THE GERTRUDE STEIN THEOREM

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Abstract. Gertrude Stein wrote a simple poem: A rose is a rose is a rose. I intend to show that a Frobenius algebra is a Frobenius algebra is a Frobenius algebra and prove that three different definitions are equivalent, facilitating the identification of other Frobenius algebras.

As we saw in the TQFT course, Frobenius algebras are important structures in the study of TQFTs. However, as it may be desirable to connect with other areas of mathematics, a broader set of definitions would be useful in showing that non-categorical structures are Frobenius algebras. So, we have three definitions of Frobenius algebras:

Definition 1. A *Frobenius algebra* is an associative algebra A over a field k with a map $\beta : A \to k$ and multiplication $\mu : A \otimes A \to A$ such that the map $\varepsilon : A \otimes A \to A$ defined as $\beta \circ \mu$ is a duality map.

Recall a duality map is an ε or an η so that the following diagram commutes:

$$\begin{tikzcd}
k \otimes A \arrow[r, "\eta\otimes 1"] & A \otimes A \otimes A \arrow[r, "1\otimes\varepsilon"] & A \otimes k \\
A \arrow[u, "\cong"] \arrow[rr, equal] & & A \arrow[u, "\cong"']
\end{tikzcd}$$

Then we say that η and ε satisfy the triangle identities.

Definition 2. A *Frobenius algebra* is an associative algebra A over a field k with a multiplication $\mu : A \otimes A \to A$ and a map $\varphi : A \otimes A \to k$ such that φ is non-degenerate and associative, i.e. $\varphi(ab, c) = \varphi(a, bc)$.

Definition 3. A *Frobenius algebra* is an associative algebra A over a field k with a unit and a coassociative coalgebra with a counit, i.e. A has maps

$$\mu : A \otimes A \to A \qquad \delta : A \to A \otimes A \qquad \alpha : k \to A \qquad \beta : A \to k$$

and the following diagram commutes:

$$\begin{tikzcd}
A \arrow[r, "\alpha\otimes 1"] \arrow[d, "1\otimes\alpha"] \arrow[dr, equal] & A \otimes A \arrow[r, "\delta\otimes 1"] \arrow[d, "\mu"] & A \otimes A \otimes A \arrow[d, "1\otimes\mu"] \\
A \otimes A \arrow[r, "\mu"] \arrow[d, "1\otimes\delta"] & A \arrow[r, "\delta"] \arrow[d, "\delta"] \arrow[dr, equal] & A \otimes A \arrow[d, "1\otimes\beta"] \\
A \otimes A \otimes A \arrow[r, "\mu\otimes 1"] & A \otimes A \arrow[r, "\beta\otimes 1"] & A
\end{tikzcd}$$

Theorem 1. *A Frobenius algebra is a Frobenius algebra is a Frobenius algebra*

Proof. I intend to show that Definition 1 holds if and only if Definition 2 does and that Definition 3 holds if and only if Definition 1does. This will show that all three definitions are equivalent.

To show Definition 1 implies Definition 2, I will first show that ε is equivalent to φ, i.e. that ε is associative and non-degenerate.

To show that ε is associative, we want to show that $\varepsilon(ab, c) = \varepsilon(a, bc)$. But

$$\varepsilon(ab, c) = \beta(\mu(ab, c))$$

and μ is associative so

$$mu(ab, c) = \mu(a, bc)$$
$$\beta(\mu(ab, c)) = \beta(\mu(a, bc))$$

and

$$\varepsilon(ab, c) = \varepsilon(a, bc))$$

So ε is associative. To show it is non-degenerate, we want to show that, given $a \in A$ such that $a \neq 0$, there is a $b \in A$ such that $\varepsilon(a, b) \neq 0$. We have from the definition that ε is a duality map, so we know that there is an η such that the triangle identites hold. We will assume that ε is degenerate and use this diagram to reach a contradiction.

Let $\{a_i\}$ be a basis for A, and suppose there exists $a \in A$ such that $\varepsilon(a, b) = 0$ for all $b \in A$. Since $b \in A$, we can write $b = \sum a_j$ Then $\eta(b, a) = \sum a_j \otimes \sum a_i = \sum a_j \otimes a_i$ since η is a map from $A \otimes A$ to $A \otimes A$ and any element of $A \otimes A$ can be written as a tensor product of linear combinations of basis elements. Fixing $a \neq 0 \in A$, we have

$$a = 1 \otimes a \mapsto^{\eta \otimes 1} \sum (a_k) \otimes \sum (a_j) \otimes \sum a_i \mapsto^{1 \otimes \varepsilon} \sum (a_k) \otimes \varepsilon(\sum (a_j), \sum a_i)$$

But $\varepsilon(\sum a_j, \sum a_i) = 0$, and $a = \sum a_i \neq 0$, so ε is not a duality map. Thus ε is non-degenerate, and we can take $\varepsilon \equiv \varphi$. To show that 1 implies 2, we need to find $\beta : A \to k$ such that $\beta \circ \mu$ is a duality map.

Since A is an algebra, we know that there is a map $\alpha : k \to A$. So then

$$A \cong A \otimes k \xrightarrow{1 \otimes \alpha} A \otimes A \xrightarrow{\varphi} k$$

Then $\beta = \varphi \circ (1 \otimes \alpha)$. It now remains to show that $\varepsilon \equiv \beta \circ \mu$ is a duality map. To do so, we need to find η so that Diagram 2 commutes.

Define a basis for A as before and say $\{a_i^*\}$ is the dual basis defined so that $a_i^* = b \in A$ such that $\varphi(a_j \otimes b) = 1$ if $i = j$ and 0 otherwise. Let $\eta(1) = \sum a_i \otimes a_i^*$. Fixing $a \in A$, we have

$$a = 1 \otimes a \mapsto^{\eta \otimes 1} \sum a_i \otimes a_i^* \otimes a \mapsto^{1 \otimes \mu} \sum a_i \otimes \varphi(a_i^* a, 1)$$

But $\varphi(a_i * a, 1) = \varphi(a_i *, a)$, since φ is associative, and $\sum a_i \otimes \varphi(a_i^*, a) = a$ by construction, so ε is a duality map. So then 1 implies 2.

Now,to show that 1 implies 3 and that 3 implies 1, we want to show that the maps η and ε given by 1 also give us the diagram in 3. By the triangle identities, we can take A to $A \otimes A \otimes A$ by $\eta \otimes 1$ and then $A \otimes A \otimes A$ to A by $1 \otimes \varepsilon$, and we will have an identity map between the starting and ending points in A. Since the diagram for the triangle identities commutes, we will also have a path from A to $A \otimes A \otimes A$ by $\varepsilon \otimes 1$ and from $A \otimes A \otimes A$ to A by $1 \otimes \eta$, giving us the exterior of the diagram:

$$\begin{tikzcd}
A \arrow[rr, bend left, "\eta\otimes 1"] \arrow[dd, bend right, "1\otimes\eta"'] \arrow[dr, equal] & & A\otimes A\otimes A \arrow[dd, bend left, "1\otimes\varepsilon"] \\
& A \arrow[dr, equal] & \\
A\otimes A\otimes A \arrow[rr, bend right, "\varepsilon\otimes 1"'] & & A
\end{tikzcd}$$

To obtain the interior of the diagram, we note that μ is the multiplication that comes with the algebra, so it suffices to find $\delta : A \to A \otimes A$ so that δ is a coassociative comultiplication satisfying the Frobenius relation, and $\beta : A \to k$ so that β is a counit.

We have from the triangle identities maps δ, δ' so that the following diagram commutes:

$$\begin{tikzcd}
A \cong k\otimes A \arrow[r, "\eta\otimes 1"] \arrow[dr, "\delta"'] & A\otimes A\otimes A \arrow[d, "1\otimes\mu"] \\
& A\otimes A \\
A \cong A\otimes k \arrow[r, "1\otimes\eta"'] \arrow[ur, "\delta'"] & A\otimes A\otimes A \arrow[u, "\mu\otimes 1"']
\end{tikzcd}$$

We first want to show that $\delta = \delta'$, because then we will have a comultiplication. Consider the following diagram:

$$\begin{tikzcd}
& & A \arrow[d, "\cong"] \arrow[ddll, bend right, "\delta"'] \arrow[ddrr, bend left, "\delta'"] & & \\
& & k\otimes A\otimes k \arrow[dl, "\eta\otimes 1"] \arrow[dr, "1\otimes\eta"] \arrow[dd, "\eta\otimes 1\otimes\eta"] & & \\
& A\otimes A\otimes A \arrow[dl, "1\otimes\mu"] \arrow[dr, "1\otimes 1\otimes 1\otimes\eta"] & & A\otimes A\otimes A \arrow[dl, "\eta\otimes 1\otimes 1\otimes 1"'] \arrow[dr, "\mu\otimes 1"] & \\
A\otimes A \arrow[dr, "1\otimes 1\otimes\eta"] \arrow[dddrr, equal, bend right] & & A\otimes A\otimes A\otimes A\otimes A \arrow[dl, "1\otimes\mu\otimes 1\otimes 1"'] \arrow[dd, "1\otimes\mu\otimes 1"] \arrow[dr, "1\otimes 1\otimes\mu\otimes 1"] & & A\otimes A \arrow[dl, "\eta\otimes 1\otimes 1"'] \arrow[dddll, equal, bend left] \\
& A\otimes A\otimes A\otimes A \arrow[dr, "1\otimes\mu\otimes 1"] \arrow[ddr, "1\otimes\varepsilon\otimes 1"'] & & A\otimes A\otimes A\otimes A \arrow[dl, "1\otimes\mu\otimes 1"'] \arrow[ddl, "1\otimes\varepsilon\otimes 1"] & \\
& & A\otimes A\otimes A \arrow[d, "1\otimes\beta\otimes 1"] & & \\
& & A\otimes A & &
\end{tikzcd}$$

If the diagram as a whole commutes, then $\delta = \delta'$. By the construction of δ, δ', the upper portion of the diagram commutes. Of the four diamond shapes in the middle, the upper three show the same composition, and thus commute. Because μ is the multiplication for A, it is associative, and this gives that the bottom diamond commutes. This leaves only the two pairs of triangles on the bottom of the figure. The outer pair is the composition of the η and the ε from the triangle identities, so each triangle is an identity and commutes. The inner pairs commute because ε is defined as the composition of β and μ. But then the entire diagram commutes, and we have $\delta = \delta'$. Further, from the bottom interior triangles, we have that β is a counit. Now consider the path along the exterior of the diagram. This is $(\delta \otimes 1) \circ \delta'$ on the right and $(1 \otimes \delta') \circ \delta)$ on the left, so we have that $\delta = \delta'$ is coassociative. It then remains to show that δ satisfies the Frobenius relation, for which we consider the following diagram:

$$\begin{tikzcd}
 & & A \otimes A \otimes A \arrow[ddrr, bend left, "\mu\otimes 1"] & & \\
 & A \otimes A \otimes A \otimes A \arrow[ur, "1\otimes\mu\otimes 1"] \arrow[rr, "\mu\otimes 1\otimes 1"] & & A \otimes A \otimes A \arrow[dr, "\mu\otimes 1"] & \\
A \otimes A \arrow[uurr, bend left, "1\otimes\delta"] \arrow[ur, "1\otimes 1\otimes\eta"] \arrow[rr, "\mu"] \arrow[dr, "\eta\otimes 1\otimes 1"'] \arrow[ddrr, bend right, "\delta\otimes 1"'] & & A \arrow[ur, "1\otimes\eta"] \arrow[rr, "\delta"] \arrow[dr, "\eta\otimes 1"'] & & A \otimes A \\
 & A \otimes A \otimes A \otimes A \arrow[rr, "1\otimes 1\otimes\mu"] \arrow[dr, "1\otimes\mu\otimes 1"] & & A \otimes A \otimes A \arrow[ur, "1\otimes\mu"'] & \\
 & & A \otimes A \otimes A \arrow[uurr, bend right, "1\otimes\mu"'] & &
\end{tikzcd}$$

Both the left-most regions and the innermost figures commute by the definitions of the maps. Since μ is associative, the rest of the the diagram commutes, and then comparing the center and exterior paths we have that δ is a left and right A-module, so it satisfies the Frobenius relation.

To show that Definition 3 implies Definition 1 we have the following claim:

Claim 1. *ε, defined as $\mu \otimes 1$ composed with $1 \otimes \beta$ where β and μ are the same as in 3 and η defined as the composition of $1 \otimes \alpha$ and $1 \otimes \delta$ satisfy the triangle identities and are thus duality maps.*

Proof of Claim 1. Consider the following diagram:

$$\begin{tikzcd}
A \arrow[r, "\alpha\otimes 1"] \arrow[rr, bend left, "\eta\otimes 1"] \arrow[d, "1\otimes\alpha"] \arrow[dr, equal] \arrow[dd, bend right, "1\otimes\eta"'] & A \otimes A \arrow[r, "\delta\otimes 1"] \arrow[d, "\mu"] & A \otimes A \otimes A \arrow[d, "1\otimes\mu"] \arrow[dd, bend left, "1\otimes\varepsilon"] \\
A \otimes A \arrow[r, "\mu"] \arrow[d, "1\otimes\delta"] & A \arrow[r, "\delta"] \arrow[d, "\delta"] \arrow[dr, equal] & A \otimes A \arrow[d, "1\otimes\beta"] \\
A \otimes A \otimes A \arrow[r, "\mu\otimes 1"] \arrow[rr, bend right, "\varepsilon\otimes 1"'] & A \otimes A \arrow[r, "\beta\otimes 1"] & A
\end{tikzcd}$$

The triangles in the upper left of the figure commute because of the definition of α and μ, and the triangles in the lower right commute because of the definition of δ and β. But then ε, η give an identity as in the following:

$$A \mapsto^{1\otimes\eta} A \otimes A \otimes A \mapsto^{\varepsilon\otimes 1} A$$

$$A \mapsto^{\eta\otimes 1} A \otimes A \otimes A \mapsto^{1\otimes\varepsilon} A$$

which are the conditions necessary for the triangle identities to be satisfied. Since $\mu \otimes 1 = \mu$ and $1 \otimes \beta = \beta$ we have a proof of the claim. $\square$

But then we have maps η and ε satisfying the triangle identities, and $\varepsilon \equiv \beta \circ \mu$, so we have shown that that 3 implies 1 and we are done. $\square$

This is important because there are a number of Frobenius algebras that are not easy to define from a categorical perspective. With these additional definitions, we can pick conditions to suit the particular structure so that it is easier to identify structures as Frobenius algebras for further study.