Topological Background to the Weil Conjectures

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Introduction In this paper we formulate some topological analogues to the Weil conjectures. In section 1, we give an informal outline of the setup of the Weil conjectures. For more information, see Sam Raskin's REU paper.

In section 2, we state a simple calculation that motivates what will come. In section 3, we give a topological setting that admits an analogue to the Weil conjectures, and give, in this setting, a cohomological interpretation of the zeta function, implying in particular that in this topological setting, the zeta function is rational.

In section 4, using our cohomological interpretation of the zeta function from section 3, we prove a functional equation for our topological zeta function.

Our final goal is an analogue of the Riemann hypothesis of Weil, which we give in section 6. This requires a different setup from that which we gave in section 3. Moreover, the proof, and even the statement, require some knowledge of Kähler manifolds. We have stated the results we need in section 5.

We should warn the reader that as our focus is on the topological analogues, we have not given careful statements of the Weil conjectures, for which the reader may consult Sam Raskin's paper mentioned above. In particular, we warn the reader that the Weil conjectures are stated for smooth connected projective varieties defined over $\mathbb{F}_q$.

1) Setup of the Weil Conjectures

Consider a closed smooth connected subvariety X of projective space $\mathbb{P}^n_{\overline{\mathbb{F}}_q}$, defined over $\mathbb{F}_q$. The Frobenius automorphism $Fr: t \mapsto t^q$ acts on X coordinatewise, and the fixed points of $(Fr)^k$ are exactly the $\mathbb{F}_{q^k}$-valued points of X. We ask the following question

. How does the number $\#(X(\mathbb{F}_{q^k}))$ of $\mathbb{F}_{q^k}$-valued points of X vary with k?

It is convenient to consider the corresponding generating function

$Z(X,t) = \sum_{k \geq 1} \#(X(\mathbb{F}_{q^k}))\, t^k$. We also define the zeta function

$$\zeta(X,t) = \prod_{\substack{Fr\text{-orbits}\\ x \text{ of}\\ X}} (1 - t^{\deg x})^{-1},$$

where $\deg x$ denotes the order of x.

A basic calculation then tells us that

$t \frac{\partial}{\partial t}(\log(\zeta(X,t))) = Z(X,t)$.

Rewriting this, we have

$$\log(\zeta(X,t)) = \sum_{k \geq 1} \frac{\#(X(\mathbb{F}_{q^k}))}{k} t^k, \quad \text{or}$$

$$\zeta(X,t) = \exp\left(\sum_{k \geq 1} \frac{\#(X(\mathbb{F}_q^k))}{k} t^k\right).$$

The Weil conjectures are based on an analogy with topology. The purpose of this paper is to tell the topological side of the story.

2) A toy example

We now imagine that X is a finite set. Let V be the $\mathbb{Q}$-vector space with basis the points of X. There is an obvious action $V \circlearrowleft F_r$. We then have a motivational

calculation $\zeta(X,t) = \det(1 - F_r \cdot t)^{-1}$

This calculation is a special case of a formula that we will prove in the next section. The interested reader may, however, wish to give a proof of the lemma directly from the definition of $\zeta(X,t)$.

3) Model example

Let X be a compact orientable smooth n-dimensional manifold, and let $f : X \to X$ be a smooth map. Although this is not necessary for now, our real analogy is with the case that X is a complex manifold; keeping this in mind will explain some of the assumptions that we will make.

Assume that each power f^k of f has finitely many fixed points, and that if x is a fixed point of f^k, then $\det((1-f)^k : T_x X \to T_x X) > 0$

By analogy with the zeta function of a variety over $\mathbb{F}_q$, we define
$\zeta(X, F, t) = \exp\left(\sum_{k\geq 1} \frac{|F^k|}{k} t^k\right)$, where
$|F^k|$ equals the number of fixed points of F^k.

<u>lemma</u> Let V be a finite dimensional vector space over a field F, and let $T: V \to V$ be a linear map. Suppose that in $\overline{F}$, $\det(1 - T\cdot t) = \prod_{i=1}^{\dim V} (1 - c_i t)$. Then $Tr(T^k) = \sum_{i=1} c_i^k$, and

$$-\log(\det(1 - Tt)) = \sum_{k\geq 1} Tr(T^k)\frac{t^k}{k}.$$

<u>proof</u> For the first statement, triangularize the matrix of T over $\overline{F}$, so

$$T = \begin{pmatrix} c_1 & & * \\ & \ddots & \\ 0 & & c_{\dim V} \end{pmatrix}. \quad \text{Then}$$

$$T^k = \begin{pmatrix} c_1^k & & * \\ & \ddots & \\ 0 & & c_{\dim V}^k \end{pmatrix}, \quad \text{so } Tr(T^k) = \sum_{i=1}^{\dim V} c_i^k.$$

Thus
$$\sum_{k\geq 1} Tr(T^k)\frac{t^k}{k} = \sum_{k\geq 1}\left(\frac{\sum_{i=1}^{\dim V} c_i^k}{k}\right)t^k$$

$$= \sum_{i=1}^{\dim V}\sum_{k\geq 1}\frac{c_i^k}{k}t^k$$

$$= \sum_{i=1}^{\dim V} -\log(1 - c_i t)$$

$$= -\log(\det(1 - Tt)).$$

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Theorem $\zeta(X,F,t) = \prod_{i=0}^{n} \det\left(1 - F^* \cdot t : H^i(X,\mathbb{Q}) \to H^i(X,\mathbb{Q})\right)^{(-1)^{i+1}}$.

Note that when the dimension of X equals 0, this gives a proof of the formula stated in the previous section.

Proof of the theorem By the Lefschetz fixed point theorem, $|F^k| = \sum_{i=0}^{n} (-1)^i \operatorname{Tr}\left((F^k)^* : H^i(X,\mathbb{Q}) \to H^i(X,\mathbb{Q})\right)$.

Substituting into the definition of $\zeta(X,F,t)$, we have

$$\zeta(X,F,t) = \exp\left(\sum_{k\geq 1}\left(\sum_{i=0}^{n} (-1)^i \operatorname{Tr}\left((F^k)^* : H^i(X,\mathbb{Q}) \to H^i(X,\mathbb{Q})\right)\right)\frac{t^k}{k}\right)$$

$$= \exp\left(\sum_{i=0}^{n}\left|\sum_{k\geq 1} (-1)^i \operatorname{Tr}\left((F^k)^* : H^i(X,\mathbb{Q}) \to H^i(X,\mathbb{Q})\right)\right)\right)\frac{t^k}{k}$$

$$= \prod_{i=0}^{n} \exp\left(\sum_{k\geq 1} (-1)^i \operatorname{Tr}\left((F^k)^* : H^i(X,\mathbb{Q}) \to H^i(X,\mathbb{Q})\right)\right)\frac{t^k}{k}$$

$$= \prod_{i=0}^{n} \exp\left(\sum_{k\geq 1} (-1)^i \operatorname{Tr}\left((F^*)^k : H^i(X,\mathbb{Q}) \to H^i(X,\mathbb{Q})\right)\right)\frac{t^k}{k}$$

by the lemma

$$= \prod_{i=0}^{n} \exp\left((-1)^{i+1} \log\left(\det\left(1 - F^* \cdot t\right)_{\text{on } H^i(X,\mathbb{Q})}\right)\right)$$

$$= \prod_{i=0}^{n} \det\left(1 - F^* \cdot t : H^i(X,\mathbb{Q}) \to H^i(X,\mathbb{Q})\right)^{(-1)^{i+1}}.$$

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By way of analogy with this theorem, Weil conjectured that the zeta function of an algebraic variety over $\mathbb{F}_q$ is always a rational function. (As in the introduction, we assume the variety to be smooth, projective, and connected.)

4) Functional equation for the model example

The above theorem, combined with poincare duality, suggest that the zeta function $\zeta(X, f, t)$ of an endomorphism of a smooth manifold should satisfy some sort of functional equation. We will derive the form of that functional equation in the case that the dimension n of X is even; this is of course always the case when X is a complex manifold.

Functional equation

Let X be a compact smooth orientable manifold of even dimension n, and let $f: X \to X$ be a smooth orientation preserving map, satisfying the conditions of section 3. Then

$$\zeta(X, f, D^{-1}t^{-1}) = t^{\chi_X} \cdot D^{\chi_X/2} \cdot \zeta(X, f, t),$$

where D is the degree of f, and χ_X is the euler characteristic of X.

<u>Obvious lemma</u> If the eigenvalues of $f^*: H^r(X, \mathbb{C}) \to H^r(X, \mathbb{C})$ are $\lambda_1, \ldots, \lambda_k$, then the eigenvalues of $f^*: H^{n-r}(X, \mathbb{C}) \to H^{n-r}(X, \mathbb{C})$ are $D/\lambda_1, \ldots, D/\lambda_k$.

<u>proof of the lemma</u> f^* is compatible with cup products, so this is immediate from Poincare duality. More specifically, if θ is an eigenvalue of $f^*: H^{n-r}(X, \mathbb{C}) \circlearrowleft$, with eigenvector β, then by poincare duality there exists $\alpha \in H^r(X, \mathbb{C})$ such that $\alpha \cup \beta \neq 0$ in $H^n(X, \mathbb{C})$; as $f^*(\alpha \cup \beta) = D \cdot \alpha \cup \beta$, and $f^*(\beta) = \theta\beta$, we must have $f^*(\alpha) = D/\theta\, \alpha$, so $D/\theta = \lambda_i$ for some $1 \le i \le k$, so $\theta = D/\lambda_i$. Conversely, let α be an eigenvector of λ_i in $H^r(X, \mathbb{C})$, then there exists $\beta \in H^{n-r}(X, \mathbb{C})$ so that $\alpha \cup \beta \neq 0$ in $H^n(X, \mathbb{C})$, which forces $f^*(\beta) = D/\lambda_i\, \beta$, so all D/λ_i occur as eigenvalues of $f^*: H^{n-r}(X, \mathbb{C}) \to H^{n-r}(X, \mathbb{C})$. //

Obvious Remark: Let $\det(1 - f^{*}\cdot t : H^i(X,\mathbb{Q}) \to H^i(X,\mathbb{Q}))$ be denoted by $P_i(t)$, and suppose $P_i(t) = \prod_{j=1}^{\dim H^i} (1 - c_{ij} t)$, $c_{ij} \in \mathbb{C}$. Then the eigenvalues of f^* on $H^i(X,\mathbb{C})$ are exactly the c_{ij}.

Proof of the functional equation

With notation as in the above remark, we have

$$\zeta(D^{-1}t^{-1}) = \frac{\prod_{\substack{i<n/2 \\ \text{odd}}} \prod_{j=1}^{\dim H^i} \left(1 - \frac{c_{ij}}{D} t^{-1}\right)\left(1 - \frac{1}{c_{ij}} t^{-1}\right)}{\prod_{\substack{i<n/2 \\ \text{even}}} \prod_{j=1}^{\dim H^i} \left(1 - \frac{c_{ij}}{D} t^{-1}\right)\left(1 - \frac{t^{-1}}{c_{ij}}\right)} \cdot \prod_{j=1}^{\dim H^{n/2}} \left(1 - \frac{c_{\frac{n}{2}j}}{D} t^{-1}\right)^{(-1)^{\frac{n}{2}+1}}$$

using the identity $1 - a^{-1}b^{-1} = -a^{-1}b^{-1}(1 - ab)$

$$= \frac{\prod_{\substack{i<n/2 \\ \text{odd}}} \prod_{j=1}^{\dim H^i} \frac{-c_{ij}}{D} t^{-1}\left(1 - D/c_{ij}\, t\right) \cdot \frac{-t^{-1}}{c_{ij}}\left(1 - c_{ij} t\right)}{\prod_{\substack{i<n/2 \\ \text{even}}} \prod_{j=1}^{\dim H^i} \frac{-c_{ij}}{D} \cdot t^{-1}\left(1 - D/c_{ij}\, t\right) \cdot \frac{-t^{-1}}{c_{ij}}\left(1 - c_{ij} t\right)} \cdot \prod_{j=1}^{\dim H^{n/2}} \left(-\frac{c_{\frac{n}{2}j}}{D} t^{-1}\left(1 - c_{\frac{n}{2}j} \cdot t\right)\right)^{(-1)^{n/2}+1}$$

$$= t^{\chi_X} \cdot D^{\chi_X/2} \cdot \zeta(X, f, t). \qquad //$$

Weil conjectured that a similar statement holds for zeta functions of algebraic varieties over $\mathbb{F}_q$, with standing assumptions on the varieties.

5) Background on Kähler Manifolds

Before we can treat another of Weil's conjectures, namely the Riemann hypothesis for varieties over $\overline{\mathbb{F}}_q$, we need to review some facts about Kähler manifolds. For our purposes, the key will be the construction of a special positive definite hermitian form on each of the cohomology groups of our Kähler manifold.

In this section, we do not give proofs. All of the results stated here can be found in André Weil's *Introduction à l'étude des variétés kählériennes*.

a) The definition of Kähler manifolds

Let X be a compact complex manifold. A hermitian metric on X is given by a smoothly varying family of maps $T'_x X \otimes_{\mathbb{C}} \overline{T'_x M} \to \mathbb{C}$, where $T'_x X$ denotes the holomorphic tangent space to X at x. Thus, a hermitian metric defines an element in $(T'_x X \otimes_{\mathbb{C}} \overline{T'_x X})^* = T_x^{*'} X \otimes_{\mathbb{C}} T_x^{*''} X$. As the imaginary part of a hermitian form is alternating, the imaginary part of a hermitian metric defines an alternating tensor in each $T_x^{*'} X \otimes_{\mathbb{C}} T_x^{*''} X$, $x \in X$; that is, it defines a $(1,1)$-form, called the associated $(1,1)$ form ω of the metric. A hermitian metric is said to be a Kähler metric if ω is a closed, and X is said to be a Kähler manifold if X admits a Kähler metric.

b) A fundamental fact

Let ω be the associated $(1,1)$ form (also known as the Kähler form) of a Kähler metric on X. Then ω^i is not exact, $1 \leq i \leq \dim X$. (This is proved with Stokes' theorem, using the fact that ω is nondegenerate.)

c) Construction of the desired hermitian form

Let X be a compact Kähler manifold of complex dimension n, and let ω be the Kähler form of some Kähler metric on X.

Def The primitive p^{th} cohomology $\mathrm{Prim}^p X$ is defined to be $H^p(X,\mathbb{C}) \cap (\mathrm{Ker}\, L^{n-p+1})$, where L is the operator of wedging with the Kähler form, $L(\alpha) = \omega \wedge \alpha$.

Lefschetz decomposition $H^p(X,\mathbb{C}) = \bigoplus_k L^k \mathrm{Prim}^{p-2k} X$

- We now define a (symmetric or alternating, depending on k) bilinear form $Q_k : H^k(X,\mathbb{C}) \times H^k(X,\mathbb{C}) \to \mathbb{C}$ by

$$Q_k(\alpha,\beta) = \int_X \omega^{n-k} \wedge \alpha \wedge \beta,$$

and a hermitian form

$$H_k(\alpha,\beta) = i^k Q(\alpha,\bar{\beta}).$$

Hodge-Riemann Bilinear Relations

$$(-1)^{k(k-1)/2}\, i^{p-q-k}\, H_k \Big|\, H^{p,q}_{prim} = H^{p,q}(X,\mathbb{C}) \cap \mathrm{Prim}^k X$$

is a positive definite hermitian form.

Furthermore, $H_k \big| L^r(\mathrm{Prim}^{k-2r} X) = (-1)^r H_{k-2r} \big| \mathrm{Prim}^{k-2r} X$.

From the bilinear relations and the Lefschetz decomposition, it is clear that we can use the H_k to define positive definite hermitian forms T_k on $H^k(X,\mathbb{C})$ with the following property: any endomorphism of $H^k(X,\mathbb{C})$ that is unitary with respect to H_k is unitary with respect to T_k.

6) The Kähler Riemann Hypothesis

Theorem Let X be a compact Kähler manifold with Kähler class u, and let $f : X \to X$ be a holomorphic map. Assume that $f^* u = q \cdot u$, $q > 0$. Then the eigenvalues of $f_r^* : H^r(X, \mathbb{C}) \to H^r(X, \mathbb{C})$ have absolute value $q^{r/2}$. Moreover, they are Weil numbers, i.e. algebraic integers all of whose conjugates have absolute value $q^{r/2}$.

proof Define $g_r : H^r(X, \mathbb{C}) \to H^r(X, \mathbb{C})$ by $g_r = q^{-r/2} \cdot f_r^*$, and define $g : H^*(X, \mathbb{C}) \to H^*(X, \mathbb{C})$ to equal g_r on $H^r(X, \mathbb{C})$. It is clear that g is an algebra homomorphism. Now $g(u) = u$, so $g(u^k) = u^k$, $1 \leq k \leq n = \dim_{\mathbb{C}} X$. In particular, as $u^n \neq 0$ in $H^{2n}(X, \mathbb{C})$ by 5.b., g is the identity on $H^{2n}(X, \mathbb{C})$.

We want to show that g_r is unitary with respect to T_r. Once we have shown this, it will follow that all eigenvalues of g_r have absolute value 1, so all eigenvalues of f_r have absolute value $q^{r/2}$. By 5.G, it suffices to show that g_r is unitary with respect to H_r. Moreover, as f_r is a real operator (i.e. f_r commutes with complex conjugation), g_r is a real operator, so to show that g_r is unitary with respect to H_r, it suffices to show that it is unitary with respect to Q_r.

Now
$$Q_r(g_r(\alpha), g_r(\beta)) = \int_X u^{n-r} \wedge g_r(\alpha) \wedge g_r(\beta)$$

Because $g(u^{n-r}) = u^{n-r}$ $\longrightarrow$
$$= \int_X g(u^{n-r} \wedge \alpha \wedge \beta)$$

Because g is the identity on $H^{2n}(X, \mathbb{C})$ $\longrightarrow$
$$= \int_X u^{n-r} \wedge \alpha \wedge \beta$$

$$= Q_r(\alpha, \beta)$$

To complete the proof, we must show that the eigenvalues of F_r^* are Weil numbers. First, given any endomorphism of a finite dimensional vector space, the conjugates of the eigenvalues are also eigenvalues, and therefore the conjugates of the eigenvalues of F_r^* also have absolute value $q^{r/2}$. Finally, F^* is already defined on $H^*(X, \mathbb{Q})$, and takes the image of $H^*(X, \mathbb{Z})$ in $H^*(X, \mathbb{Q})$ into itself. Therefore the eigenvalues of F_r^* are algebraic integers, $r = 0, \ldots, 2\dim_{\mathbb{C}} X$.

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This theorem was noticed by Serre; Weil had made his conjecture of the Riemann hypothesis for (smooth connected projective) varieties over $\mathbb{F}_q$ based on a different analogy. For more about this, see Serre's paper Analogues kählériens de Certaines Conjectures de Weil, Annals of Math., vol. 71, No. 2, 1960.

We will say a few words on why the assumption that $f^*(u) = q \cdot u$ is analogous to anything in the case of a variety over $\mathbb{F}_q$, with Fr taking the place of f. When our Kähler manifold X is projective, we may reformulate the theorem in the language of Chern classes, or of hyperplane sections. In the latter case, the assumption that $f^*(u) = q \cdot u$ becomes the assumption that there exists a hyperplane section E such that the divisor $f^{-1}(E)$ is algebraically equivalent to $q \cdot E$; this hypothesis is certainly satisfied when X is a projective variety over a finite field, with f equal to the Frobenius morphism Fr.