A COMBINATORIAL APPROACH TO PANTS DECOMPOSITIONS

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ABSTRACT. In this paper I aim to address the problem of counting the homeomorphism classes of pants decompositions for surfaces of a given genus. It can be seen (for example in [3]) that homeomorphism classes of pants decompositions are in bijective correspondence with isomorphism classes of (non-simple) trivalent graphs. Taking this bijection for granted, I will discuss some aspects of the purely combinatorial graph theoretic problem in place of studying the surfaces themselves. I conclude by constructing at least $\frac{((\lfloor\sqrt{n}\rfloor-1)!)^2}{2}$ trivalent graphs on n vertices.

CONTENTS

1. DESCRIPTION OF THE PROBLEM

I will assume the reader has some knowledge of simple closed curves on surfaces. I will also assume familiarity with the notions of homotopies of curves and homeomorphisms of surfaces. I will use $\Sigma_{g,b}$ to mean a surface of genus g with b boundary components. A pants decomposition of Σ_g is a collection of $3g-3$ pairwise non-homotopic, non-intersecting simple closed curves that cuts the surface into $2(g-1)$ copies of $\Sigma_{0,3}$ that intersect only along the boundary components. Two pants decompositions $A = \{\alpha_1, \dots, \alpha_k\}, B = \{\beta_1, \dots, \beta_k\}$ are said to be homeomorphic if there is a homeomorphism of the surface that takes the set A onto B. There is a set of moves between pants decompositions and relations between those moves, described in [1] and [2]. A complex is then created, called the pants complex, in which vertices are connected by an edge if there is a single move taking one to the other, and 2-cells are filled in according to the relations between the moves (see [1]). The 1-skeleton of the pants complex is called the pants graph. However, the moves used to define the pants graph do not always change the homeomorphism type of the pants decomposition (that is, homeomorphic pants decompositions can be adjacent vertices in the pants graph). Some moves change only the homotopy class of a pants decomposition. In particular, this means that the pants graph is infinite, since there are infinitely many homotopy classes of simple closed curves.

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However, the mapping class group acts on this complex, and the quotient by this action is a finite graph. That is, there are finitely many homeomorphism classes of pants decompositions.

The number of homeomorphism classes of pants decompositions is not known. Also, given two pants decompositions, it is hard to tell just by looking at the simple closed curves if they are homeomorphic. In order to proceed, we must either find invariants by which we may recognize non-homeomorphic pants decompositions, or we must translate this problem into a simpler one. In an attempt at the former, one notices the following invariant:

Definition 1.1 (Bounding k-set)**.** A set of simple closed curves $A = \{\alpha_1, \dots, \alpha_k\}$ is said to separate a connected surface Σ_g (or, equivalently, is said to be bounding; or is called a bounding k-set) if cutting the surface along the curves α_i creates two disjoint surfaces with boundary, and no subset of A is bounding. Let the surfaces created be denoted Σ_{g_1,b_1}, Σ_{g_2,b_2} with $g_1 \leq g_2$. Then g_1 is said to be the genus of the bounding set A.

A separating curve can be thought of as a bounding 1-set. And just as a separating curve must be mapped to a separating curve by any homeomorphism, so must a bounding k-set be mapped to a bounding k-set. Note that the genus of a bounding set must also be preserved by homeomorphism. Therefore, given two pants decompositions, if one of them has a bounding k-set of genus g for some k and g and the other does not, then the pants decompositions are non-homeomorphic.

Given a particular pants decomposition, one might go so far as to create a complete list of bounding k-sets and their genus. This list is then a complete description of the pants decomposition. Unfortunately, this procedure tells us nothing about how to create such a list. We cannot use this information in any reasonable way to create pants decompositions. I think this approach may have some merit. But since I could not find a way to turn this collection of invariants into a collection of pants decompositions, I instead translated this question into one about graphs. I will not discuss bounding k-sets further in this paper.

2. Graph Theory

Definition 2.1 (Graph)**.** A graph G is a pair (V, E) of sets, consisting of a set of vertices V and a collection of edges denoted E with elements of the form $\{v, w\}$, where $v, w \in V$. The set V is sometimes written $V(G)$ to mean the vertex set of the graph G. Similarly, the edge set is sometimes written $E(G)$. An edge having the same beginning and ending vertex is called a loop. If multiple edges have the same two endpoints, the edges are called parallel. Typically, a graph without loops or parallel edges are called simple graphs, while graphs with loops and parallel edges are called multigraphs. I will use the term graph to mean multigraph, since that is what this paper is mainly concerned with.

Definition 2.2 (Degree)**.**
The degree of a vertex is the number of edges that have an endpoint at that vertex.

Definition 2.3 (Regular Graph)**.** A graph is called regular if all of its vertices have the same degree. In the particular case of interest to this paper, a graph that is regular of degree three is called trivalent (or sometimes cubic).

In a graph, two vertices are called adjacent if they are the endpoints of an edge. Two edges are called adjacent if they share an endpoint. A vertex is adjacent to an edge if the vertex is an endpoint of the edge.

Definition 2.4 (Walk)**.** A walk in a graph is an alternating sequence of vertices and edges, beginning and ending with a vertex, such that each consecutive pair in the sequence is of adjacent elements. It may be denoted $(v_1, e_1, \ldots, e_k, v_{k+1})$, where the $v_i \in V$ and $e_i \in E$ for all i. The walk is said to be of length k, where k is the number of edges in the walk.

Definition 2.5 (Path)**.** A path is a walk in which no vertex or edge appears twice.

Definition 2.6 (Cycle)**.** A cycle is a path in which the beginning and ending vertices are equal. A cycle of length 1 is called a loop. A cycle of length 2 travels along parallel edges. A cycle of length k is a k-gon.

One can see in [3] that the homeomorphism classes of pants decompositions on a genus g surface are in bijective correspondence with trivalent graphs on $2(g-1)$ vertices. Counting (or even finding) the homeomorphism classes of pants decompositions of a surface with a given genus g is equivalent to counting (or finding) the isomorphism classes of trivalent graphs on $2(g-1)$ vertices.

3. Proof of a Lower Bound

The first step in trying to count these graphs is to find graph invariants that let us know quickly when a new graph has been created. Some typical graph invariants, such as vertex degree, are not particularly useful.

The invariant I will make use of is the presence and length of cycles. It is easy to see that if a graph G contains a cycle of length k, then any graph automorphism must preserve the length of the cycle. This can easily be used to generate graphs, whereas it was very hard to use the notion of bounding k-sets to generate pants decompositions. In the latter case, we would have needed a complete list of the bounding sets of a pants decomposition to completely identify it.

It remains to create graphs with clearly non-isomorphic cycle structures.

Theorem 3.1. *Let $T(n)$ be the number of isomorphism classes of trivalent graphs on n vertices. Then*

$$T(n) \geq \frac{((\lfloor\sqrt{n}\rfloor - 1)!)^2}{2}$$

Briefly, the method of proof will be this:

1 Find many small pieces of a graph that can be easily put together.
2 Find many ways to put the pieces together. graph.
3 Fill in the graph with edges until it is trivalent.

Proof. The simplest way to make use of cycles is to use them as the pieces from which our graphs will be formed. Let C_j denote the cycle with j vertices. Consider the collection $C_2, C_4, \ldots, C_{2k}$ of cycles. This collection has

$$2 + 4 + 6 + \cdots + 2k = k(k+1)$$

total vertices. We must choose k, then, in relation to n such that this sum has not exceeded the n total vertices that we are allowed. In particular, note that if $k = \lfloor\sqrt{n}\rfloor - 1$, then:

$$k(k+1) < (k+1)(k+1) \leq n$$

If we were to arrange these k graph pieces in a line from left to right, there would be $k!$ ways to order them. Given an ordering, we must connect the pieces together. (Let us think of the polygons in order from least to greatest number of vertices, so that I may index them from left to right and have a relationship between the index and the number of vertices. It should be clear that it is not necessary for the proof. It will just be cleaner and easier to follow with a single subscript. I will state when I want the index to be thought of as left-to-right or as least-to-greatest if it is not clear from context.) In order to make the connections, first choose two vertices from each cycle. If the polygon has $2i$ vertices, then there are i ways of choosing pairs that are mutually inequivalent under automorphisms of the polygon. In particular, the distance between the pair of vertices in the polygon is invariant under automorphisms of the polygon. We may choose a pair of vertices that are at distance $1, 2, 3, \dots, i$, thus there are i possible choices. Let the chosen vertices in the i-th polygon be denoted v_{i_1} and v_{i_2}. Add a vertex with a loop to the left of the leftmost polygon, and connect it by an edge to v_{1_1}. Then, for $i = 2, 3, \dots, k$, connect $v_{(i-1)_2}$ to v_{i_1} (this is meant to connect the polygons from left to right in a general ordering). Then add another vertex with a loop to the right of the rightmost polygon and connect it to v_{k_2}. So far $k(k+1)+2$ vertices have been used. Note that by the choice of k there certainly were at least those two available vertices that created the loops at either end of this graph. If $n - k(k+1) - 2 > 0$ (i.e. if there are remaining vertices), they may be added to the largest polygon without harm. (They may NOT be added to an arbitrary polygon. Adding extra vertices to a small polygon may cause it to become the same size as another polygon already used, possibly creating new automorphisms of the graph.)

There were $k!$ orderings of the pieces, but now, having connected them, we may see that reverse orderings create isomorphic graphs. So there are really $\frac{k!}{2}$ pairwise non-isomorphic orderings. Then, for $i = 1, 2, \dots, k$ we made k independent choices of i possibilities, yielding $\prod_{i=1}^{k} i = k!$ ways to connect each different ordering. So we have $\frac{(k!)^2}{2}$ possible graphs so far. It remains to "fill out" the graph, i.e. to add edges until every vertex is of degree three.

Within the i-th polygon, begin with v_{i_1} (which is already of degree 3). Going around the cycle clockwise, whenever a vertex of degree two is found, connect it by an edge to the next vertex of degree two. This will add a sequence of parallel edges around the polygon, skipping over only the vertices connected by bridges to other polygons. In the i-th polygon, the only two paths between v_{i_1} and v_{i_2} are still those along the boundary of the polygon (i.e. adding these parallel edges did not add smaller cycles in the i-th polygon or new $2i$-cycles that do not travel along the boundary), so no isomorphic graphs were created by this process.

We have now constructed $\frac{((\lfloor\sqrt{n}\rfloor-1)!)^2}{2}$ non-isomorphic trivalent graphs on n vertices, thus proving the theorem. □

References

[1] Allen Hatcher. Pants Decompositions of Surfaces. June 12, 1999.
[2] Dan Margalit. Automorphisms of the Pants Complex. Duke Mathematical Journal. 2004.
[3] Andrew Putman. A Note on the Connectivity of Certain Complexes Associated to Surfaces. December 26, 2006.
[4] Numerous discussions with Benson Farb and Matthew Day.