A Prelude to Euler's Pentagonal Number Theorem

August 15, 2007

Preliminaries: Partitions

In this paper we intend to explore the elementaries of partition theory, taken from a mostly graphic perspective, culminating in a theorem key in the proof of Euler's Pentagonal Number Theorem.

A *partition* λ= $(\lambda_1\lambda_2\ldots\lambda_r)$ of a positive integer n is a nonincreasing sequence of positive integers whose sum is n. As each n is finite, it follows trivially that each sequence is finite. Elements of a partition are called *parts.* As it is common for a given partition of n to have an inconvenient amount of repetition (for exmple, n may be written as a sum of n 1's), it will sometimes be useful to write a partition as a series of distinct integers, with subscripts denoting the number of repetitions of each integer. Thus, for example,$3+3+2+2+2+2+1+1 = 16$ will be written $(3_2 2_4 1_2)$.

Many problems in partition theory deal with the number of partitions of n. In this spirit, we define the partition function $p(n)$ as the number of partitions of n. As an illustration of the aforementioned concepts, we list the first six values of $p(n)$, and enumerate the coresponding partitions in lexographical order.

p(1) = 1: 1=(1);

p(2) = 2: 2=(2), 1+1=(1_{2});

p(3) = 3: 3=(3), 2+1=(21), 1+1+1=(1_3);

p(4) = 5: 4=(4), 3+1=(31), 2+2=(2_2), 2+1+1=(21_{2}), 1+1+1+1=(1_4);

p(5) = 7: 5=(5), 4+1=(41), 3+2=(32), 3+1+1=(31_2), 2+2+1= ($2_2 1$), 2+1+1+1=(21_3), 1+1+1+1+1=(1_5);

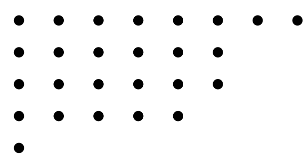
p(6) =11: 6=(6), 5+1=(51), 4+2= (42), 4+1+1=(41_2), 3+3=(3_2), 3+2+1=(321), 3+1+1+1=(31_3), 2+2+2=(2_3), 2+2+1+1=($2_2 1_2$), 2+1+1+1+1=(21_4), 1+1+1+1+1+1= (1_6) ;

The partition function continues to increase rapidly with n. For example, p(10) = 42, p(20) = 627, p(50) = 204,226, p(100) = 190,596,292, p(200) = 3,972,999,029,388 (!).

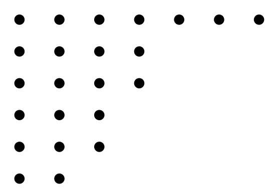
One useful tool for analysing partitions is a generating function for p(n), however in this paper we will endeavor to use a graphical means of analysis, as it is simple, yet achieves non-intutive and supprising results- and is hence most elegant.

Graphical Representation of Partitions

We introduce the Ferrers diagram (or graph, as it is also called), named for British mathematician Norman Macleod Ferrers (1829-1903), who created them to uniquely represent partitions of n. A Ferrers diagram consists of a stack of rows of dots or squares, each row representing a part of the partition, and the number of dots in the row the size of the part. For example, the Ferrers diagram of $(86_2 51)$ is:



Similarly, we have $(74_2 3_2 2)$

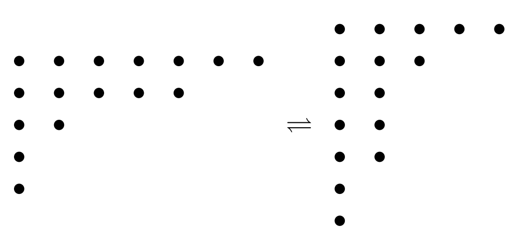


To illustrate the usefulness of the Ferrers diagram, and to further explore partitions, we introduce some problems along with their solutions.

Examples

Claim 1

The number of partitions of n into no more than k parts is equal to the number of partitions of n in which no part is greater than k. Integral to the proof of this claim is the quite natural concept of the *conjugate* of a Ferrers diagram, which is easily described if we think of the diagram as a matrix. Let us consider the Ferrers diagram as analagous to an NxM matrix with a_{ij} equal to 1 if the corresponding location in the diagram contains a dot, and 0 otherwise for all $1 \leq i \leq N$,$1 \leq j \leq M$. This being the case, the conjugate of a diagram would be analogous to the transposed matrix. We note it is fairly obvious that $D^{c^c}=D$, since for all matricies $M, M^{t^t} = M$.



Proof of claim 1

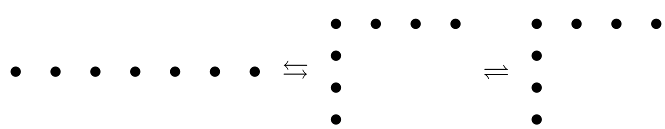
Now, using conjugates, the proof is relatively trivial. Suppose we let S be the set of all partitions of n with no more than k parts, and T be the set of all partitions of n with parts at most k (in value). We now form a bijection between the two sets, associating with each member of S the partition represented by the conjugate of its Ferrers diagram, which we claim is a member of T. Since a partition with no more than k parts will be represented by a diagram of no more than k rows, the conjugate of which will have no more than k columns, and hence represent a partition whose greatest part is at most k, and noting the process is reversable, the bijection is valid.■

Claim 2

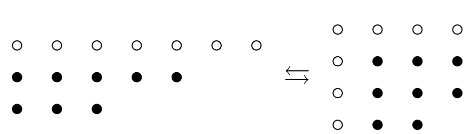
The number of partitions of n that are composed of distinct odd parts is equal to the number of partitions of n whose Ferrers diagrams are self-conjugate.

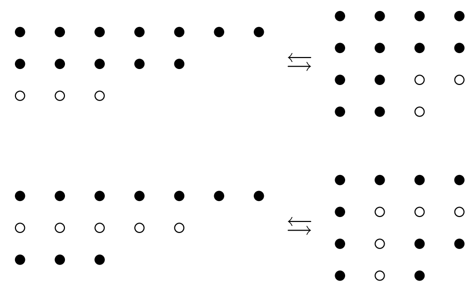
Proof of Claim 2

We again shall attempt to create a bijection between the two sets, S containing all partitions of with distinct odd parts, T containing all partitions whose Ferrers diagrams are self-conjugate. Note the elements of S and T are not asociated with any particular n- we simply are creating a bijection between the two kinds of partitions. We proceed by induction on the number of distinct odd parts in a partition λ in S. First, we let λ be composed of one odd part. In this case, we take the ferrers diagram, and fold it, like so:



This process is clearly reversable, and so we insert the double sided arrow. We use the $\rightleftharpoons$to show the second diagram is the conjugate of the third, and vice a versa. By the inductive hyothesis, we may do so with k odd distinct parts. Suppose now that we add one more distinct odd part, λ_i to λ, making λ'. Either λ_i is the largest part of λ', the smallest, or neither. In all three cases, the diagram of λ' is still self-conjugate:





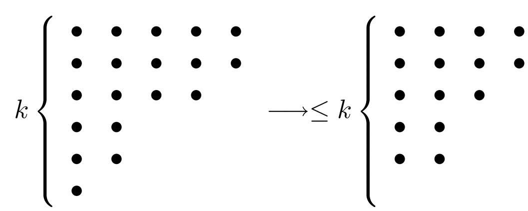
Let $H(n)$ be the set of all partitions of n. Since we have a bijection between S and T, we clearly have a bijection between $S \cap H(n)$ and $T \cap H(n)$ ■

Claim 3

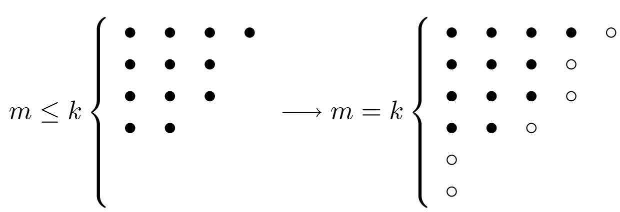
The number of partitions of n with number of parts at most k is equal to the number of partitions of $n+k$ with number of parts exactly k.

Proof of Claim 3

Once again, we will form a bijection between the two sets of partitions. Let S be the set of all partitions of n with number of parts at most k, and let T be the set of partitions of $n+k$ with number of parts exactly k. Suppose we begin with $\lambda \in T$. We will then take its ferrers diagram, and subtract one dot from each row (part) ofλ.



Since λ has k parts, its diagram has k rows, and hence we may subtract k from the partition by taking a dot form each row. The resulting diagram has at most k rows (it will have less if it had any rows with only one dot), and hence represents a partition of $(n+k)-k=n$, since the diagram now has only n dots in it. Therefore the resulting diagram represents an element of S. In the other direction, we may take $\lambda \in$S with $m \leq k$ parts. We add one dot to each of its rows, plus $k-m$ rows, with one dot in each row.

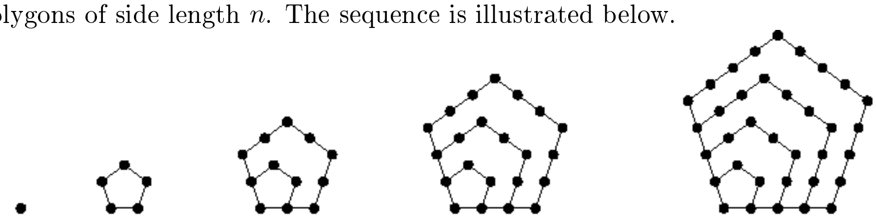


We have now added to the original m rows $m-k$ rows, making k rows, and k dots to the original n, making the new diagram representative of a partition of $n+k$ with exactly k parts.■

Now that we are thoroughly comfortable with ferrers diagrams, we move on the pentagonal number theorem.

The Pentagonal numbers

The pentagonal numbers are polygonal numbers, i.e. generated by constructing polygons of side length n. The sequence is illustrated below.



A B C D E

The sequence begins, as we observe form the illustration, $1, 5, 12, 22, 35...$(Sloane's A000326), and is generated by

$$P_m = \frac{m(3m-1)}{2}$$

It is interesting though of little use to this paper to note that $3P_n$ is the nth triangular number. The generalised pentagonal numbers, which seem to have little geometric interpretation, are obtained by letting the minus sign in the formula above become a $\pm$. This sequence is $1, 2, 5, 7, 12, 15, 22...$(Sloane's A001318). We now present a theorem.

Theorem

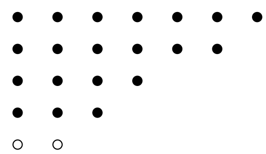
Let S be the set of all partitions of n into an even number of disinct parts. Let T be the set of all partitions of n into an odd number of disinct parts. Then

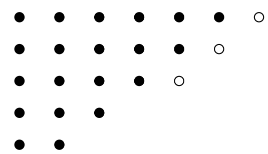
$$\mid S \mid - \mid T \mid = \begin{cases} (-1)^m & if \quad n = \frac{1}{2}m(3m \pm 1) \\ 0 & otherwise \end{cases}$$

Proof

For this proof, we will again attempt to form a bijection betwen S and T. The bijection will hold rather well, except of course when n is a generalised pentagonal number, where there will be one more of one kind than the other.

To begin with, we note that since for all partitions $\lambda \in T, S$ λ is composed of distinct parts, there is a smallest part of λ. We let the smallest part of λ be $s(\lambda)$. Also, the largest part of λ is the first in a subsequence of parts a_i,where $a_{i+1}= a_i - 1$. We denote the length of this sequence (which may be =1) $\sigma(\lambda)$. To further illustrate what each of these numbers represent, we provide the following figures:

$s(\lambda) = 2$ 

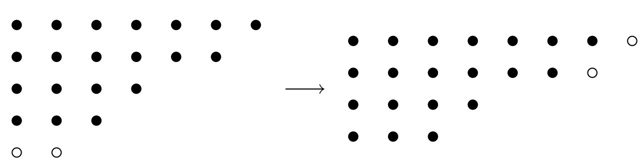
$\sigma(\lambda) = 3$ 

We will now describe a transformation on these diagrams that will establish a correspondence between partitions of n with evan and odd number of parts, respectively.

Case 1: $s(\lambda) \leq \sigma(\lambda)$ In this case, we delete the smallest part of λ, and add one dot to each of the $s(\lambda)$ largest parts of λ. Thus

$$\lambda = (76432) \longrightarrow \lambda' = (8743)$$

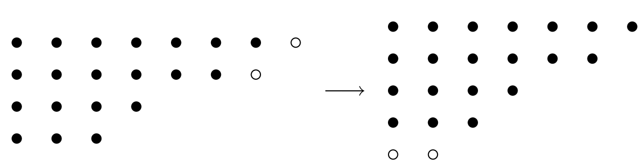
or



Case 2: $s(\lambda) > \sigma(\lambda)$ This this case, we perform the reverse operation- we delete one dot from the $\sigma(\lambda)$ largest parts of λ, and add a smallest part of size $\sigma(\lambda)$ to the partition. Thus

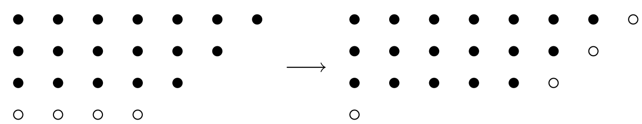
$$\lambda = (8743) \longrightarrow \lambda' = (76432)$$

or



In either case, the party of the number of parts in \lambda is changed by the transformation, yet the number of dots stays the same. Hence, we still have a partition of n. Also, its is simple to see that this transformaiton is an involution, and that for any \lambda, exactly one case is applicable. Hence, it would seem we have a bijection between S and T. However, both cases break down for certain classes of partitions. Case 1 breaks down when λ has r parts, $\sigma(\lambda) = r = s(\lambda)$. In this case, The partition is no longer composed of distince parts:

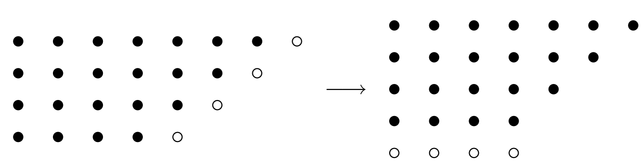
$$\lambda = (7654) \longrightarrow \lambda' = (8761)$$



In this case, $n = r + (r+1) + (r+2) \cdots (2r-1) = \frac{1}{2}r(3r-1)$ by Gauss' formula for arithmetic sums.

On the other hand, Case 2 breaks down when λ has r parts, $\sigma(\lambda) = r$, and $s(\lambda) = r+1$. In this case, the parity of the number of parts in λ is not changed by the transformation:

$$\lambda = (8765) \longrightarrow \lambda' = (76544)$$



Now, the number being partitioned is of the form $n = (r+1) + (r+2) \cdots (2r) = \frac{1}{2}r(3r+1)$.

Consequently, we have that if n is not a generalised pentagonal number, $\mid S \mid - \mid T \mid = 0$. On the other hand, if $n = \frac{m(3m-1)}{2}$, $\mid S \mid - \mid T \mid = (-1^m)$. ■

Corollary: Euler's Pentagonal Number Theorem

$$\prod_{n=1}^{\infty}(1-q^n) = 1 + \sum_{m=1}^{\infty}(-1)^m q^{\frac{1}{2}m(3m-1)}(1+q^m) = \sum_{-\infty}^{\infty}(-1)^m q^{\frac{1}{2}m(3m-1)}$$

We will not prove this theorem, as it takes an explanation of the generating function for $p(n)$ to truly understand, but we present it here to satisfy the curiosity this paper has undoubtedly aroused in the reader.