A Peter May Picture Book, Part 1

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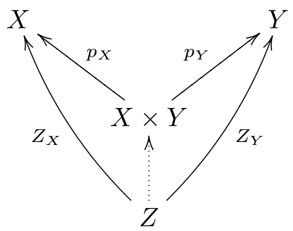
This is the beginning of a larger project, a notebook of sorts intended to clarify, elucidate, and/or illustrate the principal ideas in *A Concise Course in Algebraic Topology*. Many of the "worthwhile exercises" suggested in the text will be done in relatively complete detail, one-line proofs expanded, constructions illustrated, and definitions given as assertions that a certain diagram commutes (which, when I try to explain to my friends, I am walked away from) given explanations anyway. The bulk of this paper focuses on the categorical intuition that the book assumes, with the concepts illustrated in the category of topological spaces. Following that is a careful illustration of a proof that well illustrates the utility of such intuition, which I first presented as part of a directed reading program during the previous academic year. This paper concludes with a list of currently intended illustrations for chapters 5-10, which will be added to a working version of this document as they are done.

p. 16, on limits and colimits in Top and Grp

At this point, anyone who hasn't seen category theory before will probably be utterly despondent. Some pictures will help. We'll do the product and coproduct, initial and terminal object, equalizer and coequalizer, pullback and pushout in the categories of topological spaces and of groups – these are the most important examples, and in practice the only ones that come up all that often.

Let's start with the product in **Top**. Remember, in this category the objects are topological spaces, and the arrows are continuous maps between spaces. Effectively, with the product we start with a collection of unrelated topological spaces. We want the limit of this collection, and assert that it is their Cartesian product endowed with the product topology, along with the canonical projection maps onto each coordinate.

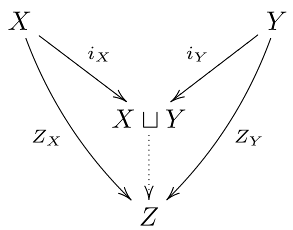
The following is a diagram indicating this for two spaces X and Y; see also Figure 1.



To say that this diagram commutes is to say that the Cartesian product endowed with the product topology has two special properties: first, it is itself endowed with projection maps onto each coordinate; second, that any other space with such projections has a unique projection onto the Cartesian product. We shall check this assertion in general.

Let I be an indexing set; we assert that the product of $X_i, i \in I$ is $\Pi_{i\in I} X_i$ endowed with the product topology, along with its projection maps $p_j : \Pi_{i\in I} X_i \to X_j$ for $j \in I$. On the set-theoretic level, this is relatively clear: given any other space Q with maps $Q_i : Q \to X_i$, we get a factorization through the Cartesian product. Specifically, define the map $f : Q \to \Pi_{i\in I} X_i$ by $q \mapsto (Q_i(q))_{i\in I}$; uniqueness up to bijection is forced by definition. We need only impose a topology on the Cartesian product as a set; since the product topology is homeomorphic to the weak topology generated by the projection maps, and the weak topology is the weakest topology relative to which the projection maps are continuous, since Q_i are actually continuous maps, we are done.

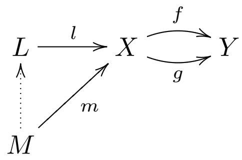
Dually, we assert that the coproduct of topological spaces is the disjoint union endowed with the topology generated by its disjoint subspaces, along with the standard inclusion maps. The diagram



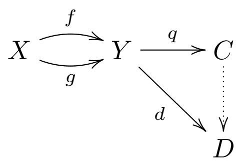
says that each coordinate embeds in the disjoint union, and also that the disjoint union embeds uniquely in any other space endowed with embedding maps. See also Figure 2.

A terminal object is the specific case of the product on the empty set; its dual, the coproduct on the empty set, is called an initial object. It should be clear now that for topological spaces, the terminal object is the one-point space with its only topology and the initial object is the empty set with its only topology.

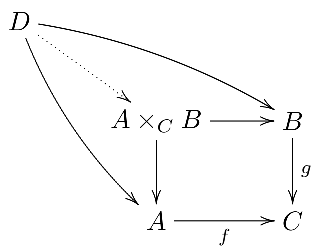
Now we restrict ourselves to a pair of spaces X and Y, but we also have a pair of continuous maps f, g between them. We call the limit of this diagram the equalizer of f and g. As in the case of the product, its name is suggestive – in **Top**, it is the subspace of X on which f and g are equal. In the following, we set $L = \{x \in X | f(x) = g(x)\}$ and l to be the standard inclusion map.



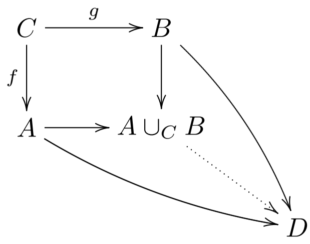
Dually, the coequalizer of X and Y with maps f and g will be a quotient space; specifically, the quotient by the smallest equivalence relation for which, for all $x \in X, f(x) \sim g(x)$. In the following, we call it C, with q the quotient map.



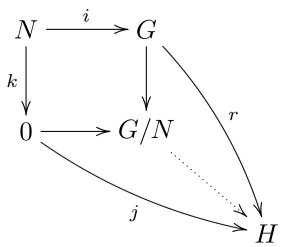
The pullback is the limit of $A \xrightarrow{f} C \xleftarrow{g} B$. For topological spaces, it is the subspace of $A \times B$ given by $\{(x, y) | x \in X, y \in Y, f(x) = g(y)\}$ with its standard maps (projection to each factor followed by inclusion), and is usually written as $A \times_C B$.



The dual to the pullback is the pushout; it is the colimit of $A \xleftarrow{f} C \xrightarrow{g} B$. For a topological space, it is a quotient of $A \cup B$, and as with the coequalizer it is the quotient induced by the weakest equivalence relation such that $f(x) \sim g(y)$. It is usually written as $A \cup_C B$. An illustration provides far more intuition than pages of formality; thus, see Figure 3.



In general, the intuition used for calculating the limiting objects in **Top** carry through to **Grp** with minor modifications. We assert the following for groups: the product is again the Cartesian product with its natural group structure, the coproduct is the free product amalgamated over the identity, an equalizer is the largest subgroup contained in $\{a | f(a) = g(a)\}$, a coequalizer is the quotient by the largest normal subgroup contained in $\{a | f(a) = g(a)\}$, and similarly for the pullback and pushout. We prove only the following (which will be useful for the theorem on page 36, which we later illustrate): if G is a group and N is normal in G, the pushout of $G \hookleftarrow_i N \to 0$ in the category of groups is isomorphic to G/N.



We must show that any object making the diagram commute factors uniquely through G/N and its associated maps. Assume that $H, j : 0 \to H, r : G \to H$ does so, and consider $n \in N$; we trace all of its possible paths. Following i first, we see that $n \mapsto n$, where the latter is in G. On the other hand, for jk to be a group homomorphism, we must have that $jk(n) = 0$. Since the diagram commutes, $jk(n) = ri(n) = 0$, which immediately implies that $\ker(r) \geq N$. Identifying H by the first isomorphism theorem as $G/\ker(r)$, the above gives us a unique factorization through G/N, which verifies the universal property of the pushout.

p. 17, completeness is equivalent to the existence of products and equalizers

I'm relatively convinced that in this case, "the proof is a worthwhile exercise" should be interpreted to mean "the proof is an awful diagram-chasing mess." The idea, however, is well worth seeing. We present the basic idea of the standard proof, and refer the reader to Borceux for the gorey details; effectively, we generalize the idea above that a pullback is a subset of the cartesian product (a product and an equalizer). Thinking dually, a pushout is a quotient of a disjoint union (a coproduct and a coequalizer), and by reversing all the arrows, the following will also prove that cocompleteness is equivalent to the existence of coproducts and coequalizers.

Let $\mathfrak{C}$be a category. If $\mathfrak{C}$is complete, it contains in particular all products and equalizers, so that's fine. So assume that $\mathfrak{C}$has all of its products and equalizers, let $\mathfrak{D}$be a small category (that is, a *set* of objects equipped with a collection of arrows between objects), and let F be a functor from $\mathfrak{D}$to $\mathfrak{C}$. This is just a way of organizing information. For example, the equalizer of $f, g : A \to B$ written in this way would be the limit of $Fx, Fy : F1 \to F2$ where $\mathfrak{D}$is the category with objects 1 and 2 and arrows $x, y : 1 \to 2$, $F1 = A, F2 = B, Fx = f, Fy = g$. What we want is the equalizer of a pair of maps α, β.

$$L \xrightarrow{l} \prod_{D \in \mathfrak{D}} FD \rightrightarrows \prod_{f \in \mathfrak{D}} F(t(f))$$

Here $s(f)$ is the source (domain, corange) of f, $t(f)$ is its target (range, codomain), $\alpha((x_D)_{D\in\mathfrak{D}}) = (x_{t(f)})_{f\in\mathfrak{D}}, \beta((x_D)_{D\in\mathfrak{D}}) = (Ff(x_{s(f)}))_{f\in\mathfrak{D}}$, and L, l is the equalizer of α, β. We assert that L is the limit of F. In our example above, $\alpha(x_A, x_B) = (x_B, x_B), \beta(x_A, x_B) = (f(x_A), g(x_A))$, and equalizing α, β is the assertion that $f(x_A) = g(x_A)$; that is, L is the equalizer of f, g. From here the proof is fairly straight-forward: show that L, l gives a cone on F, and then that any other cone factors uniquely through it. I can offer no more intuition than that gained by staring at the diagrams which prove this, so I leave the remainder to Borceux.

p. 36, an illustration of the proof that every group is π_1 of some space

See Figures 4 and 5.

p. 39, on the adjunction of product and Hom in CGHTop

See Figure 6.

To be continued...

- p. 41, some intuition for the HEP
- p. 42, a mapping cylinder
- p. 42, a cofibration is an inclusion with closed image
- p. 43, illustrating the retraction which a cofibration effects
- p. 47, on duality, intuition for the CHP
- p. 47, a mapping path space
- p. 49, a fiber bundle is a fibration, with picture
- p. 55, $S^m \wedge S^n \cong S^{m+n}$, with picture
- p. 56, $[\Sigma^2 X, Y]$ is an Abelian group
- p. 58, an illustration that $Ci \to Ci/CA$ is a homotopy equivalence
- p. 58, an illustration that "up to homotopy equivalence, each pair of maps in our cofiber sequence is the composite of a map and the inclusion of its target in its cofiber"
- p. 59, a homotopy fiber
- p. 61, illustration of unit and counit between loops and suspension
- p. 63, illustration of the maps in the long exact sequence of pairs
- p. 64, a few calculations actually calculated
- p. 65, the Hopf map is a fiber bundle
- p. 66, a word and picture on change of basepoint
- p. 67, homotopy equivalence induces weak equivalence
- p. 68, the point of the technical lemma, with picture, explanation and condolances
- p. 73, how the lemma implies the HELP
- p. 74, the Whitehead theorem, and what it is not

References

- J. Peter May, *A Concise Course in Algebraic Topology.* University of Chicago Press, Chicago, 1999. http://www.math.uchicago.edu/∼may/
- F. Borceux, *Handbook of categorical algebra 1. Basic category theory.* Cambridge University Press, Cambridge, 1994.