$ and $f(a)^{-1} = f(a^{-1})$ for group homomorphisms.

$$\begin{array}{ccc} G & \xrightarrow{f} & H \\ {\scriptstyle \eta_G}\downarrow & & \downarrow{\scriptstyle \eta_H} \\ G^{op} & \xrightarrow{f^{op}} & H^{op} \end{array}$$

Showing naturality. Now, to show that this is not only a natural transformation but an isomorphism, we must show that each η_G actually has an inverse. Consider the morphism $\eta_{G^{op}}$. This must be defined on group elements as $\eta_{G^{op}}(b) = b^{-1}$. But by our definition of G^{op}, $b^{-1} \in G$. Thus, $\eta_G^{-1} = \eta_{G^{op}}$ for all G, showing that this transformation is an isomorphism.

5 Products of Categories

Suppose we have two categories, $\mathscr{C}$ and $\mathscr{D}$. The product, $\mathscr{C} \times \mathscr{D}$, gives us another category, defined as follows:

The objects $Ob(\mathscr{C} \times \mathscr{D})$ are $Ob(\mathscr{C}) \times Ob(\mathscr{D})$

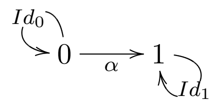
For all pairs of objects $(C_1, D_1), (C_2, D_2)$, the set $(\mathscr{C} \times \mathscr{D})[(C_1, D_1), (C_2, D_2)] = \mathscr{C}(C_1, C_2) \times \mathscr{D}(D_1, D_2)$

The identity morphism $Id(C, D)$ we define as (Id_C, Id_D)

Composition is defined componentwise as expected.

6 Arbitrary Categories, Applications

We can define categories indiscriminately, by representing objects as graph vertices and morphisms as edges. Recall that in our definition of a monoid as a category, we had a category with one object and morphisms from that object to itself. We can represent this graphically as one vertex, with many arrows pointing from that vertex to itself (necessarily including the identity arrow), plus an additional composition definition. Now consider the following arrangement:



This certainly defines a category $\mathscr{E}$: we have two objects, each with an identity morphism, and a set of morphisms $\mathscr{C}(0,1) = \{\alpha\}$, and the set $\mathscr{C}(1,0) = \{\emptyset\}$. This graph also imposes all of the compositions of morphisms: for example, $Id_1 \circ \alpha$ is forced to be α.

Proposition: Let $\mathscr{C}, \mathscr{D}$ be categories. The following are equivalent:

1. Two functors, $\mathscr{C} \underset{G}{\overset{F}{\rightrightarrows}} \mathscr{D}$, and a natural transformation $\eta : F \to G$

2. $\mathscr{C} \times \mathscr{E} \xrightarrow{H} \mathscr{D}$, for H a functor, and $\mathscr{E}$ the category with two objects defined above.

Proof: Let $H(C,0) = F(C)$ and $H(C,1) = G(C)$ for all $C \in \mathscr{C}$. This is all we need to define operations of objects; as we can see, all the information contained in those two functors $F, G : \mathscr{C} \to \mathscr{D}$ is represented by this definition; for F, look at what H does with any C together with 0 object, and for G, look at C with 1.

Now, make the following equalities, for $f : C \to C'$:

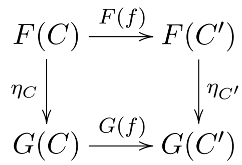
1. $H(C,0) = F(C)$
2. $H(C',0) = F(C')$
3. $H(f, Id_0) = F(f)$
4. $H(C,1) = G(C)$
5. $H(C',1) = G(C')$
6. $H(f, Id_1) = G(f)$
7. $H(Id_C, \alpha) = \eta_C$

With these definitions in hand, we shall first show that (2) implies (1), and then that (1) implies (2).

Based on the definition of a functor, the following diagram commutes in (2):

$$\begin{array}{ccc} H(C,0) & \xrightarrow{H(f,Id_0)} & H(C',0) \\ {\scriptstyle H(Id_C,\alpha)}\downarrow & & \downarrow{\scriptstyle H(Id_{C'},\alpha)} \\ H(C,1) & \xrightarrow{H(f,Id_1)} & H(C',1) \end{array}$$

But observe that, simply by definition, those diagram is exactly:



Which is what we wanted. To see that (1) implies (2), simply start with the bottom diagram, look at our definitions table, and see that it is exactly the top diagram.

7 Another Natural Isomorphism

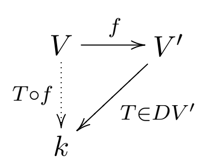
Proposition: Every finite dimensional vector space is naturally isomorphic to its double dual. (The dual of a vector space is the set of linear transformations from the vector space to its ground field. This actually forms a vector space, the dual of which is the double dual for the original space.) In categorical terms, this states that there is a natural isomorphism between the identity functor $Id : Vect_k \to Vect_k$ and the functor $DD : Vect_k \to Veck_k$, where k is the ground field of V and DD denotes the function sending V to its double dual, DDV (which is also a vector space over k).

Let us define a function $\eta_V : Id(V) \to DD(V)$, i.e. $\eta_V : V \to DDV$. Given a $\lambda \in DV$ (a linear transformation from V to its ground field k) and $v \in V$, define

$$\eta_V(v)(\lambda) = \lambda(v)$$

For convenience, denote $\eta_V(v) = eval_v$. First, we will show that this is an isomorphism, and then we will show that it is natural.

To get a better grasp of the types of functors we are looking at, suppose we have a $D : V \mapsto DV$. The natural way to define the composition is:



Thus $D(f(T)) = T \circ f$. This shows that functors $D : V \mapsto DV$ are contra-variant. Taking that same functor twice, into DDV, yields a covariant functor.

Let $e_1 ... e_k$ be a basis of V, and define T_{e_i} by

$$T_{e_i}(e_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j. \end{cases}$$

The set of T_{e_i} forms a basis for DV, which shows that V and DV have the same dimension. Suppose $v \neq 0$. We can write v in terms of a chosen basis: $v = a_1e_1 + ... + a_ne_n$. At least one of these coefficients must be nonzero, since $v \neq 0$. So if $a_i \neq 0$, let $T = T_{e_i}$. Then $T_{e_i}(v) = T(a_1e_1 + ... + a_ne_n) = a_i \neq 0$. Thus $eval_v$ is nonzero. This also shows that $\eta_V : V \to DDV$ is injective since every nonzero vector gets sent to something nonzero. We have shown that V and DDV are finite dimensional with the same dimension, so η_V must be an isomorphism.

To show that η is natural, suppose we have $f : V \to V'$. Now the consider the diagram

$$\begin{array}{ccc} V & \xrightarrow{f} & V' \\ {\scriptstyle \eta_V}\downarrow & & \downarrow{\scriptstyle \eta_{V'}} \\ DDV & \xrightarrow{DDf} & DDV' \end{array}$$

We must show that this commutes. Suppose, then, that we have a $v \in V$. We must show that whichever way we go round the diagram, we get the same element of DDV'. Going clockwise, we obtain $f(v) \in V'$ and then $eval_{f(v)}$. Going counterclockwise, we obtain first $eval_v$, and then $DDf(eval_v)$. But $DDf(eval_v) = eval_v \circ Df$. For any $T \in DV'$, $(eval_v \circ Df)(T) = eval_v(Df(T)) = eval_v(T \circ f) = T \circ f_v = T_{f(v)}$.

Note that for any $T \in DV'$, $eval_{f(v)}(T) = T_{f(v)}$. Therefore we obtain the same element of DDV' going either way around the diagram; the diagram commutes. Amen.

8 Further Possibilities

This paper has given the merest of introductions to category theory, with a few examples and statements. We can accomplish much more, relating more sophisticated concepts in mathematics, using this branch; for that, however, you shall have to consult the textbooks, as I have none of the required background.

References

[1] Berrick and Keating, "Categories and Modules", 2000, 2-34.

[2] Guillou and Skiadas, "WOMP 2004: Category Theory", 2004.

[3] Peter May's lectures, class notes.