# Recursion and Random Walks

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October 3, 2007

**Abstract**

This paper examines conditions for recurrence and transience for random walks on discrete surfaces, such as $\mathbb{Z}^d$, trees, and random environments.

## 1 Random Walks on Non-Random Environments

### 1.1 Definitions

**Definition 1.** A class of subsets $\mathcal{F}$ of a set $\Omega$ is an *algebra* if the following hold:

1. $\emptyset \in \mathcal{F}$ and $\Omega \in \mathcal{F}$.
2. $A_1, A_2, \ldots, A_n \in \mathcal{F} \Longrightarrow \bigcup_{i=1}^n A_i \in \mathcal{F}$.
3. $A \in \mathcal{F} \Longrightarrow A^c \in \mathcal{F}$.

If additionally $A_1, A_2, \ldots \in \mathcal{F} \Longrightarrow \bigcup_{i=1}^\infty A_i \in \mathcal{F}$, then $\mathcal{F}$ is called a *$\sigma$-algebra*.

**Definition 2.** A *Probability Measure* is a function $\mathbb{P} : \mathcal{F} \to [0, 1]$, where $\mathcal{F}$ is a $\sigma$-algebra, satisfying:

1. $\mathbb{P}\{\emptyset\} = 0$
2. If $A_1, A_2, \ldots$ is a collection of disjoint members of $\mathcal{F}$, so that $A_i \cap A_j = \emptyset \ \forall i \neq j$, then
$$\mathbb{P}\Big\{\bigcup_{i=1}^{\infty} A_i\Big\} = \sum_{i=1}^{\infty} \mathbb{P}\{A_i\}.$$

**Definition 3.** A *Probability Space* is represented by the triple $(\Omega, \mathcal{F}, \mathbb{P})$, where $\Omega$ is a set, called a *Sample Space*, containing all possible outcomes, $\mathcal{F}$ is a $\sigma$-algebra of subsets of $\Omega$ containing events, and $\mathbb{P}$ is a probability measure that assigns a measure of 1 to the whole space. $\mathcal{F}$ will be a $\sigma$-algebra on $\Omega$ throughout the paper.

**Definition 4.** A *Random Variable* is a function $X : \Omega \to \mathbb{R}$ with the property that
$$\{\omega \in \Omega : X(\omega) \leq x\} \in \mathcal{F} \ \forall x \in \mathbb{R}.$$

Two events $A = \{X_i \leq x_i\}, B = \{X_j \leq x_j\}$ are *independent* when $\mathbb{P}\{A \cap B\} = \mathbb{P}\{A\}\mathbb{P}\{B\}$. More generally, random variables $X_1, X_2, \ldots$ are independent if $\forall k \in \mathbb{N}, x_1, x_2, \ldots \in \mathbb{R}$,

$$\mathbb{P}\{X_1 \leq x_1, X_2 \leq x_2, \ldots X_k \leq x_k\} = \prod_{i=1}^{k} \mathbb{P}\{X_i \leq x_i\}.$$

If $X$ is a random variable, then for every Borel subset $B$ of $\mathbb{R}$, $X^{-1}(B) \in \mathcal{F}$. We define a measure $\mu_X$, called the *distribution* of the random variable, on Borel sets by
$$\mu_X(B) := \mathbb{P}\{X \in B\} = \{X^{-1}(B)\}.$$

If $\mu_X$ takes values only for countable subsets of the real numbers, $X$ is a *discrete* random variable; these are the only random variables we consider in this paper. If random variables $X_1, X_2, \ldots$ are all independent and have the same distribution, we say they are *independent and identically distributed* (i.i.d.).

**Definition 5.** Intuitively, a *discrete stochastic process* is characterized by a state space $V$, and transitions between states which occur randomly according to some probability distribution. The process is *memoryless* if the probability of a transition $i \to j$ does not depend on the history of the process, i.e. $\forall i, j, u_o, \ldots, u_{t-1} \in V$,

$$\mathbb{P}\{X_{t+1} = j | X_t = i, X_{t-1} = u_{t-1}, \ldots, X_0 = u_0\} = \mathbb{P}\{X_{t+1} = j | X_t = i\}.$$

If additionally $p_{ij} = \mathbb{P}\{X_{t+1} = j | X_t = i\} = \mathbb{P}\{X_t = j | X_{t-1} = i\}$, so the transition probability does not depend on time, then the process is *homogenous.*

**Definition 6.** A *Markov chain* is a memoryless homogeneous discrete stochastic process.

Roughly, this means that, conditional upon knowing the state of the process up to the $n^{\text{th}}$ step, the values after $n$ steps do not depend on the values before the $n^{\text{th}}$ step.

## 1.2 Simple Random Walks on Integer Lattices

First we consider a simple random walk on $\mathbb{Z}$ where a walker starts at some $z \in \mathbb{Z}$ and always moves to either adjacent point with probability 1/2. Let $S_n$ denote the position of the walker after $n$ steps. The model immediately implies that the random walk is a Markov Chain, i.e.

$$\begin{aligned} &\mathbb{P}\{S_{n+1} = i_{n+1} | S_n = i_n, S_{n-1} = i_{n-1}, \ldots, S_1 = i_1, S_0 = i_0 = z\} \\ &\qquad = \mathbb{P}\{S_{n+1} = i_{n+1} | S_n = i_n\} = \frac{1}{2}, \end{aligned} \tag{1}$$

where $i_0 = z, i_1, i_2, \ldots, i_{n+1}$ is a sequence of integers with $|i_1 - i_0| = |i_2 - i_1| = \cdots = |i_{n+1} - i_n| = 1$.

Alternatively, work on the probability space $[0, 1]$ with Lebesgue measure. Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables with $\mathbb{P}\{X_1 = 1\} = \mathbb{P}\{X_1 = -1\} = 1/2$, where $X_i$ corresponds to the value of the $i^{\text{th}}$ coin flip (that $X_1, X_2, \ldots$ exist with these properties is beyond the scope of this paper). Then the position of the random walk after $n$ steps is given by:

$$S_n = z + X_1 + X_2 + \cdots + X_n \qquad n = 1, 2, \ldots.$$

**Lemma 7.** *Let $S_n$ denote the position of a random walker at time $n$. Let*

$$p_n = \mathbb{P}\{S_n = S_0\}$$

*Then*

$$p_n \sim \frac{1}{\sqrt{2\pi n}}.$$

*Proof.* When $n$ is odd, $p_n = 0$, so we only consider the case where the walker takes an even number of steps $2n$. To return to the starting point in that number of steps, the walk must consist of exactly $n$ steps to the right and $n$ to the left, and there are $\binom{2n}{n}$ways to choose such a move. Each particular choice of $2n$ steps occurs with probability $2^{-2n}$. Using Stirling's formula, given by

$$n! \sim \sqrt{2\pi n}\; e^{-n} n^n$$

We have

$$p_{2n} = \binom{2n}{n} \frac{1}{2^{2n}} = \frac{(2n)!}{n!n!} \frac{1}{2^{2n}} \sim \frac{\sqrt{4\pi n}\; e^{-2n}(2n)^{2n}}{(\sqrt{2\pi n}\; e^{-n}n^n)^2 2^{2n}} = \frac{1}{\sqrt{2\pi n}}.$$

□

We can form an analogous construction for simple random walks on $\mathbb{Z}^d$. The random walk $S_n$ now has $2d$ choices of moves for each $n$. The probability of moving in any one direction at time $n$ ($\frac{1}{2d}$) is equal to the probability of moving in any other direction, and is independent of the particular path the walker followed to arrive at $n$.

Another way to to think of the simple random walk in $\mathbb{Z}^d$ is to think of $d$ different simple random walks in each of the $d$ directions. To choose each step, one of the one-dimensional random walks is picked at random to make a move, and the walker moves in the direction indicated by that move, keeping the position in other directions constant.

**Lemma 8.** *[6] Let $S_n$ denote the position of a two-dimensional walker after $n$ steps, and $p_n = \mathbb{P}\{S_n = S_o\}$. Then,*

$$p_n \sim \frac{1}{2\pi n}.$$

*Proof.* To return to the starting point in 2 dimensions, the walk must again comprise an even number of steps $2n$, and there are $4^{2n}$ ways to choose such walks of length $2n$ returning to the starting point. All the walks returning to the starting point consist of $k$ steps to the north and south, and $n-k$ to the east and west. So we have

$$p_{2n} = 2^{-2n} \sum_{k=0}^{n} \binom{2n}{k\ k\ n-k\ n-k} = 2^{-2n} \sum_{k=0}^{n} \frac{(2n)!}{k!k!(n-k)!(n-k)!}$$

$$= 2^{-2n} \sum_{k=0}^{n} \frac{(2n)!}{n!n!} \frac{n!n!}{k!k!(n-k)!(n-k)!} = 2^{-2n} \binom{2n}{n} \sum_{k=0}^{n} \binom{n}{k}^2.$$

However, $\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$, so

$$p_{2n} = \left( \binom{2n}{n} \frac{1}{2^{2n}} \right)^2,$$

which is the square of the one-dimensional result. Therefore we have

$$p_{2n} \sim \left( \frac{1}{\sqrt{2\pi n}} \right)^2 = \frac{1}{2\pi n}.$$

□

**Lemma 9.** *[6] Let $S_n$ denote the position of a three-dimensional random walk, and $p_n$ as before. Then,*

$$p_{2n} \leq \frac{C}{n^{3/2}}.$$

*Proof.* As before, the three-dimensional random walk must make an equal number of steps in each direction in order to return to its starting point. There are now $6^{2n}$ ways to choose walks of length $2n$ returning to the origin. If we let $n_1$ denote the number of steps in the $x$ direction, $n_2$ the steps in the $y$ direction, and $n_3$ the steps in the $z$ direction, we have

$$p_{2n} = 6^{-2n} \sum_{n_1+n_2+n_3=2n} \binom{2n}{n_1\ n_1\ n_2\ n_2\ n_3\ n_3}$$

$$= 6^{-2n} \sum_{n_1+n_2+n_3=2n} \frac{n!}{n_1!n_2!n_3!} = \frac{1}{3^{2n}} \binom{n}{n_1\ n_2\ n_3}$$

$\frac{1}{3^{2n}} \binom{n}{n_1\ n_2\ n_3}$ represents the probability that when 3 balls are able to fall into 3 separate bins, $n_1$ of them fall into the first, $n_2$ into the second, and $n_3$ into the third. Informally, we see that if the balls fall randomly, this probability is maximized by letting $n_1$, $n_2$, $n_3$ be as close as possible to $n/3$. Replacing one of the terms by this fact, we have

$$p_{2n} \leq 2^{-2n}\binom{2n}{n}\frac{1}{3^n}\frac{n!}{\lfloor\frac{n}{3}\rfloor!\,\lfloor\frac{n}{3}\rfloor!\,\lfloor\frac{n}{3}\rfloor!}\sum_{n_1+n_2+n_3=2n}\frac{1}{3^n}\frac{n!}{n_1!n_2!n_3!}.$$

Here $\lfloor\frac{n}{3}\rfloor$ denotes the largest integer less than or equal to $\frac{n}{3}$. The last sum represents the sum of all probabilities for the outcome that $n$ balls fit into three bins, and so is just one. Therefore, we have

$$p_{2n} \leq \frac{1}{2^{2n}}\binom{2n}{n}\frac{1}{3^n}\frac{n!}{\lfloor\frac{n}{3}\rfloor!^3}.$$

Applying Stirling's formula, we have

$$p_{2n} \leq \frac{C}{n^{3/2}}.$$

□

In addition to the probability of the random walk returning to its starting point, we are also interested in the frequency of return.

**Definition 10.** Let $A_n$ be the event that a random walk returns to its starting point on the $n^{\text{th}}$, $\{S_n = S_0\}$. Then the event that the random walk returns to its starting point infinitely often is given by:

$$\{A_n \text{ infinitely often (i.o.)}\} := \limsup_{n\to\infty} A_n = \bigcap_{n=1}^{\infty}\bigcup_{m=n}^{\infty} A_m.$$

If $\mathbb{P}\{A_n \text{ i.o.}\} = 1$, the walk is called *recurrent*. If $\mathbb{P}\{A_n \text{ i.o.}\} = 0$, the walk is called *transient*.

**Theorem 11** (Recurrence Theorem)**.** *[3] A random walk $S_n$ on $\mathbb{Z}^d$ is recurrent if $d \leq 2$. If $d \geq 3$, then the walk is transient, and*

$$\mathbb{P}\{S_n \neq S_0 \ \forall n > 0\} > 0.$$

*Proof.* Let $J_n = 1\{A_n\}$ be an indicator variable for the event that the random walk returns to its starting point. The total number of visits to the origin is given by:

$$V = \sum_{n=0}^{\infty} J_{2n}.$$

Using the linearity of the expected value,

$$\mathbb{E}[V] = \sum_{n=0}^{\infty}\mathbb{E}[J_{2n}] = \sum_{n=0}^{\infty}\mathbb{P}\{S_{2n} = S_0\}.$$

We know $\sum n^{-a}$ converges only when $a > 1$. So when $d = 1, 2$, by Lemmas 7 and 8, the sum is divergent. When $d = 3$, by Lemma 9, the sum is convergent. For $d > 3$, we see that to return, the random walk in four dimensions must at least return in three dimensions. Since the sum of returning probabilities converges in three dimensions, it must also converge in any greater number of dimensions. Therefore, we have

$$\mathbb{E}[V] = \begin{cases} \infty & d = 1, 2 \\ < \infty & d \geq 3 \end{cases}$$

Let $q$ be the probability that the random walker ever returns to its starting point. Assuming $q < 1$, its distribution is given by:

$$\mathbb{P}\{V = k\} = q^{k-1}(1 - q), \qquad k = 1, 2, \ldots$$

Again assuming $q < 1$, we find:

$$\mathbb{E}[V] = \sum_{k=1}^{\infty} k\mathbb{P}\{V = k\} = \sum_{k=1}^{\infty} kq^k(1 - q) = \frac{1}{1 - q} < \infty.$$

For $d = 1, 2$ we know $\mathbb{E}[V] = \infty$, so by contradiction $q = 1$, and the walk returns to the origin for some $n$ with probability 1. When $d \geq 3$, $\frac{1}{1-q} < \infty$, so $q < 1$, and with non-zero probability the random walk may not return to the starting point at all.

Now from the distribution of $q$,

$$\mathbb{P}\{A_n \text{ finitely often}\} \geq \mathbb{P}\{V = k\} = 1 - q^k$$

So if $d = 1, 2$, as $k \to \infty$, $\mathbb{P}\{A_n \text{ finitely often}\} \to 0$, so $\mathbb{P}\{A_n \text{ i.o.}\} = 1$. When $d \geq 3$, as $k \to \infty$, $\mathbb{P}\{A_n\} \to 1$, so $\mathbb{P}\{A_n \text{ i.o.}\} = 0$.

$\square$

## 1.3 The Zero-One Law

The proof of the Recurrence Theorem is based on the fact that

$$\mathbb{P}\{S_n = S_0 \text{ at least for one n}\} = 1 \Longrightarrow \mathbb{P}\{S_n = S_0 \text{ infinitely often (i.o.)}\} = 1$$

And

$$\mathbb{P}\{S_n = S_0 \text{ at least for one n}\} < 1 \Longrightarrow \mathbb{P}\{S_n = S_0 \text{ i.o.}\} = 0.$$

That is, if the random walk returns once to the origin with probability 1, it will return again and again with probability 1. Similarly, if the random walk stays away from the origin with non-zero probability for every $n$, then it cannot possibly return infinitely often. With infinite sequences of independent random variables, the probabilities of certain events can only be 0 or 1. This observation is formalized in Kolmogorov's Zero-One Law, which can be used to provide another proof of the Recurrence Theorem.

**Definition 12.** Let $(\Omega, \mathbb{P})$ be a probability space. Two $\sigma$-algebras $\mathcal{A}$, $\mathcal{B}$ are independent if for all $A \in \mathcal{A}$, $B \in \mathcal{B}$,

$$\mathbb{P}\{A \cap B\} = \mathbb{P}\{A\}\mathbb{P}\{B\}.$$

**Definition 13.** Assume $X_1, X_2, \ldots$ are independent random variables on $(\Omega, \mathcal{F})$. Then the $\sigma$-algebra *generated* by $X_1, X_2, \ldots$ is the smallest $\sigma$-algebra on which all the $X_i$ are measurable.

**Definition 14.** Let $\mathcal{F}_n$ be the $\sigma$-algebra generated by $X_1, \ldots, X_n$, and let $\mathcal{G}_n$ be the $\sigma$-algebra generated by $X_{n+1}, X_{n+2}, \ldots$ with $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots$, and $\mathcal{G}_1 \supset \mathcal{G}_2 \supset \cdots$. Let $\mathcal{F}$ be the $\sigma$-algebra generated by $X_1, X_2, \ldots$, the smallest $\sigma$-algebra containing the algebra $\mathcal{F}^0 = \cup_{n=1}^{\infty} \mathcal{F}_n$, and $\mathcal{G}$ defined similarly. The *tail $\sigma$-algebra* $\mathcal{T}$ is defined by

$$\mathcal{T} = \bigcap_{n=1}^{\infty} \mathcal{G}_n.$$

**Lemma 15.** *[3] Suppose $\mathcal{F}$ and $\mathcal{G}$ are two algebras of events that are independent. Then $\mathcal{F}^1 = \sigma(\mathcal{F})$ and $\mathcal{G}^1 = \sigma(\mathcal{G})$ are independent $\sigma$-algebras.*

*Proof.* Let $B \in \mathcal{G}$ and define the measure

$$\nu_B(A) := \mathbb{P}\{A \cap B\}, \quad \overline{\nu}_B(A) := \mathbb{P}\{A\}\mathbb{P}\{B\}$$

Since $\nu_B, \overline{\nu}_B$ are finite measures, they agree on $\mathcal{F}$. Using the Carathéodory extension (which states that $\mathbb{P}$ can be extended uniquely to a measure space $(\Omega, \overline{\mathcal{F}}, \mathbb{P})$ where $\mathcal{F} \subset \overline{\mathcal{F}}$), they must agree on $\mathcal{F}$. So $\mathbb{P}\{A \cap B\} = \mathbb{P}\{A\}\mathbb{P}\{B\}$. Supposing $A \in \mathcal{F}^1$, $B \in \mathcal{G}^1$, we take the measures

$$\nu_A(B) := \mathbb{P}\{A \cap B\}, \quad \overline{\nu}_A(B) := \mathbb{P}\{A\}\mathbb{P}\{B\},$$

completing the proof. $\square$

**Theorem 16** (Zero-One Law)**.** *( [1], p. 290) Let $X_1, X_2, \ldots$ be independent random variables on $(\Omega, \mathcal{F}, \mathbb{P})$. If $A \in \mathcal{T}$,*

$$\mathbb{P}\{A\} = 0 \qquad or \qquad \mathbb{P}\{A\} = 1.$$

*Proof.* By Lemma 14, any event in $\sigma(X_n, X_{n+1}, \ldots)$ is independent of any event in $\sigma(X_1, \ldots, X_n)$. Defining $\mathcal{F}^0$ and $\mathcal{T}$ as above, any event in $\mathcal{F}^0$ is independent of any event in $\cup_{n=1}^{\infty} \mathcal{H}_n$. So any event in $\mathcal{H}_\infty$ is independent of any event in $\sigma(\cup_{n=1}^{\infty} \mathcal{H}_n)$. However, $\mathcal{H}_\infty \subset \sigma(X_1, \ldots) = \sigma(\cup_{n=1}^{\infty} \mathcal{H}_n)$, so any tail event is independent of itself. That is, $\mathbb{P}\{A\} = \mathbb{P}\{A \cap A\} = \mathbb{P}\{A\}\mathbb{P}\{A\}$, so $\mathbb{P}\{A\} = 0, 1$. $\square$

**Lemma 17** (Borel-Cantelli 1)**.** *( [2], p. 27)*

*Let $A_1, A_2, \ldots$ be a sequence of events for which $\sum_{n=1}^{\infty} \mathbb{P}\{A_n\} < \infty$. Then*

$$\mathbb{P}\{\limsup_{n \to \infty} A_n\} = \mathbb{P}\{A_n \ i.o.\ \} = 0.$$

So with probability 1, only a finite number of the events $A_n$ occur.

*Proof.* Supposing that $\sum_{n=1}^{\infty} \mathbb{P}\{A_n\} = 0$ and using Definition 9, one has

$$\mathbb{P}\{\limsup_{n\to\infty} A_n\} = \mathbb{P}\{\bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m\} = \lim_{n\to\infty} \mathbb{P}\{\bigcup_{m=n}^{\infty} A_n\} \le \lim_{n\to\infty} \sum_{m=n}^{\infty} \mathbb{P}\{A_m\} = 0,$$

where the second equality follows from the fact that if $\nu$ is a measure, $A_1, \ldots$ are measurable set, $A_{n+1} \subset A_n, n = 1, , \ldots$, and $\nu\{A_n\} < \infty$, then $\nu\{\cap_{n=1}^{\infty}\} = \lim_{n\to\infty} \nu\{A_n\}$. The third inequality comes from the fact that the $A_n$ are not necessarily disjoint.

□

**Lemma 18** (Borel-Cantelli 2)**.** *( [4], p. 319) Let $A_1, A_2, \ldots$ be a sequence of events for which*

$$\sum_{n=1}^{\infty} \mathbb{P}\{A_n\} = \infty \quad \textit{and} \quad \liminf_{n\to\infty} \frac{\sum_{k=1}^{n} \sum_{i=1}^{n} \mathbb{P}\{A_k A_i\}}{(\sum_{k=1}^{n} \mathbb{P}\{A_k\})^2} \le C \quad C \ge 1$$

*. Then,*

$$\mathbb{P}\{\limsup_{n\to\infty} A_n\} \ge C^{-1}.$$

*Proof.* Let $J_n = 1\{A_n\}$ and $N_k = \sum_{n=1}^{k} J_n$. $\forall \epsilon > 0$, let $B_{n,\epsilon}$ denote the measurable set defined by

$$N_k \ge \epsilon \mathbb{E}[N_k] \quad \text{for some } k \ge n.$$

We pick the measurable functions given by $\mathbf{f}(t) = \mathbb{E}[e^{itX}]$, the characteristic function of the set $B_{n,\epsilon}$ and $\mathbf{g} = \mathbf{f}(N_n)$. By the Cauchy-Schwartz Inequality, given by $\mathbb{E}[\mathbf{fg}]^2 \le \mathbb{E}[\mathbf{f}^2]\mathbb{E}[\mathbf{g}^2]$, we have

$$\mathbb{P}\{B_{n,\epsilon}\} = \mathbb{E}[\mathbf{f}^2] \ge \frac{\mathbb{E}[\mathbf{fg}]^2}{\mathbb{E}[\mathbf{g}^2]} \ge \frac{\mathbb{E}[\mathbf{fg}]^2}{\mathbb{E}[\mathbf{N_n}^2]} \ge \frac{(\mathbb{E}[N_n] - \mathbb{E}[N_n(1-\mathbf{f})])^2}{\mathbb{E}[N_n^2].}$$

But $\mathbb{E}[N_n(1-\mathbf{f})] \le \epsilon \mathbb{E}[N_n]$, so

$$\mathbb{P}\{B_{n,\epsilon} \ge (1-\epsilon)^2 \frac{\mathbb{E}[N_n]^2}{\mathbb{E}[N_n^2]} \qquad n = 1, 2, \ldots.$$

Since $\mathbb{E}[N_n] \to \infty$ as $n \to \infty$,

$$\mathbb{P}\{\bigcap_{n=1}^{\infty} \bigcup_{n=m}^{\infty} A_m\} \ge \lim_{n\to\infty} \mathbb{P}\{B_{n,\epsilon}\}.$$

Since this is true for every $\epsilon > 0$,

$$\mathbb{E}\{\bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m\} \ge \limsup_{n\to\infty} \frac{\mathbb{E}[N_n]^2}{\mathbb{E}[N_n^2]} = \limsup_{n\to\infty} \frac{\sum_{k=1}^{n} \sum_{i=1}^{n} \mathbb{P}\{A_k A_i\}}{(\sum_{k=1}^{n} \mathbb{P}\{A_k\})^2} \le C.$$

□

**Lemma 19.** *( [2], p. 25) For any $-\infty < a \le b < +\infty$ we have*

$$\mathbb{P}\{\liminf_{n\to\infty} S_n = a\} = \mathbb{P}\{\limsup_{n\to\infty} S_n = b\} = 0$$

*Proof.*

$$\limsup_{n\to\infty} S_n = \lim_{n\to\infty} \big(\sup_{m\ge n} S_m\big) = \lim_{n\to\infty} n = \infty$$

So $\mathbb{P}\{\liminf_{n\to\infty} S_n = a\} = 0$ and similarly for the lim inf.

□

**Lemma 20.** *( [2], p. 196) When $d = 2$,*

$$\lim_{n\to\infty} \frac{\sum_{j=1}^n \sum_{k=1}^n \mathbb{P}\{S_{2j} = S_0, S_{2k} = S_0\}}{(\sum_{k=1}^n \mathbb{P}\{S_{2k} = S_0\})^2} = 2$$

*Proof.*

$$\sum_{j=1}^n \sum_{k=1}^n \mathbb{P}\{S_{2j} = S_0, S_{2k} = S_0\}$$

$$= 2\sum_{j=1}^n \sum_{k=1}^{n-j} \mathbb{P}\{S_{2j} = S_0, S_{2j+2k} = S_0\} + \sum_{j=1}^n \mathbb{P}\{S_{2j} = S_0\}$$

$$= 2\sum_{j=1}^n \sum_{k=1}^{n-j} \mathbb{P}\{S_{2j} = S_0\}\mathbb{P}\{S_{2k} = S_0\} + \sum_{j=1}^n \mathbb{P}\{S_{2j} = S_0\}$$

$$\sim \frac{2}{\pi^2}\sum_{j=1}^n \sum_{k=1}^{n-j} j^{-1}k^{-1} \sim 2\Big(\frac{\log n}{\pi}\Big)^2 \sim 2\Big(\sum_{k=1}^n \mathbb{P}\{S_{2k} = S_0\}\Big)^2.$$

□

We use the above facts to provide another proof of Theorem 11.

**Proof of Recurrence Theorem for $d = 1$.**

*Proof.* Contained in the tail $\sigma$-algebra generated by random variables $X_1, X_2, \dots$ are events such as

$$\{A_n > 0 \text{ i.o.}\}, \quad \{\limsup_{n\to\infty} A_n = 0\}$$

which do not refer to finite subcollections such as $X_1, \dots X_n$. Our random walk $S_n$, by the Markov Property, has a value after $n$ steps which depend on the $n^{\text{th}}$ step, but not previous steps. The symmetric nature of the random walk implies that $\mathbb{P}\{\liminf_{n\to\infty} S_n = -\infty\} = \mathbb{P}\{\limsup_{n\to\infty} S_n = +\infty\} = C$, while Lemma 18 guarantees that $C > 0$. Since the Zero-One Law guarantees that $C$ can be only zero or one, $C = 1$.

□

**Proof of Recurrence Theorem for $d = 2$.** ( [2], p. 197) Lemma 15 shows that the random walk on two dimensions satisfies the hypotheses of the second Borel-Cantelli Lemma, which in turn asserts that

$$\mathbb{P}\{S_n = S_0 \text{ i.o.}\} \geq 1/2.$$

By the Zero-One law, since this probability is not zero, it must be one.

**Proof of the Recurrence Theorem for $d \geq 3$.** ( [2], p. 197)

Lemma 7 asserts that the probability of return when $d \geq 3$ converges in sum. This satisfies the hypothesis of the first Borel-Cantelli lemma, which immediately shows that the probability of return is zero.

## 1.4 Time

Let $\rho_1(k) = \min\{n : S_n = k\} \quad k = 1, 2, \ldots.$

**Lemma 21.** *( [2], p. 97) Let* $\rho_0 = 0$ *and* $\rho_k = \min\{j : j > \rho_{k-1}, S_j = 0\}$ $k = 1, 2, \ldots.$ *Then,*

$$\mathbb{P}\{\rho_1 = 2k\} = 2^{-2k+1} k^{-1} \binom{2k-2}{k-1} \quad k = 1, 2, \ldots$$

*and*

$$\mathbb{P}\{\rho_1 > 2n\} = 2^{-2n} \binom{2n}{n} = \mathbb{P}\{S_{2n} = 0\}.$$

*Proof.*

$$\mathbb{P}\{\rho = 2k\} = \frac{1}{2}\mathbb{P}\{\rho_1 = 2k | X_1 = +1\} + \frac{1}{2}\mathbb{P}\{\rho_1 = 2k | X_1 = -1\}$$

Clearly,

$$\mathbb{P}\{\rho_1 = 2k | X_1 = +1\} = \mathbb{P}\{\rho_1 = 2k | X_1 = -1\} = \mathbb{P}\{\rho_1(1) = 2k - 1\}.$$

□

**Remark**

$$\mathbb{P}\{\rho_1 < \infty\} = \sum_{k=1}^{\infty} \mathbb{P}\{\rho_1 = 2k\} = \sum_{k=1}^{\infty} 2^{2k+1} k^{-1} \binom{2k-2}{k-1} = 1$$

So the particle returns to the origin with probability one, i.e. we have a new proof of the Recurrence Theorem for $d = 1$.

While it is guaranteed that a random walker will return to the origin, the mean waiting time of the recurrence is infinite.

**Theorem 22.** *The expected time to hit the origin is infinite.*

*Proof.*

$$\mathbb{E}[\rho_1] = \sum_{k=1}^{\infty} 2^{-2k+2} \binom{2k-2}{k-1} = \infty.$$

□

## 1.5 Biased Random Walk

A biased random walk is a random walk which tends to move in some directions with greater probability than others. More formally, let $S_n$ denote the position after $n$ steps of a a *biased* one-dimensional random walker. Let $p > 1/2$ and $S_n = X_1 + \dots X_n$ , where $X_1, \dots, X_n$ are independent with

$$\mathbb{P}\{x_j\} = 1 - \mathbb{P}\{x_j = -1\} = p$$

**Theorem 23.** *[3] Let $S_n$ denote the position of a biased one-dimensional random walker. Then there is a $\rho < 1$ such that as $n \to \infty$*

$$\mathbb{P}\{S_{2n} = 0\} \sim \rho^n \frac{1}{\sqrt{\pi n}},$$

*and the random walk is transient.*

*Proof.* The number of choices available for a random walk to return to its starting point remains the same as in the case of the simple random walk. The probability of such paths, however, is now given by $p^n(1-p)^n$, since out of $2n$ steps, $n$ must be taken in either direction, and the probability of moves is $p$ in one direction and $1-p$ in the other.

$$\mathbb{P}\{S_{2n} = 0 | P_0 = 0\} = \binom{2n}{n} p^n (1-p)^n = \frac{(2n)!}{(n)!(n)!} p^n (1-p)^n.$$

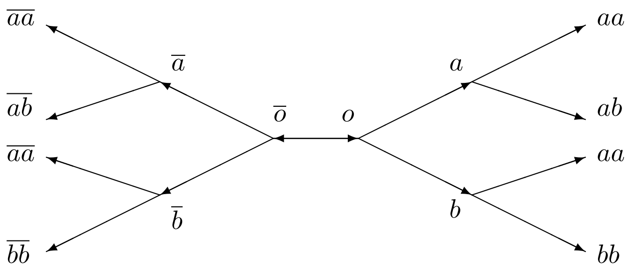
Using Stirling's formula, we find that

$$\frac{(2n)!}{n!n!} p^n (1-p)^n \sim \frac{\sqrt{4\pi n} e^{-2n} (2n)^{2n}}{(\sqrt{2\pi n} e^{-n} n^n)^2} = \frac{2^{2n}}{\sqrt{2\pi n}} (p(1-p))^n = \rho^n \frac{1}{\sqrt{\pi n}}.$$

This sum converges, so by the first Borel-Cantelli lemma we have transience. □

## 1.6 Random Walks on Graphs

We construct a tree $\mathcal{T}_1$ as follows. The vertices of $\mathcal{T}_1$ are the empty word, denoted by $o$, and all finite sequences of the letters $a, b$, i.e. words $x_1 \dots x_n$ where $x_1, x_2, \dots x_n \in \{a, b\}$. Both words of one letter are adjacent to $o$. We say that a word of length $n-1$ and of length $n$ are adjacent if they have the exact same letters, in order, in the first $n-1$ positions. Note that each word of positive length is adjacent to three words and the root is adjacent to only two words. We construct another tree $\mathcal{T}_2$ similarly, calling the root $\overline{o}$ and using the letters $\overline{a}, \overline{b}$. Finally we make a tree $\mathcal{T}$ by taking the union of $\mathcal{T}_1$ and $\mathcal{T}_2$ and adding one more connection: we say that o and $\overline{o}$ are adjacent.




**Lemma 24.** *[3] $\mathcal{T}$ is a connected tree*

*Proof.* Suppose $x, y \in \mathcal{T}_1$. We construct a unique path between them as follows: If $x \sim y$ we are done. If not, consider the *common word* consisting of the first k letters in which $x$ and $y$ agree (this may be the empty word). Any word of length $n$ in this tree is adjacent to only one word of length $n-1$. Call the edge between them a *reduction*. Using reductions, we can find a unique path from $x$ to the common word, and similarly for $y$. Uniting the path from $x$ to the common word, and from $y$ to the common word provides the path from $x$ to $y$.

If $x \in \mathcal{T}_1$ and $y \in \mathcal{T}_2$, then we use reductions to find a unique path from $x$ to $o$ and from $y$ to $\overline{o}$. Then we use the fact that $o \sim \overline{o}$ to find a unique path between $x$ and $y$. □

**Theorem 25.** *[3] Let $S_n$ denote simple random walk on the tree, where the walker goes to one of the three neighbors each with probability 1/3, and each choice is independent of the previous choices. Then $S_n$ is transient.*

*Proof.* Let $p_n = \mathbb{P}\{S_n = S_0\}$. For the random walk to return to the starting point, the walk must still make $n$ steps to the left, and $n$ steps in the other direction. Steps to the left are made with probability $\frac{1}{3}$ while steps to the right occur with probability $\frac{2}{3}$. The total probability is given by:

$$p_{2n} = \binom{2n}{n}\left(\frac{1}{3}\right)^n\left(\frac{2}{3}\right)^n = \frac{(2n)!}{n!n!}\left(\frac{2}{9}\right)^n$$

Using Stirling's formula, we arrive at:

$$p_{2n} = \frac{(2n)!}{n!n!}\left(\frac{2}{3}\right)^n \sim \frac{\sqrt{4\pi n}e^{-2n}(2n)^{2n}}{(\sqrt{2\pi n}e^{-n}n^n)^2 2^{2n}}\left(\frac{2}{9}\right)^n = \frac{2^{2n}}{\sqrt{2\pi n}}\left(\frac{2}{9}\right)^n = \frac{1}{\sqrt{2\pi n}}\left(\frac{8}{9}\right)^n$$

Since the sum

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{2\pi n}}\left(\frac{8}{9}\right)^n$$

converges, the first Borel-Cantelli lemma guarantees that the walk is transient. □

**Theorem 26.** *[3] With probability 1 the random walk does one of the two things: either the random walk visits $\mathcal{T}_1$ only finitely often, or it visits $\mathcal{T}_2$ finitely often.*

*Proof.* In order for a random walk on $\mathcal{T}$ to visit either $\mathcal{T}_1$ or $\mathcal{T}_2$ finitely often, it must at least visit the points $o, \overline{o}$ (the 'bridge' points) finitely often. Then we have:

$$\mathbb{P}\{\text{The random walk visits } \mathcal{T}_1 \text{ or } \mathcal{T}_2 \text{ only finitely often}\} \geq$$

$$\mathbb{P}\{\text{the random walk visits the bridge points finitely often}\}$$

However, transience establishes that the probability of visiting any point finitely often is one, so the random walk must visit either $\mathcal{T}_1$ or $\mathcal{T}_2$ only finitely often. □

# 2 Random Walks on Random Environments

## 2.1 The Creation of the Universe

The random walk considered in the first section is a mathematical model of linear Brownian motion in a homogenous, or non-random, environment. We now consider a random environment (in one dimension) encountered, for instance, in a magnetic field. Instead of always moving to the right or left with equal probability (as in the simple random walk) or always moving to the right with one probability and the left with another (as in the biased random walk), the chance of moving to the right or left now vary randomly depending on where the walker is on a one-dimensional integer lattice. Our model is given in two steps:

**God creates the Universe** With a sequence of i.i.d. random variables $\mathcal{E} = \{\dots E_{-2}, E_{-1}, E_0, E_1, E_2, \dots\}$ with distribution $\mathbb{P}_1\{E_0 < x\} = \mu_{\mathcal{E}}(x), \mu_{\mathcal{E}}(0) = 0, \mu_{\mathcal{E}}(1) = 1$, God visits the integers and to $i \in \mathbb{Z}$ randomly assigns a value $E_i$ (this is a random number between zero and one).

**Life in the Universe** Into a random environment $\mathcal{E}$, a particle is born on the origin and begins a random walk. The particle moves a step to the right with probability $E_0$, and left with probability $1 - E_0$. If after $n$ steps the particle is at point $i$, then the probability of a step to the right is $E_i$, while the probability of a step to the left is $1 - E_i$. We have defined a random walk $R_n$ with $R_0 = 0$ and

$$\begin{aligned}&\mathbb{P}_{\mathcal{E}}\{R_{n+1} = i + 1 | R_n = i, R_{n-1}, R_{n-2}, \dots, R_1\} \\ &= 1 - \mathbb{P}_{\mathcal{E}}\{R_{n+1} = i - 1 | R_n = i, R_{n-1}, R_{n-2}, \dots, R_1\} = E_i. \end{aligned} \tag{2}$$

Note that when $\mathbb{P}_1\{E_0 = 1/2\} = \mu_X(1/2+0) - \mu_X(1/2) = 1$, we have our usual simple random walk.

Formally, let $\{\Omega_1, \mathcal{F}_1, \mathbb{P}_1\}$ be a probability space and let $(\omega_1 \in \Omega_1)$ be a sequence of i.i.d. random variables with $\mathbb{P}_1\{E_1 < x\} = F(x), F(0) = 1 - F(1) = 0$ and

$$\mathcal{E} = \mathcal{E}(\omega_1) = \{\ldots E_{-1} = E_{-1}(\omega_1), E_0 = E_0(\omega_1), E_1 = E_1(\omega_1), \ldots\}$$

Let $\{\Omega_2, \mathcal{F}_2\}$ be the measurable space of the sequences $\omega_2 = \{\epsilon_1, \epsilon_2, \ldots\}$ where $\epsilon_i = 1$ or $-1$ for $i = 1, 2, \ldots$ and $\mathcal{F}_2$ is the natural $\sigma$-algebra (the $\sigma$-algebra generated by collections of all product sets). Define the random variable $Y_1, Y_2, \ldots$ on $\Omega_2$ by $Y_i(\omega_2) = \epsilon_i$ for $i = 1, 2, \ldots$ and let $R_0 = 0, R_n = Y_1 + Y_2 + \cdots + Y_n, n = 1, 2, \ldots$. Then we construct a probability measure $\mathbb{P}$ on the measurable space $\{\Omega = \Omega_1 \times \Omega_2, \mathcal{F} = \mathcal{F}_1 \times \mathcal{F}_2\}$ as follows: for any given $\omega_1 \in \Omega_1$ we define a measure $\mathbb{P}_{\omega 1} = \mathbb{P}_{\mathcal{E}(\omega 1)} = \mathbb{P}_{\mathcal{E}}$ on $\mathcal{F}_2$ satisfying (2).
Note the difference between $\mathbb{P}_1$ and $\mathbb{P}_{\mathcal{E}}$: the former is a probability measure used when determining the probability that any $i \in \mathbb{Z}$ takes on some value $E_i$. With the creation of the random environment, the probability measure $\mathbb{P}_{\mathcal{E}}$ refers to the probability applicable for the random walk, referring to the probability of moving to the right or left.

Let $U_k = (1 - E_k)/E_k$, $V_j = \log U_j$, and $F(x)$ as above. We assume the following conditions:

$$\exists\ 0 < \beta < 1/2 \text{ s.t. } \mathbb{P}\{\beta < E_0 < 1 - \beta\} = 1 \tag{3}$$

$$\mathbb{E}_1[V_0] = \int_{\infty}^{\infty} x d\mathbb{P}_1\{V_0 < x\} = \int_{\beta}^{1-\beta} \log \frac{1-x}{x} dF(x) = 0 \tag{4}$$

$$0 < \sigma^2 = \mathbb{E}_1[V_0^2] = \int_{\beta}^{1-\beta} \big(\log \frac{1-x}{x}\big)^2 dF(x) < \infty. \tag{5}$$

Note that for the simple random walk, $\mathbb{P}\{U_0 = 1\} = \mathbb{P}_1\{V_0 = 0\} = 1$, so (3) and (4) are satisfied. However, (5) is not satisfied since $\mathbb{E}_1[V_0^2] = \sigma^2 = 0$ (that is, we expect the simple random walk to be at its starting point).

## 2.2 Recursion in a Random Environment

We can state a theorem of recurrence in random environments analogous to the result in non-random environment. With probability one, God creates an environment in which the random walk is recurrent. First, two Lemmas:

**Lemma 27.** *( [2], p. 311) Let*

$$p(a,b,c) = \mathbb{P}_{\mathcal{E}}\{\min\{j : j > m, R_j = a\} < \min\{j : j > m, R_j = c\} S_m = b\} \quad a \le b \le c$$

*Where $p(a,b,c) = p(a,b,c,\mathcal{E})$ is the probability that a particle starting from $b$ hits $a$ before $c$ given the environment $\mathcal{E}$. Then*

$$p(a,b,c) = 1 - \frac{D(a,b)}{D(a,c)}$$

*where*

$$D(a,b) = \begin{cases} 0 & b = a \\ 1 & b = a+1 \\ 1 + \sum_{j=1}^{b-a-1} \prod_{i=1}^{j} U_{a+i} & b \geq a+2 \end{cases}$$

*and in particular,*

$$p(0,1,) = 1 - \frac{1}{D(n)}$$

*where* $D(b) = D(0,b) + 1 + U_1 + U_1U_2 + \cdots + U_1U_2 \ldots U_{b-1}$

*Proof.* Obviously $p(a,a,c) = 1$, $p(a,c,c) = 0$, and $p(a,b,c) = E_b p(a,b+1,c) + (1-E_b)p(a,b-1,c)$. Therefore, $p(a,b+1,c) - p(a,b,) = \frac{1-E_b}{E_b}(p(a,b,c) - p(a,b-1,c)$. By iteration, we have

$$\begin{aligned} p(a,b+1,) - p(a,b,c) &= U_bU_{b-1}\cdots U_{a+1}p(a,a+1,c) - p(a,a,c)) \\ &= U_bU_{b-1}\cdots U_{a+1}(p(a,a+1,c) - 1) \end{aligned} \tag{6}$$

Adding the above equations for $b = a, a+1, \ldots, c-1$ we have

$$-1 = p(a,c,c) - p(a,a,c) = D(a,c)(p(a,a+1,c) - 1)$$

or

$$p(a,a+1,c) = 1 - \frac{1}{D(a,c)}$$

Two equations above imply

$$p(a,b+1,c) - p(a,b,c) = -\frac{1}{D(a,c)}U_bU_{b-1}\cdots U_{a+1}$$

Adding these equations gives

$$p(a,b_1,c) - 1 = p(a,b+1,c) - p(a,a,c)$$

$$= \frac{-1}{D(a,c)}(1 + U_{a+1} + U_{a+1}U_{a+2} + \cdots + U_{a+1}U_{a+2}\cdots U_b) = \frac{-D(a,b+1)}{D(a,c)}.$$

So we have the lemma $\square$

**Consequence**

$$\mathbb{P}\{\lim_{n\to\infty} p(0,1,n;\mathcal{E}) = \lim_{n\to\infty}(1 - \frac{1}{D(n)}) = 1\} = 1 \tag{7}$$

The consequence follows from the previous Lemma and a claim, left unproven, that $\lim_{n\to\infty} D(n) = \infty$ in probability (with respect to $\mathbb{P}_1$.)

**Lemma 28.** *For any $-\infty < a \leq b < \infty$ we have*

$$\mathbb{P}\{\limsup_{n\to\infty} R_n = a\} = \mathbb{P}\{\limsup_{n\to\infty} R_n = b\} = 0.$$

*Proof.* Analogue of Lemma 19. $\square$

**Theorem 29.** *( [2], p.311) Assuming conditions 4, 5, 6 we have*

$$\mathbb{P}\{R_n = 0 \ i.o.\} = \mathbb{P}_1\{\mathbb{P}_{\mathcal{E}}\{R_n = 0 \ i.o.\} = 1\} = 1$$

*Assuming 4, 6, and $\mathbb{E}_1 V_0 \neq 0$ we have*

$$\mathbb{P}_1\{\mathbb{P}_{\mathcal{E}}\{R_n = 0 \ i.o.\} > 0\} = 0.$$

*Proof.* Assume that the walk begins with a move to the right. Then Lemma 28 implies that the walker either returns to 0 or else goes to $+\infty$ before it returns. But by (7), for any $\epsilon > 0$ there is a $n_0 = n_0(\epsilon, \mathcal{E})$ such that $p(0,1,n) = 1 - 1/D(n) \geq 1 - \epsilon$ when $n \geq n_0$ Therefore the probability that the walker returns to zero is greater than $1 - \epsilon$ for any $\epsilon > 0$, and so is one.

$\square$

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