DIFFERENTIABILITY OF BROWNIAN MOTION

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Abstract. In this paper I will show that a Brownian Motion is with probability one nowhere differentiable on the interval (0,1). The proof follows from an outline given in Billingsley, 1995.

1. Preliminaries

Proposition 1.1. *If* $f\colon (0,1) \to \boldsymbol{R}$. *is differentiable at* $x \in (0,1)$*, then* $\exists C > 0, \delta > 0$ *such that* $|f(x) - f(s)| \leq C|x - s|$ *for* $s \in [x - \delta, x + \delta]$.

Proof. Since $\lim_{s\to x} \frac{f(x)-f(s)}{x-s} = f'(x) < \infty$, we can choose $\delta' > 0$ such that $|\frac{f(x)-f(s)}{x-s} - f'(x)| < 1$ for $s \in (x - \delta', x + \delta') - \{x\} \equiv D$. Choose $\delta > 0$ s.t. $[x - \delta, x + \delta] - \{x\} \subset D$. Set $C = |1 + |f'(x)||$. Trivially, when x = s $|f(x) - f(s)| = C|x - s| = 0$. Thus for $s \in [x - \delta, x + \delta], |f(x) - f(s)| \leq C|x - s|$. □

Proposition 1.2. *For* $A_1, A_2, \ldots,$ *we have* $P(liminf_n A_n) \leq liminf_n P(A_n)$.

Proof. Follows from Fatou's lemma with $\mathbf{I}_{A_{nk}} \to \mathbf{I}_{A_n}$, $\mathbf{I}_{A_{nk}} \geq 0$. □

2. Main Result

Definition 2.1. A Brownian motion is a stochastic process $[W_t\colon t \geq 0]$ on some (Ω, F, P), with three properties:

(1) $P[W_0 = 0] = 1$
(2) The increments are independent: If $0 \leq t_0 \leq t_1 \leq \cdots \leq t_k$, then $P[W_{t_i} - W_{t_{i-1}} \in H_i, i \leq k] = \prod_{i\leq k} P[W_{t_i} - W_{t_{i-1}} \in H_i]$
(3) For $0 \leq s < t$ the increment $W_t - W_s$ is normally distributed with mean 0 and variance t-s.

Theorem 2.2. *A Brownian Motion* W_t *is with probability one nowhere differentiable on* $(0,1)$.

Proof. Let $M(k,n) = max\{|W_{\frac{k}{n}} - W_{\frac{k-1}{n}}|, |W_{\frac{k+1}{n}} - W_{\frac{k}{n}}|, |W_{\frac{k+2}{n}} - W_{\frac{k+1}{n}}|\}$.

Let $M_n = min\{M(1,n), \ldots, M(n,n)\}$.

Let $B = \{\omega \in \Omega|\ \exists t \in (0,1)\ s.t.\ W(t,\omega)\ is\ differentiable\ at\ t\}$. Take $\omega' \in B$. Then by Proposition 1.1 we have $C > 0, \delta > 0\ s.t.\ |W(t,\omega') - W(s,\omega')| \leq C|t - s|$. Choose $n_0 \in \mathbf{N}$ such that $n_0 > t$ and $\frac{4}{n_0} < \delta$, which holds $\forall n \geq n_0$. Then choose k s.t. $\frac{k-1}{n_0} \leq t \leq \frac{k}{n_0}$. Then $|\frac{i}{n_0} - t| < \delta$, i=k-1, k, k+1, k+2.

So $|W_{\frac{i+1}{n_0}} - W_{\frac{i}{n_0}}| \leq |W_{\frac{i+1}{n_0}} - W_t| + |W_{\frac{i}{n_0}} - W_t| \leq C|\frac{i}{n_0} - t| + C|\frac{i+1}{n_0} - t| \leq 2C\frac{4}{n_0} = \frac{8C}{n_0}$. Thus $M(k,n_0)(\omega') \leq \frac{8C}{n_0}$, and consequently $M_{n_0}(\omega') \leq \frac{8C}{n_0}$. Furthermore, since we can choose a k such that the above inequalities hold for each $n \geq n_0$, $M_n(k,n) \leq \frac{8C}{n}\ \forall n \geq n_o$.

Note that increments $W_{\frac{k+1}{n}} - W_{\frac{k}{n}}$ of a Brownian motion are independent and distributed normally with mean 0 and variance $\frac{1}{n}$, which is the distribution of $\frac{1}{\sqrt{n}}W_1$. So $P\{M(k,n) \leq \frac{8C}{n}\} = [P\{|W_1| \leq \frac{8C}{\sqrt{n}}\}]^3$. Since the normal distribution is bounded above by one and is symmetric, $P\{|W_1| \leq \frac{8C}{\sqrt{n}}\} \leq \frac{16C}{\sqrt{n}} \Rightarrow P\{|W_1| \leq \frac{8C}{\sqrt{n}}\} \leq (\frac{16C}{\sqrt{n}})^3$.

Also, $P\{M_n \leq \frac{8C}{n}\} \leq nP\{M(n,k) \leq \frac{8C}{n}\} \leq n(\frac{16C}{\sqrt{n}})^3$, since $P\{M_n \leq \frac{8C}{\sqrt{n}}\} \leq P\{\{M(1,n) \leq \frac{8C}{\sqrt{n}}\} \cup \cdots \cup \{M(n,n) \leq \frac{8C}{\sqrt{n}}\}\ \}$. But $\forall C > 0\ P\{M_n \leq \frac{8C}{n}\} \leq \frac{16^3C^3}{\sqrt{n}} \to 0$. Let $A_n = \{M_n \leq \frac{8C}{n}\}$. If $\omega' \in B$, then $M_n(\omega') \leq \frac{8C}{n}$ for all $n \geq n_0$, so $\omega' \in liminf A_n$. We have shown $\lim P\{A_n\} \to 0$. But $P(liminf_n A_n) \leq liminf_n P(A_n) = \lim P\{A_n\} \to 0$ by Proposition 2. Hence $P\{liminf A_n\} = 0$. Since $B \subset liminf A_n$ by the above arguments, $P\{B\} = 0$. Thus, the set of ω with W'(t) for some t has measure 0.

□

References

[1] Patrick Billingsley. Probability and Measure. John Wiley and Sons, Inc. 1995.