COLLECTIONS OF SIMPLE CLOSED CURVES INTERSECTING AT MOST ONCE

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Abstract. Given a closed genus g surface in two dimensions, how many simple closed curves can be placed on it so that no two intersect more than once? This question was posed by Farb in 2006 and the precise answer is not known for genus greater than 2. This paper presents a quadratic lower bound and an exponential upper one, and also explains the result of Farb and Leininger that 12 is the maximum number of curves for genus 2.

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1. Introduction

In graph theory, one deals with a collection of objects (vertices) which may be related (if connected by an edge), and one often is concerned with questions of forced patterns – any coloring (with a fixed number of colors) of a sufficiently large complete graph is forced to contain a complete monochromatic subgraph. In a rough analogy, one can talk about forced patterns in topology. We have collections of objects (simple closed curves on closed orientable 2-manifolds) which may be related (if they intersect more than once) and a similar condition forcing order on these collections: a sufficiently large number of simple closed curves, not homotopic to each other, is bound to contain two which intersect more than once. This paper explains a few results towards trying to determine the size of a "best collection." The only cases for which the exact size of a best collection is known are the torus (for which 3 curves is the maximum) and the genus 2 surface (12 curves.) There is a well-known quadratic lower bound, which does not attain 12 for genus 2 – a new, larger quadratic bound is also shown here, which does reach 12 curves in the case of genus 2. Also presented is an exponential upper bound, and an explanation of the result of Farb and Leininger that 12 is the maximum number of curves for genus 2.

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2. Lower Bounds

To find a lower bound, one need only produce an example of a collection with $i(a, b) \leq 1$. It's immediate to check whether a given collection satisfies the intersection condition, so finding lower bounds is a matter of exhibiting large collections. The first quadratic bound comes from splitting up the curves into two parts. If the surface has genus 2g, consider the curve separating it into two manifolds with boundary, each of genus g, connected by a small cylindrical annulus. On each of these submanifolds, choose arcs that go from the boundary to itself, encircling between 1 and g holes. (There are 2g -1 of these: g-1 on "top," g -1 on the "bottom," and one that encircles all g holes. In terms of the fundamental group, these are a_1, a_1a_2, through $a_1a_2\ldots a_g$ and $b_1a_1b_1^{-1}$ through $b_1a_1a_2\ldots a_gb_1^{-1}$.) Now 2g-1 curves intersect each boundary, and any pair of them can be connected to form a simple closed curve. These curves intersect only on the annulus (by construction they are disjoint elsewhere.) The intersection of one arc with the annulus is homotopic to a pair of straight lines. Connect each arc to all the arcs on the other submanifold that are "interior" to it, and we have a collection of simple closed curves intersecting at most once, with $\frac{(2g-1)(2g-2)}{2}$ curves. (See Fig. 1).

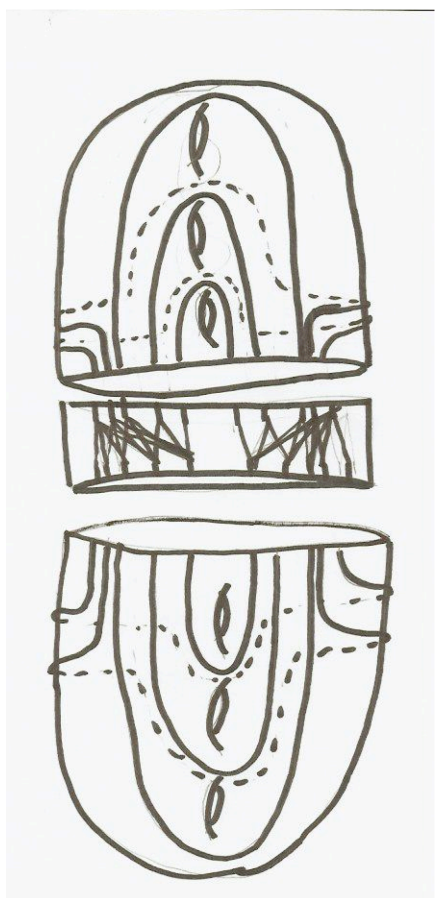

Figure 1.

The argument is similar for surfaces of odd genus, except that the two subsurfaces have unequal genus. For a genus g surface, the lower bound is $\frac{(g-1)(g-2)}{2}$.

A better lower bound, also quadratic, can be found using the 4g+2 polygonal presentation. Any genus g surface is homeomorphic to a polygon with 4g+2 sides, with opposite sides identified in an orientation-preserving direction. When identifications are made, only two vertices of the polygon are actually distinct: the odd-numbered and the even-numbered. So any curve connecting a vertex to an identical vertex is closed; if it doesn't cross itself it is simple. Connecting the odd vertices into a (2g+1)-gon makes 2g+1 loops sharing only one point. There are 2g-2 diagonals emanating from one odd vertex (which don't intersect each other except at that single point.) Then there is one "all-round curve" which circles the polygon's perimeter. Finally, there are 2g(g-1) "bounces" which follow the perimeter of the polygon for at least three sides, circle an even vertex, and return to the odd vertex. (See Fig. 2 for an example with g = 2).

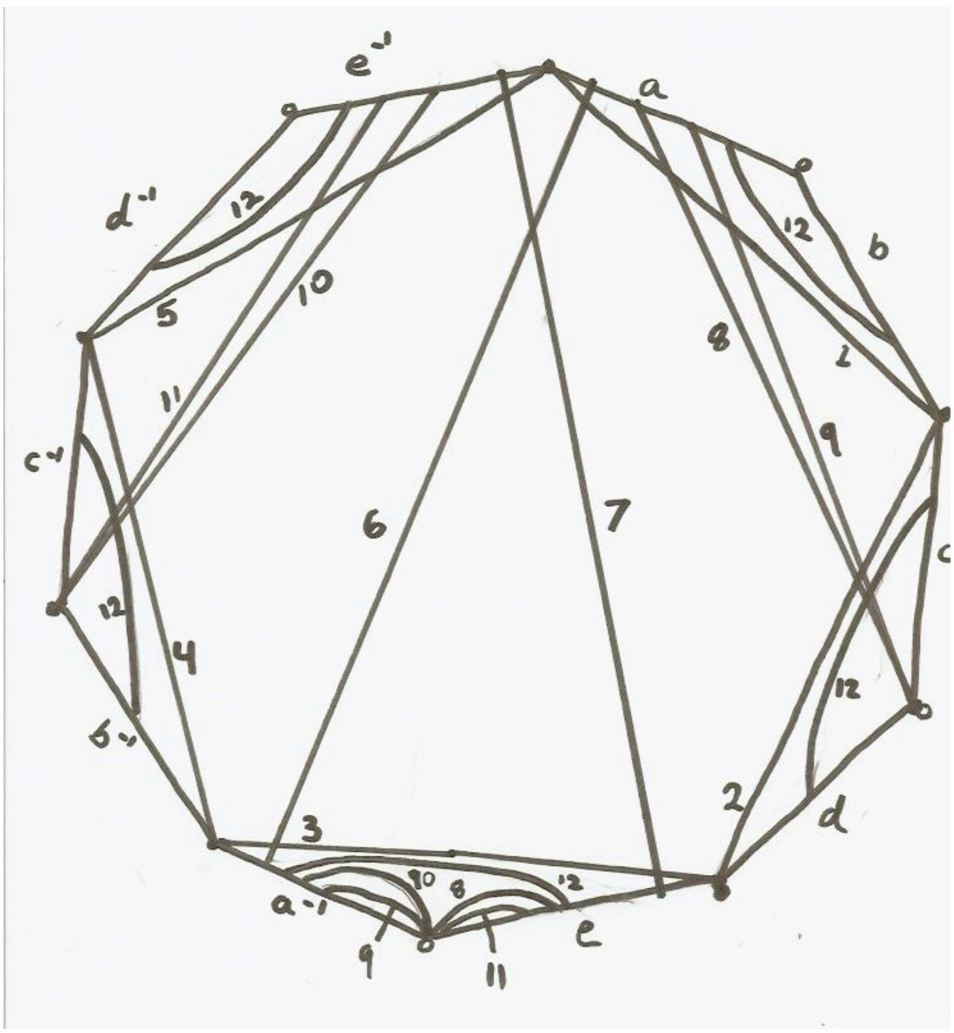

Figure 2.

This gives a total of $2g^2 + 2g$ curves. The formula does not apply to the torus; its 4g+2 presentation is a hexagon, and the odd vertices form a triangle, which is too small for additional diagonals or bounces. The $2g^2 + 2g$ lower bound produces 12 for genus 2, which is the only case for which the size of the best collection is known.

3. The Genus 2 Case

The maximum number of curves that can be placed on a genus 2 surface, such that no two intersect more than once, is 12.

We begin by clarifying the kind of curves in question with a lemma.

Lemma 3.1. *In a best collection of curves, in any genus, all curves are nonseparating.*

Proof. Suppose there was a separating curve γ. We claim that it can be replaced with two other nonseparating curves, which do not intersect with any other curve more than once. This would show that the original collection was not a best collection. All the other simple closed curves on the surface are confined to one of the subsurfaces separated by γ; if a curve crossed γ once, it would have to cross twice, and so would not be permissible. Pick a simple closed curve on each subsurface, and create a new curve that follows around one on its left side (well-defined since this surface is orientable), crosses γ, follows the second closed curve on the left, and returns to its starting point. Then create a second curve that follows around one simple closed curve on the left, then follows the second one on the right. The two new curves only intersect each other in one place. They cannot intersect any other curves in the collection more than once, since they can follow the original curves as closely as necessary to avoid collisions. (See Fig. 3.)

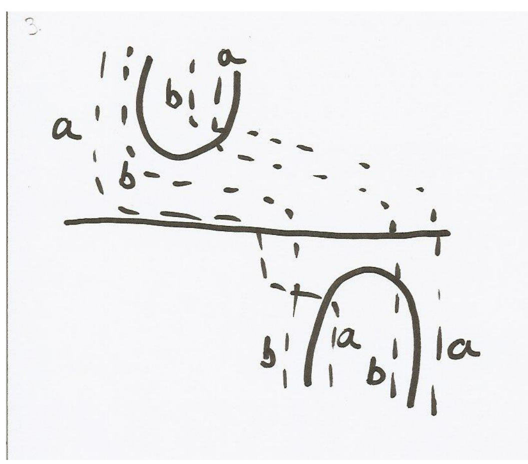

Figure 3.

We have replaced a separating curve with two permissible nonseparating curves; thus, there cannot be a separating curve in the best collection. ☐

Because a nonseparating curve is the same as a curve which is not trivial in the first homology group $H_1(\Sigma_g, \mathbb{Z})$, this lemma allows us to apply homology and use linear algebra (as in Lemma 3.2). Now return to the genus 2 surface. First we claim that any best collection of curves includes a pants decomposition as in Fig. 4.

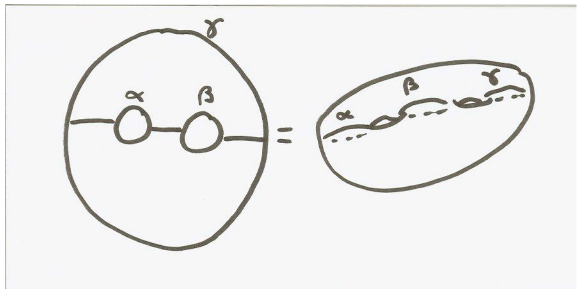

Figure 4.

Then, because there are limitations on surfaces on pairs of pants, we show that there can be no more than twelve curves with such a decomposition. Since all curves are nonseparating, we can assume without loss of generality that the first curve is curve α in Fig. 4, by the change of coordinates principle. The next curve must be nonseparating, and it may be disjoint from α or intersect it once. If it is disjoint, then without loss of generality we can identify it as β.

Lemma 3.2. *If there are at least 2g + 1 curves in a best collection, on any surface, then there exists a pair of disjoint curves in that collection.*

Proof. We will show that there can be no more than 2g curves, all of whom have intersection number exactly 1. Consider the first homology group $H_1(\Sigma_g, \mathbb{Z})$ as a vector space $\mathbb{Z}^{2g}$ spanned by the usual symplectic basis $(a_1, b_1, \ldots, a_g, b_g)$. (FM 119)The coordinates of a curve here are the numbers of times it winds around each given basis element. The classification of surfaces tells us that a genus g surface can be divided into g punctured tori. The number of intersections between two curves $(a_1, b_1, \ldots, a_g, b_g)$ and $(c_1, d_1, \ldots, c_g, d_g)$ is the sum of the intersection numbers (in absolute value) on each punctured torus, where $\hat{\imath}((a_i, b_i)(c_i, d_i)) = a_i c_i - b_i d_i$. Now construct a set of curves, each of which intersects every other exactly once. The first, without loss of generality, can be set to

$$(1, 0, 0, \ldots, 0).$$

The next, in order to intersect once, must be

$$(0, 1, 0, \ldots, 0).$$

All subsequent curves must begin with two 1's so as to intersect each of the first two curves. The next two must be

$$(1, 1, 1, 0, 0 \ldots, 0).$$

and

$$(1, 1, 0, 1, 0, \ldots, 0).$$

The list continues, but it must end at 2g curves: two for each punctured torus. □

This shows that both α and β are members of any best collection, since any best collection contains at least $g^2 + 2g$ curves, as shown above. Now it remains to be shown that there is another disjoint nonseparating curve in the collection, which can be identified as γ. Again the argument uses vectors and proceeds by method of contradiction. We now have

$$\alpha = (0, 1, 0, 0)$$

and

$$\beta = (0,0,0,1).$$

The only vectors that intersect either of them once (without intersecting any others more than once) are of the form

$$(1, a, 0, b),$$

$$(1, c, 1, d),$$

or

$$(0, e, 1, f).$$

But since none of these can intersect each other more than once, b and e are each ± 1 and, without loss of generality (since the difference of two f's, a's, c's, or d's is at most ± 1) the other coordinates are each 1 or 0. This yields only 10 curves, and we know from the lower bound that 12 are possible, so this couldn't be a best collection. That means that all three of (α, β, γ) are in any best collection. This decomposes the genus 2 surface into two pairs of pants.

Lemma 3.3. *If the genus 2 surface is divided into two pairs of pants connected on their boundaries, then any best collection has at most 12 curves.*

Proof. Any simple closed curve is either a boundary curve (one of α, β, or γ) or consists of an arc on one pair of pants and an arc on the other, beginning and ending on boundary components. Simple closed curves on the interior of the pair of pants are homotopically trivial. (FM 154) Arcs from one boundary component to itself cannot be part of a permissible collection since they intersect that boundary component twice. So we are concerned with arcs that go from one boundary component to another. Between two boundary components there are infinitely many distinct arcs since we can include any number of Dehn twists about any boundary component; however, since a pair of pants is homeomorphic to a thrice-punctured sphere, any curve with a Dehn twist is self-intersecting (see Fig. 5).

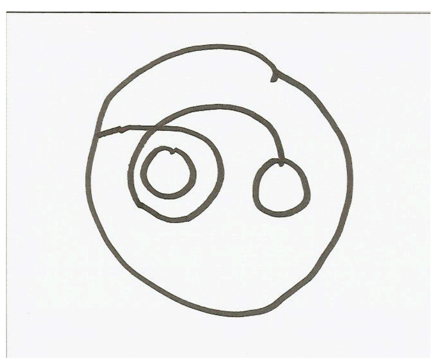

Figure 5.

So, neglecting twists, the only arcs available are those without twists going from one boundary component to the other: there are three of these. So there are nine ways to connect any one with any other. Adding the three boundary curves, we obtain a maximum of 12. □

4. An Exponential Upper Bound

Theorem 4.1. *The number of simple closed curves on a genus g surface such that no two intersect more than once is $O(2^{3g})$.*

The proof is by induction on genus. Suppose we had a best collection of simple closed curves on a genus g surface. Since by definition genus is the number of disjoint closed curves which do not separate the surface, and since all curves are nonseparating, cutting along one curve in the best collection reduces the genus by one. Now what remains is a collection of curves and arcs. Any closed curves (which thus are disjoint from the curve that was cut) are identically curves that were in the best collection. However, some of the arcs may be duplicates; several distinct curves in the best collection may have produced homotopically equivalent arcs in the cut surface. How many such duplications are there? If there are several different such curves, they must differ only in an annular neighborhood of the cut curve (otherwise they would produce different arcs.) On this neighborhood (see fig. 6) the only distinct arcs going from one boundary component to the other are the compositions of the "straight" arc a with Dehn twists about the only simple closed curve b, arcs of the form $T_b^k(a)$, or the curve which is k Dehn twists about b applied to a.

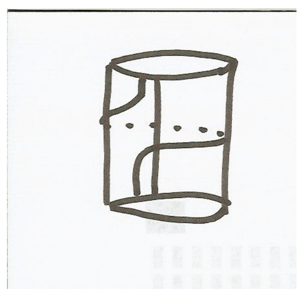

Figure 6.

How many such arcs can be placed on the annulus simultaneously so that no two intersect more than once? We have

$$i(T_b^k(a), a) = |k| i(a, b)^2$$

and since $i(a, b) = 1$, two arcs satisfy the intersection condition if and only if the number of Dehn twists differs by one; this means there can be only two arcs placed on the annulus at a time. This means that the number of curves on the genus g surface can be at most twice the number of curves on the genus (g-1) surface, since no arc represents more than two duplicated curves.

Proceeding inductively, one can continue cutting along closed curves, reducing genus each time and leaving arcs and boundary components. The only curves that are cut during this process are a collection of mutually disjoint curves; all other curves in the collection become arcs going from one boundary component to another.

When no further disjoint curves remain to be cut, there is a punctured surface, possibly with positive genus, and all remaining elements of the best collection present as arcs. The boundary components were curves in the best collection, so no arc can intersect one twice; every arc begins on one boundary component and ends on a different one. (see fig. 7 for an example of such an arc.)

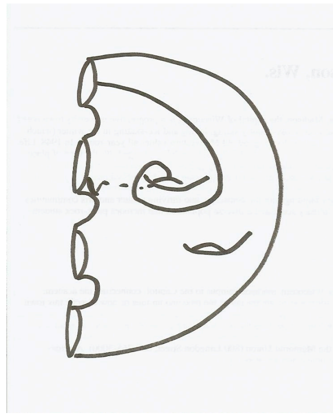

Figure 7.

To prove an exponential upper bound, it would now suffice to show that the number of arcs on this punctured surface is exponential in genus. Let g be the genus of the punctured surface. Consider the inclusion map from this punctured surface to the closed genus g surface obtained by taking the connect sum with a disc for every boundary component, and map each arc to a curve by connecting its two endpoints (in a unique way up to homotopy, since the sphere is simply connected.) If we fix a basepoint on one of these curves (an image of an arc) then the curve is an element of the fundamental group of the genus g surface, and this curve is identical to the arc that was its preimage outside an annular neighborhood of the boundary components.

That implies that every arc on the punctured surface is determined by the boundary components at its endpoints, and the word in the fundamental group that defines the corresponding curve. Now we can bound the number of possible curves satisfying the intersection condition. There are 2g generators for the fundamental group, two at each genus; call them a_i and b_i for $1 \leq i \leq g$. Since

$$i(T_b^k(a), a) = |k| i(a, b)^2,$$

by a similar argument as above, for two curves to intersect at most once, the exponents of any a_i or b_i must differ by at most one (unless one has zero exponent) so there can be at most three choices at each generator of the fundamental group, giving an exponential bound of $3^2 g$. We multiply this by the number of choices of two boundary components. But as shown above, there can be no more than 3g - 3 curves on a surface, all pairwise disjoint, so there are no more than 3g - 3 boundary components. The number of choices of two among these is quadratic in g, so the exponential bound still holds.

5. Acknowledgments

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