# Two-Dimensional Orbifolds

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## 1 Definitions

An orbifold is a Hausdorff space $X_0$ along with an open cover $\{U_i\}$ that is closed under finite intersection with the following properties. For each $U_i$ there is an open set $\tilde{U_i} \subseteq \mathbb{R}^n$ and a finite group $\Gamma_i$ which acts on $\tilde{U_i}$. In addition there is a homeomorphism

$$\varphi_i : U_i \approx \tilde{U_i}/\Gamma_i$$

and whenever $U_i \subset U_j$ there is an injective homomorphism

$$f_{ij} : \Gamma_i \hookrightarrow \Gamma_j$$

and $\tilde{U_i}$ can be embedded in $\tilde{U_j}$ such that the following diagram commutes.

$$\begin{array}{ccc}
\tilde{U_i} & \xrightarrow[g_{ij}]{} & \tilde{U_j} \\
\downarrow & & \downarrow \\
\tilde{U_i}/\Gamma_i & \xrightarrow[g_{ij}/\Gamma_i]{} & \tilde{U_j}/\Gamma_i \\
\Big\downarrow \varphi_i & & \downarrow \\
 & & \tilde{U_j}/\Gamma_j \\
 & & \downarrow \varphi_j \\
U_i & \subset & U_j
\end{array}$$

Where $g_{ij}$ is the embedding $\tilde{U}_i \hookrightarrow \tilde{U}_j$. This embedding must also be equivariant with respect to $f_{ij}$, that is for all $\gamma \in \Gamma_i, g_{ij}(\gamma(x)) = f_{ij}(\gamma)(g_{ij}(x))$. We will only consider smooth orbifolds, that is orbifolds where for all $i, \Gamma_i$ is a smooth action.

**Definition 1.** An orbifold $O$ has the geometric structure of $S^2$ ($\mathbb{R}^2$ or $\mathbb{H}^2$ respectively) if for every open set $U_i \subset O$, $\tilde{U}_i \subset S^2$, $\Gamma_i \subset Isom(S^2)$ ($\mathbb{R}^2$ or $\mathbb{H}^2$ respectively) and when $U_i \subset U_j$, $g_{ij} \in Isom(S^2)$ ($\mathbb{R}^2$ or $\mathbb{H}^2$ respectively).

**Definition 2.** An orbifold *with boundary* is an orbifold $O$ and a boundary $\partial O$ such that $\partial O$ is has an open cover $\{V_i\}$ where each $V_i$ is homeomorphic to $\tilde{V}_i/\Gamma_i$ and $\tilde{V}_i$ is an open subset of $\mathbb{R}^2_+$, the upper half plane. As before, $\Gamma_i$ is a finite group acting on $\mathbb{R}^2_+$. We say $O \cup \partial O$ is an orbifold with boundary.

An orbifold with boundary can be defined have a geometric structure in the obvious way.

**Example 1.** Any manifold $M$ in an orbifold. For all $i$, $\Gamma_i$ is the trivial group, so it is clear that all the properties above hold.

**Example 2.** Let $O = \{(x, y) : y \geq 0\} \subset \mathbb{R}^2$. Then $O$ has an orbifold structure where every point of the form $(x, 0)$ has a neighborhood homeomorphic to $\mathbb{R}^2/\mathbb{Z}_2$ where $\mathbb{Z}_2$ acts on $\mathbb{R}^2$ by reflection along the line $y = 0$.

**Example 3.** Let $O = \mathbb{R}^2/\mathbb{Z}_3$ where $\mathbb{Z}_3$ acts on $\mathbb{R}^2$ by rotation by $2\pi/3$. In this case, the point $(0, 0)$ gets mapped to a "cone point." $\mathbb{R}^2$ induces a metric on $O$, but notice that $\{x \in O : d(x, (0, 0)) \leq 1\}$ (the unit ball around (0,0)) has area $\frac{1}{3}\pi$, one third the area in $\mathbb{R}^2$.

**Definition 3.** The *underlying space* $X_O$ of an orbifold $O$ is the the topological space associated with $O$.

It is important to note that often in 2 dimensions $X_O$ is a manifold or a manifold with boundary even when $O$ is not. This is possible because $O$ really consists of a topological space along with the cover $\{U_i\}$, the homeomorphisms $\varphi_i$ and the groups $\Gamma_i$ acting on $\mathbb{R}^n$. It is possible that a point in $O$ has a neighborhood homeomorphic to $\mathbb{R}^n$, but it is not a manifold if the $\Gamma_i$ associated with the point is not $\{1\}$.

In example 2, $X_O$ is the upper half plane, which is a manifold with boundary, while $O$ is an orbifold without boundary. In example 3, $X_O$ is an infinite cone, which is homeomorphic to $\mathbb{R}^2$, so it is a manifold while $O$ is not.

For each point $x \in O$, there is associated $\tilde{U} \in \mathbb{R}^2$ and a group $\Gamma$ acting on $\tilde{U}$. Let $\Gamma_x$ be the stabilizer subgroup of the pre-image of $x$ in $\Gamma$.

**Definition 4.** The *singular locus* of $O$ is $\Sigma_O = \{x : \Gamma_x \neq \{e\}\}$ (Where $\{e\}$ is the trivial group).

If the singular locus of $O$, $\Sigma_O = \phi$, then $O$ is a manifold.

**Definition 5.** An orbifold $\tilde{O}$ along with a projection $p$ is a covering of $O$ if $p : X_{\tilde{O}} \to X_O$ such that $\forall x \in X_O$, there is a neighborhood $U$ of $x$ that is homeomorphic to $\tilde{U}/\Gamma$ ($\tilde{U} \subset \mathbb{R}^n$) and every $v_i \in p^{-1}(U)$ has a neighborhood $U_i \approx \tilde{U}_i/\Gamma_i$ where $\tilde{U}_i \subset \mathbb{R}^n$ and $\Gamma_i \subset \Gamma$.

**Example 4.** Let $O_n = \mathbb{R}^2/\mathbb{Z}_n$ where $\mathbb{Z}_n$ acts on $\mathbb{R}^2$ by rotations of the form $a\pi/n, a \in \mathbb{Z}$. If $p, q, r \in \mathbb{Z}$ and $pq = r$ then $O_p$ and $O_q$ can be projected onto $O_r$ as orbifold covers.

**Definition 6.** A *good* orbifold is an orbifold which has a covering that is a manifold. A *bad* orbifold cannot be covered by a manifold.

**Definition 7.** An orbifold $O$ is *orientable* if the underlying space $X_O$ is orientable and for all $x \in O$ the group $\Gamma$ associated with $x$ has no orientation reversing elements.

# 2 Classification

**Theorem 1.** *The singular locus of a 2-dimensional orbifold consists of three types of points:*
*(i) mirror points: homeomorphic to* $\mathbb{R}^2/\mathbb{Z}_2$*,* $\mathbb{Z}_2$ *acts by reflection along a line.*
*(ii) cone points: homeomorphic to* $\mathbb{R}^2/\mathbb{Z}_n$*,* $\mathbb{Z}_n$ *is generated by rotation of* $2\pi/n$*.*
*(iii) reflector corners: homeomorphic to* $\mathbb{R}^2/D_n$*,* $D_n$ *is the dihedral group of order 2n, that is generated by reflection about two lines that meet at an angle of* $\pi/n$*.*

*Proof.* If $\Gamma$ is a finite group acting on $\mathbb{R}^2$ then it must be a finite subgroup of $O(2)$. Every element of $O(2)$ consists of a rotation around the origin possibly composed with a reflection.

(i) If $\Gamma$ consists of just the identity and one reflection, then we have a mirror point.

(ii) If $\Gamma$ has no orientation-revesing elements, then it has only rotations. Suppose it contains an element $\gamma$ that is a rotation by $\alpha$ which is not a rational multiple of $2\pi$. Then $\gamma^n \neq e$ for any $n \neq 0$ but $\gamma^n$ can be arbitrarily close to $e$, so $\Gamma$ is not finite. Let $\gamma$ be rotation by $\frac{m}{n}2\pi$ where $\frac{m}{n}$ is in lowest terms. Then there exists $q, p \in \mathbb{Z}$ s.t. $mq - np = 1$, so $\gamma^q$ is rotation by $\frac{1}{n}2\pi$, which generates all rotations of the form $\frac{a}{n}2\pi$. Since $\Gamma$ is finite, there must be a maximum $n$ such that $\Gamma$ contains rotation my $\frac{1}{n}2\pi$ and this element generates $\Gamma$. Then we have a cone point.

(iii) If $\Gamma$ contains rotations and reflections, then the orientation preserving elements of $\Gamma$ are generated by an element $\gamma$, which is rotation by $\frac{2\pi}{n}$. Without loss of generality, we may assume that $\Gamma$ contains an element $\alpha$ which is reflection about the $x$-axis. The point $(\cos(\frac{\pi}{n}), \sin(\frac{\pi}{n}))$ is fixed by $\gamma \circ \alpha$. However, $(\cos(\frac{n+1\pi}{n}), \sin(\frac{n+1\pi}{n}))$ is not fixed, so $\gamma \circ \alpha = \beta$ must be reflection along the line that intersects the origin at angle $\frac{\pi}{n}$. $\gamma = \beta \circ \alpha$, so $\Gamma$ is generated by $\alpha$ and $\beta$ and we have the dihedral group $D_n$. $\square$

Any closed 2-dimensional orbifold can then be defined by its underlying space the cone points and corner reflectors where $(n_1, \ldots, n_k; m_i, \ldots, m_j)$ denotes cone points of orders $n_1, \ldots, n_k$ and corner reflectors of orders $2m_1, \ldots, 2m_j$.

**Theorem 2.** *There are only four 2-dimensional bad orbifolds without boundary. They are:*

*(i)* $X_O = S^2 : (n;)$ *(tear drop)*
*(ii)* $X_O = S^2 : (n_1, n_2;), n_1 > n_2$ *(spindle)*
*(iii)* $X_O = D^2 : (; n_1)$
*(iv)* $X_O = D^2 : (; n_1, n_2), n_1 > n_2$

*Proof.* Note that (iii) can be covered by (i) and (iv) can be covered by (ii). Let $O$ be a bad orbifold with no proper orbifold covering. First, consider the $X_O = S^2 : (n_1, n_2, n_3;)$. This orbifold can be generated by a group acting on $S^2$, $\mathbb{R}^2$ or $\mathbb{H}^2$ where the group is generated by reflections across the edges of a triangle with corners of angles $\pi/n_1, \pi/n_2, \pi/n_3$. (This actually forms the orbifold with three corner reflectors, but we achieve the orbifold with three cone points by taking the subgroup of orientation preserving elements). If $1/n_1 + 1/n_2 + 1/n_3 > 1$ we can form a triangle in $S^2$ if $1/n_1 + 1/n_2 + 1/n_3 = 1$ the triangle is in $\mathbb{R}^2$ and if $1/n_1 + 1/n_2 + 1/n_3 > 1$ the triangle is in $\mathbb{H}^2$. Therefore, if $O$ has at least three cone points, we can take a suborbifold $D$ that has underlying space of the disc and contains exactly 3 cone points. We can then find a proper cover $\tilde{Y}$ of $Y$ by finding a proper cover of $\tilde{D}$ of $D$ using the method above. Then we glue a copy of $Y - D$ to each boundary component of $\tilde{D}$. If $O$ contains reflector lines, then there is a proper orbifold covering which can be achieved by taking two copies of $O$ and gluing them along the reflector lines. Also, if $X_O$ is not simply connected, then it has a proper covering $\tilde{X_O}$ which naturally inherits an orbifold structure to become $\tilde{O}$ an orbifold cover of $O$. □

**Definition 8.** Given an orbifold $O$, if $\mathcal{K}$ is a CW-complex decomposition of $O$ s.t. the group associated with each point is constant along each cell, then the Euler characteristic of $O$ is defined to be

$$\chi(O) = \sum_{\Delta}(-1)^{dim(\Delta)}\frac{1}{|\Gamma_\Delta|}$$

where the sum is taken over all cells in $\mathcal{K}$ and $|\Gamma_\Delta|$ is the order of the group associated with the cell $\Delta$.

For a manifold, $\frac{1}{|\Gamma_\Delta|} = 1$ for all $\Delta$, so this reduces to the usually definition of Euler characteristic.

**Theorem 3.** *If $O$ is an orbifold that has a degree $n$ covering $\tilde{O}$ then $\chi(O) = \frac{1}{n}\chi(\tilde{O})$*

*Proof.* Let $O$ be an orbifold that is finitely covered by an orbifold $\tilde{O}$ such that $O \approx \tilde{O}/G$ for some finite group $G$ and $p$ is the projection. Then for a given cell $\Delta \in \mathcal{K}$, the CW-complex decomposition of $O$, there are $m$ pre-images of $\Delta$ in $\tilde{\mathcal{K}}$, the CW-complex decomposition of $\tilde{O}$ where $n$ divides$|G|$. Then there are $\frac{|G|}{m}$ elements of $G$ which fix $\Delta$. Thus, the group $\Gamma$ associated with $\Delta$ has order $\frac{|G|}{n}|\Gamma_{\tilde{\Delta}}|$ where $\Gamma_{\tilde{\Delta}}$ is the group associated to each $\tilde{\Delta}$ in the pre-image of $\Delta$. Therefore,

$$\frac{1}{|\Gamma_\Delta|} = \frac{m}{|G||\Gamma_{\tilde{\Delta}}|} = \frac{1}{|G|}\sum_{\tilde{\Delta}\in p^{-1}(\Delta)}\frac{1}{|\Gamma_{\tilde{\Delta}}|}$$

and

$$\chi(O) = \frac{1}{|G|}\chi(\tilde{O})$$

□

We will now develop a formula for the Euler character of a 2-orbifold based only on the underlying surface, cone points and corner reflectors. We will assume that every good, compact 2-orbifold without boundary is finitely covered by a manifold. Let $O$ be a 2-orbifold with n cone points of order $q_i, 1 \leq i \leq n$ and no reflector corners. Let $D_1, D_2, \ldots, D_n$ be disjoint discs in $X_O$ such that the $i$th cone point is on the interior of the $i$th disc. Let $Y$ be the complement of the discs, so that $X_O = Y \cup \left( \bigcup_i D_i \right)$. Since $\chi(D_i) = 1$, $\chi(X_O) = \chi(Y) + n$. We know $O$ is $d$-covered by some manifold $\tilde{O}$. If $\tilde{Y}$ is the pre-image of $Y$, then $\chi(\tilde{Y}) = d\chi(Y)$. But for each disc $D_i$, there are only $d/q_i$ pre-images. Therefore, $\chi(\tilde{O}) = d\chi(Y) + \sum_i d/q_i$. However, we have shown that $\chi(\tilde{O}) = d\chi(O)$. Thus $\chi(O) = \chi(Y) + \sum_i 1/q_i$, but $\chi(Y) = \chi(X_O) - n$ so we have

$$\chi(O) = \chi(X_O) - \sum_{i=1}^{n} \left(1 - \frac{1}{q_i}\right)$$

Now suppose $O$ has $m$ reflector corners of order $2p_j, 1 \leq j \leq m$ in addition to $n$ cone points of order $q_i, 1 \leq i \leq n$. Let $DO$ be the orbifold achieved from doubling $O$ along its reflector curves. Now each cone point has two pre-images and the pre-image of each corner reflector is a cone point of order $p_j$. We have shown

$$\chi(DO) = \chi(X_{DO}) - 2\sum_i \left(1 - \frac{1}{q_i}\right) - \sum_j \left(1 - \frac{1}{p_j}\right)$$

. Since $\chi(DO) = 2\chi(O)$ and $\chi(X_{DO}) = \chi(X_O)$, we have:

**Theorem 4.** *A 2-orbifold with $m$ reflector corners of order $2p_j, 1 \leq j \leq m$ and to $n$ cone points of order $q_i, 1 \leq i \leq n$ has Euler characteristic*

$$\chi(O) = \chi(X_O) - \sum_i \left(1 - \frac{1}{q_i}\right) - \frac{1}{2}\sum_j \left(1 - \frac{1}{p_j}\right)$$

This is called the Riemann-Hurwitz formula. It makes it especially clear that the teardrop has no proper covering. The teardrop has Euler characteristic $1 + 1/n$, so any cover would have characteristic greater than 2.

**Theorem 5.** *Every orientable closed 2-dimensional orbifold other than those mentioned in Theorem 2 have the geometric structure of $S^2$, $\mathbb{R}^2$, or $\mathbb{H}^2$.*

**Lemma 1.** *The only closed orientable orbifolds with positive Euler characteristic are:*

$X_O = S^2 : (), (n, n), (2, 2, n), (2, 3, 3), (2, 3, 4), (2, 3, 5).$

*and each has the geometric structure of* $S^2$*.*

*Proof.* Clearly $S^2 : (), (n, n)$ are covered by the sphere.
$S^2 : (2, 2, n), (2, 3, 3), (2, 3, 4), (2, 3, 5)$ are all covered by tessellating the sphere with triangles with angles $\pi/n_i$ and then taking the orientation preserving elements of the group generated by reflections along each of the lines. Since $S^2$ is the only 2-manifold with positive Euler characteristic, any orbifold with positive Euler characteristic must have $X_O = S^2$. Also, if $O$ has $k$ cone points then $\sum_1^k 1 - \frac{1}{n_i} < 2$. However,

$$1 - \frac{1}{n_i} \geq \frac{1}{2} \Rightarrow \sum_1^k 1 - \frac{1}{n_i} \geq \frac{k}{2}$$

so $O$ can have at most 3 cone points. It is also clear that if $O$ has three cone points, then it must have at least one of order 2. Assuming neither of the other two cone points has order 2, then

$$(1 - \frac{1}{n_2}) + (1 - \frac{1}{n_3}) < \frac{1}{2} \Rightarrow n_2 = 3 \text{ or } n_3 = 3$$

Assuming it is $n_2$, then $(1 - 1/n_3) < 5/6 \Rightarrow n_3 = 4$ or $n_3 = 5$. □

**Lemma 2.** *The only closed orientable orbifolds with Euler characteristic* 0 *are:*

*(i)* $X_O = S^2 : (2, 3, 6), (2, 4, 4), (3, 3, 3), (2, 2, 2, 2)$

*(ii)* $X_O = T^2 : ()$

*and each has the geometric structure of* $\mathbb{R}^2$*.*

*Proof.* $X_O = S^2 : (2, 3, 6), (2, 4, 4), (3, 3, 3)$ are covered by tessellating the plane with Euclidean triangles. $S^2 : (2, 2, 2, 2)$ covered by laying out a grid in the plane and taking the group generated by rotations of $\pi$ around the vertices. $T^2$ is covered by taking the group generated by two linearly independent translations. The torus and the sphere are the only 2-manifolds with non-zero Euler characteristic, so any orbifold with non-zero Euler characteristic must have $X_O = S^2$ or $T^2$. Since $\chi(T^2) = 0$, the orbifold cannot have any cone points if the underlying space is the torus. If $X_O = S^2$ then $O$ cannot have more than four coin points and can only have four cone points if they all have order 2. If $O$ has three cone points, they must all have order 3 if none has order 2. If one has order 2, then

$$1 - \frac{1}{n_2} + 1 - \frac{1}{n_3} = \frac{3}{2} \Rightarrow n_2 = 3, n_3 = 6 \text{ or } n_2 = 4, n_3 = 4$$

If $n_2 = 5$ then $1 - \frac{1}{n_3} = 7/10$ which is not possible for $n_3 \in \mathbb{N}$. If $n_2 > 6$, then $1/2 < 1 - \frac{1}{n_3} < 2/3$ which is also not possible for $n_3 \in \mathbb{N}$. If $O$ has two, one or zero cone points, then its Euler characteristic will be positive. □

**Theorem 6.** *Let $O$ be a closed orientable 2-orbifold with $\chi(O) < 0$. Then $O$ has a hyperbolic structure.*

The proof will only be outlined. The method is similar to the pants decomposition used for 2-manifolds. However, instead of using just pants ($P$), we will also use $A_{(n)}$ (the annulus with one cone point) and $D^2_{(n_1,n_2)}$ (the disk with two cone points where $n_1 \neq 2$ or $n_3 \neq 2$). Each of these "generalized pants", which are orbifolds with boundary are covered by generalized triangles in $\mathbb{H}^2$.

**Lemma 3.** *Each of the generalized pants $P$, $A_{(n)}$, and $D^2_{(n_1,n_2)}$ ($n_1 \neq 2$ or $n_3 \neq 2$) have a hyperbolic structure.*

*Proof.* A pair of pants can be decomposed into two right hexagons by cutting along geodesics from one boundary to the next that meet at right angles. Since pants are paramaterized by the length of their boundaries, we need to show that for any $a, b, c \in \mathbb{R}$ there is a hexagon in $\mathbb{H}^2$ with geodesic edges and edge lengths $l_1, l_2, l_3$ (on disjoint edges). This hexagon can be constructed in the upper half plane model as shown in *Fig 1*. We first fix $z_1$ on the imaginary axis and draw a line of length $l_1$ along the perpendicular geodesic. Then draw side $C$ along the geodesic perpendicular to side $a$ at the point $z_2$. A side of length $l_2$ can be drawn perpendicular to side $C$ at the point $z_3$. As $z_3$ moves along its geodesic, the other endpoint of $b$, $z_4$, will trace out a Euclidean circle $d$ in the plane. Side $A$ can be constructed perpendicular to $b$ and tangent to $d$. For sufficiently small $C$, the geodesic along which $A$ is drawn will lie entirely right half of the plane and it will be possible to draw $c$ perpendicular to $A$ and $B$. As $|C|$ increases, the Euclidean circle on which $c$

is drawn will increase in radius, which means $|B|$ will shrink. The hexagonal law of sines ([3], page 82) says

$$\frac{sinh|a|}{sinh|A|} = \frac{sinh|b|}{sinh|B|} = \frac{sinh|c|}{sinh|C|}$$

but

$$\lim_{|C|\to\infty} \frac{sinh|b|}{|B|} = \infty \Rightarrow \lim_{|C|\to\infty} \frac{sinh|c|}{|C|} = \infty$$

so

$$\lim_{|C|\to\infty} |c| = \infty$$

Which means that as $|C|$ varies from the smallest value for which $c$ exists to $\infty$, $|c|$ ranges over all positive values and there exists right hexagon with edge lengths $l_1, l_2, l_3$.

The annulus with one cone point can be constructed in a similar way. Instead of allowing $|C|$ to be large enough for $c$ to exist, we instead shrink $|C|$ until $A$ and $B$ meet at infinity. We can then shrink $C$ further so that the angle between $A$ and $B$ is the angle of the cone point.

To get the disc with two cone points we shorten $B$ from the bottom (keeping $z_1$ fixed). Eventually $b$ will disappear when $C$ and $A$ meet at infinity. We can then continue shrinking $B$ until the correct angle is achieved between $A$ and $C$. This will not work if both angles are $\frac{\pi}{2}$ because then the proper angle will not be reached until $A = a$, so it is necessary for one of the angles to be strictly less that $\frac{\pi}{2}$. □

First, if $X_O = S^2 : (n_1, n_2, n_3)$ with $(1-1/n_1)+(1-1/n_2)+(1-1/n_3) > 2$ then $O$ can be covered by tessellating hyperbolic plane with hyperbolic triangles. If $O$ has pairs of cone points with order 2, then we can cut along a line between the cone points, which will replace them with a geodesic boundary. Therefore we will assume that $O$ has at least 3 cone points with at most one cone point of order 2. We can then remove $D^2_{(n_1,n_2)}$ by cutting along simple closed curves that separate two cone points from the rest of the orbifold. Each time we do this, the Euler characteristic increases by $1 - \frac{1}{n_1} - \frac{1}{n_2} < 1$. If there is a lone cone point remaining, we may cut off $A_{(n)}$ by cutting along a simple closed curve that separates the cone point and a boundary from the rest of the orbifold.

After performing these surgeries, we are left with an orientable 2-manifold with at least 2 boundaries. If it is the annulus, then we did one surgery too many and by undoing the last surgery, we have either $D^2_{(n_1,n_2)}$ or $A_{(n)}$. Otherwise, our remaining surface must have negative Euler characteristic. This means we can cut off pairs of pants by pairing off boundaries until there are only three boundaries left. This will give us a generalized pants decomposition of the orbifold.

If $O$ has only one or two cone points, then the underlying surface must be $T^2$ or have negative Euler characteristic. If $X_O = T^2$ with two cone points then we have two copies of $A_{(n)}$ glued together. If $X_O = T^2$ with one cone point, then we have one copy of $A_{(n)}$ with its boundaries glued together. If the underlying surface has negative Euler characteristic, then we can cut off one or two copies of $A_{(n)}$ and still be left with a surface with negative Euler characteristic, which can be decomposed into pants.

This decomposition into generalized pants gives $O$ a hyperbolic geometry.

# 3 Relations to Groups

This demonstrates one of the main reasons that orbifolds are studied. Not only does each orbifold have a universal cover by $S^2$, $\mathbb{R}^2$ or $\mathbb{H}^2$ where the deck transformations are isometries of the covering space, but any group $\Gamma$ of isometries that acts properly discontinuously on $S^2$, $\mathbb{R}^2$ or $\mathbb{H}^2$ will define an orbifold. The group associated with a point $x \in O$ will be the stabilizer subgroup of a pre-image of $x$ in $S^2$, $\mathbb{R}^2$ or $\mathbb{H}^2$. If the group also acts freely, then the orbifold will be a manifold. For example, take the subgroup of $Isom(\mathbb{H}^2)$ generated by reflecting across the geodesics that form the the triangle as shown in *Fig 2*. The resulting orbifold is achieved by taking a fundamental region of the group action and associating the group $\{1\}$ with every point on the interior, $D_2$, $D_3$, $D_7$ to each of the vertices and $\mathbb{Z}_2$ with the edges.

This also allows us to use the classification above to classify certain subgroups of $Isom(\mathbb{R}^2)$ and $Isom(S^2)$. Since we have a list of all closed orientable orbifolds with positive or zero Euler characteristic, we have also made a list of all subgroups of $Isom(\mathbb{R}^2)$ and $Isom(S^2)$ that act proper discontinuously, consist of only orientation preserving elements and have a compact fundamental region.

## References


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