Dirichlet's Theorem

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Abstract

This paper provides a proof of Dirichlet's theorem, which states that when $(m, a) = 1$, there are infinitely many primes p such that $p \equiv a \pmod{m}$. The proof explained here mirrors Serre's proof in [1]. However, one distinction is that the Riemann zeta function is used as motivation to obtain results for the Dirichlet L-functions, which yields the above result. In fact, this paper can be read from the viewpoint of asking how do simple results of the zeta function depend on basic properties.

For the rest of the paper, m is a fixed integer as given in the above theorem. Let $\mathbb{P}$ deonote the set of primes, $\mathbb{P}_{m,a}$ denote the set of primes that are congruent to a modulo m and $G_m = (\mathbb{Z}/m\mathbb{Z})^\times$. We will use the Euler-phi function $\phi(m)$ which counts the numbers of positive integers less than m that are coprime to m. In particular, $|G_m| = \phi(m)$. A sequence of complex numbers $\{a_n\}$ is called strictly multiplicative if $a_n \cdot a_m = a_{nm}, \forall n, m \in \mathbb{N}$. The concept of analytic continuation will be useful in getting a deeper understanding of the proofs. However, prior knowledge is not necessary, and comments relating to analytic continuation can be ignored. Basic group theory is assumed. In particular, the result that abelian groups can be written as the product of cyclic groups is used without proof.

1 Riemann-zeta Function

In this section, we recall and derive certain basic properties of the Riemann zeta function. For each $s \in \mathbb{C}, \mathrm{Re}(s) > 1$, the Riemann zeta function is defined as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

Proposition 1. *For* $\mathrm{Re}(s) > 1$*, the zeta function converges.*

Proof. Using the integral test,

$$\int_1^{\infty} \frac{dt}{t^s} = \sum_{n=1}^{\infty} \int_n^{n+1} \frac{dt}{t^s} \leq \sum_{n=1}^{\infty} \frac{1}{n^s} = 1 + \sum_{n=2}^{\infty} \frac{1}{n^s} \leq 1 + \sum_{n=2}^{\infty} \int_{n-1}^{l} \frac{dt}{t^s} = 1 + \int_1^{\infty} \frac{dt}{t^s}$$

The result follows as we know that $\frac{1}{s-1} = \int_1^{\infty} \frac{dt}{t^s}$ within the domain. □

Proposition 2.

$$\zeta(s) = \prod_{p \in \mathbb{P}} \frac{1}{1 - p^{-s}}$$

Proof. Let N be any natural number, A_N be the set of natural numbers whose prime factors are not larger than N. Then we have

$$\sum_{n \in A_N} \frac{1}{n^s} = \prod_{p \in \mathbb{P}, p \leq N} \sum_{j=0}^{\infty} \frac{1}{p^{js}} = \prod_{p \in \mathbb{P}, p \leq N} \frac{1}{1 - p^{-s}}$$

The result follows by letting N tend to infinity. □

Comment. *Notice that the numerators are* $\{1, 1, 1, \ldots\}$ *and form a strictly multiplicative sequence, which allows the function to be written in a product form.*

Proposition 3. $\zeta(s) = \frac{1}{s-1} + \rho(s)$*, where* $\rho(s)$ *is holomorphic for* $\mathrm{Re}(s) > 0$.

Proof. Recall that

$$\frac{1}{s-1} = \int_1^{\infty} \frac{dt}{t^s} = \sum_{n=1}^{\infty} \int_n^{n+1} \frac{dt}{t^s}$$

Hence,

$$
\begin{aligned}
\zeta(s) &= \sum_{n=1}^{\infty} \frac{1}{n^s} = \sum_{n=1}^{\infty} \int_n^{n+1} \frac{dt}{n^s} \\
&= \sum_{n=1}^{\infty} \int_l^{n+1} \frac{dt}{n^s} + (\frac{1}{s-1} - \sum_{n=1}^{\infty} \int_l^{n+1} \frac{dt}{t^s}) \\
&= \frac{1}{s-1} + \sum_{n=1}^{\infty} \int_n^{n+1} \frac{1}{n^s} - \frac{1}{t^s}\, dt \qquad (*)
\end{aligned}
$$

Let $\rho_n(s) = \int_{n=1}^{l+1} \frac{1}{n^s} - \frac{1}{t^s}$, and $\rho(s) = \sum_{n=1}^{\infty} \rho_n(s)$. It is clear that each $\rho_n(s)$ is well defined and holomorphic for $\mathrm{Re}(s) > 0$. Hence, it remains to show that the series of functions converges uniformly on compact sets of $\mathrm{Re}(s) > 0$.
Observe that $|\rho_n(s)| \leq \sup_{n \leq t \leq n+1} |n^{-s} - t^{-s}|$, and $n^{-s} - t^{-s} = 0$ when $t = n$.
By considering the derivative of $n^{-s} - t^{-s}$ which is $st^{-(s+1)}$, and letting $x = \mathrm{Re}(s)$,

$$
\begin{aligned}
|\rho_n(s)| &\leq \sup_{n \leq t \leq n+1} \left| \int_n^{n+1} \frac{s}{t^{s+1}}\, dt \right| \leq |s| \sup_{n \leq t \leq n+1} \int_n^{n+1} \left| \frac{1}{t^{s+1}} \right| dt \\
&= |s| \sup_{n \leq t \leq n+1} \int_n^{n+1} \frac{dt}{t^{x+1}} \leq |s| \frac{1}{n^{x+1}} \leq \frac{|s|}{n^2}
\end{aligned}
$$

Thus, $\sum \rho_n(s)$ converges uniformly for $|s| \leq k$, $\mathrm{Re}(s) \geq \epsilon$, $\forall \epsilon > 0$. □

Corollary. *$\zeta(s)$ has a simple pole at $s = 1$.*

Comment. *The equation $(*)$ gives the analytic continuation of $\zeta(s)$ in* $\mathrm{Re}(s) > 0, s \neq 1$.

Theorem 1. *For $\mathrm{Re}(s) > 1$, as $s \to 1$, $\sum_{p \in \mathbb{P}} \frac{1}{p^s} \sim \log \frac{1}{s-1}$.*

Proof. From proposition 3, since $\rho(s) = \zeta(s) = \frac{1}{s-1}$ is holomorphic hence bounded in a neighborhood of 1, thus $\log \frac{1}{s-1} \sim \log \zeta(s)$. Replacing $\zeta(s)$ by its product form in Proposition 2,

$$\log \zeta(s) = \log \prod_{p\in\mathbb{P}} \frac{1}{1-p^{-s}} = \sum_{p\in\mathbb{P}} \log \frac{1}{1-p^{-s}} = \sum_{p\in\mathbb{P},k\geq 1} \frac{1}{k\cdot p^{ks}} = \sum_{p\in\mathbb{P}} \frac{1}{p^s} + \eta(s)$$

where $\eta(s) = \sum_{p\in\mathbb{P},k\geq 2} \frac{1}{k\cdot p^{ks}}$. To show that $\eta(s)$ is bounded, we majorize it by the series, with $x = \mathrm{Re}(s)$

$$\begin{aligned} |\eta(s)| &\leq \sum_{p\in\mathbb{P},k\geq 2} \left|\frac{1}{kp^{ks}}\right| \leq \sum_{p\in\mathbb{P},k\geq 2} \left|\frac{1}{p^{ks}}\right| \leq \sum_{p\in\mathbb{P},k\geq 2} \left|\frac{1}{p^{kx}}\right| \\ &\leq \sum_{p\in\mathbb{P}} \left|\frac{1}{p^x(p^x-1)}\right| \leq \sum_{p\in\mathbb{P}} \frac{1}{p(p-1)} \leq \sum_{n=2}^{\infty} \frac{1}{n(n-1)} \leq 1 \end{aligned}$$

Hence,

$$\log \frac{1}{s-1} \sim \log \zeta(s) \sim \sum_{p\in\mathbb{P}} \frac{1}{p^s}$$

. □

2 Characters of finite abelian groups

In this section, we obtain a basis for complex valued functions on $G_m = (\mathbb{Z}/m\mathbb{Z})^\times$ and study their properties. These arise naturally as an application of characters of G_m. Let G be an abelian group with the operation written multiplicatively.

Definition. A character of a group G is a homomorphism $\chi : G \to \mathbb{C}^\times$.

Example. *The function that maps every element of G identically to 1 is clearly a homomorphism. It is call the trivial character and denoted by χ_1.*

It is easy to check that the characters of G form a group $\mathrm{Hom}(G, \mathbb{C}^\times)$ which is denoted by $\hat{G}$, and called the dual of G.

Proposition 4. *$\hat{G}$ is a finite abelian group of the same order as G. Moreover, they are isomorphic as groups.*

Proof. We will first prove this for cyclic groups $|G| = n$. Let x be a generator for the cyclic group G, then each $\chi \in \text{Hom}(G, \mathbb{C}^\times)$ is uniquely determined by $\chi(x)$. Moreover, $1 = \chi(1) = \chi(x^n) = \chi(x)^n$ shows that there are at most n possible characters of G. Conversely, given any n^{th} root of unity ω, the map $\chi_\omega : G \to \mathbb{C}^\times$ given by $\chi_\omega(x^a) = \omega^a$ is a homomorphism.

We will now prove the general statement for abelian groups $|G|$. By the classification of abelian groups, $G \equiv H_1 \times \ldots \times H_k$, where each H_i is a cyclic group of prime power h_i . Let H_i be generated by x_i, then each $\chi \in \text{Hom}(G, \mathbb{C}^\times)$ is uniquely determined by $(\chi(x_1), \chi(x_2), \ldots, \chi(x_k))$. Since $\chi(h_i)$ can take on at most $|H_i|$ values, $|\hat{G}| \leq \prod |H_i| = |G|$.
Conversely, let $\pi : G \to \prod H_i$ be the projection map sending g to $({x_1}^{g_1}, {x_2}^{g_2}, \ldots, {x_k}^{g_k})$. Then, given any $(n_1, n_2, \ldots, n_k) \in \mathbb{N}^k, 1 \leq a_i \leq h_i$, consider $\chi(n_1, n_2, \ldots, n_k)$ which takes g to $\prod e^{2\pi i \frac{n_i}{h_i} g_i}$ clearly defines a character. Thus, $|\hat{G}| = |G|$, with the isomorphism given by the generators (hence not canonically)

□

Proposition 5. *Let G be a finite abelian group. Then G is canonically isomorphic to $\hat{\hat{G}}$.*

Proof. Given $x \in G$, the map $\phi_x : \chi \to \chi(x)$ is a character of $\hat{G}$. By the previous proposition, since $|G| = |\hat{\hat{G}}|$, it remains to show that the map $x \to \phi_x$ is injective.
With notation as in the previous proposition, consider the projection map $G \to \prod H_i$ sending g to $(x_1^{a_1}, \ldots, x_k^{a_k})$. For $g \neq 1$, $\exists a_j \neq 0$. Without loss of generality, $j = 1$. Then, $\chi(1, 0, \ldots, 0)(g) \neq 1$, which proves injectivity. □

Proposition 6. *Let $|G| = n$, $\chi \in \hat{G}$. Then,*

$$\sum_{x \in G} \chi(x) = \begin{cases} n & \chi = 1 \\ 0 & \chi \neq 1 \end{cases}$$

Proof. The first formula follows from adding $n\, 1's$. For the second, choose $y \in G$ such that $\chi(y) \neq 1$. Then

$$\chi(y) \sum_{x \in G} \chi(x) = \sum_{x \in G} \chi(xy) = \sum_{x \in G} \chi(x)$$

which implies that $\sum_{x \in G} \chi(x) = 0$.

□

Corollary. *Let* $|G| = n$, $\chi \in \hat{G}$. *Then*

$$\sum_{\chi \in \hat{G}} \chi(x) = \begin{cases} n & x = 1 \\ 0 & x \neq 1 \end{cases}$$

Proof. This follows from the previous proposition applied to the dual group, and using the isomorphism given in proposition 5. □

Comment. *Now consider functions from* G *to* $\mathbb{C}$. *Since* $\chi \in \hat{G}$ *form a linearly independent set of functions from* G *to* $\mathbb{C}$ *and* $|G| = |\hat{G}|$, *we get a basis set of multiplicative functions which satisfy* $\chi(xy) = \chi(x)\chi(y)$.
For G_m, *we can extend the domain of these characters from* G_m *to* $\mathbb{N}$, *by defining*

$$\chi(n) = \begin{cases} 0 & (n, m) \neq 1 \\ \chi(a) & n = a \,(mod\ m) \end{cases}$$

Furthermore, it is clear that $\{\chi(n)\}$ *is strictly multiplicative. These* χ *will be used to define the Dirichlet L-functions in Section 3.4.*

3 Dirichlet series

3.1 Lemmas

The following lemmas are useful for dealing with convergence of summations over complex numbers.

Lemma 3.1. *Let* $\{a_n\}, \{b_n\}$ *be two sequences of complex numbers and define* $A_{j,k} = \sum_{n=j}^{k} a_n$, *and* $S_{j,j'} = \sum_{n=j}^{j'} a_n b_n$. *Then,*
$S_{j,j'} = A_{j,j'} \cdot b_{j'} + \sum_{n=j}^{j'-1} A_{j,n}(b_n - b_{n+1})$.

Proof. The statement follows by replacing a_n with $A_{j,n} - A_{j,n-1}$ and regrouping terms. □

Lemma 3.2. *$\alpha, \beta \in \mathbb{R}$ such that $0 < \alpha < \beta$. Let $z = x + iy$, $x, y \in \mathbb{R}$ and $x > 0$, then*

$$|e^{-\alpha z} - e^{-\beta z}| \leq |\frac{z}{x}|(e^{-\alpha x} - e^{-\beta x})$$

Proof.

$$e^{-\alpha z} - e^{-\beta z} = z \int_{\alpha}^{\beta} e^{-tz}\, dt$$

hence by taking absolute values, with $x = \mathrm{Re}(z)$,

$$|e^{-\alpha z} - e^{-\beta z}| \leq |z| \int_{\alpha}^{\beta} \left|e^{-tz}\right| dt = |z| \int_{\alpha}^{\beta} e^{-tx}\, dt = |\frac{z}{x}|(e^{-\alpha x} - e^{-\beta x})$$

□

The following is a standard lemma from complex analysis.

Lemma 3.3. *Let U be an open subset of $\mathbb{C}$ and f_n a sequence of holomorphic functions on U which converge unifromly on every compact set to a function f. Then, f is holomorphic in U and the derivatives f_n' converge uniformly on all compact subsets to the derivative f' of f.*

Proof. Let D be a closed disc contained in U and ∂D be its boundary with usual orientation. By Cauchy's formula, $\forall z$ in the interior of D,

$$f_n(z_0) = \frac{1}{2\pi i} \int_{\partial D} \frac{f_n(z)}{z - z_0}\, dz$$

Taking limits, uniform convergence allows us to exchange the integral at lmits, hence we get

$$\begin{aligned} f(z_0) = \lim f_n(z_0) &= \lim \frac{1}{2\pi i} \int_{\partial D} \frac{f_n(z)}{z - z_0}\, dz \\ &= \frac{1}{2\pi i} \int_{\partial D} \lim \frac{f_N(z)}{z - z_0}\, dz = \frac{1}{2\pi i} \int_{\partial D} \frac{f(z)}{z - z_0}\, dz \end{aligned}$$

which shows that f is holomorphic in the interior of D. The statement for derivatives is proved similarly using

$$f'(z_0) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(z)}{(z - z_0)^2} \, dz$$

□

3.2 Dirichlet series

Let (λ_n) be a increasing sequence of positive real numbers tending to infinity. A Dirichlet series with exponents (λ_n) is a series of the form

$$f(z) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n z}, a_n \in \mathbb{C}, z \in \mathbb{C}$$

Example. *The zeta function is a Dirichlet series with $a_n = 1$ and $\lambda_n = \log n$. The Dirichlet series with these exponents will be studied in Section 3.3*

Proposition 7. *If the Dirichlet series $f(z)$ converges for $z = z_0$, then it converges uniformly in every domain of the form $Re(z - z_0) \geq 0$, $Arg(z - z_0) \leq \alpha$, where $\alpha < \frac{\pi}{2}$.*

Proof. Without loss of generality, $z_0 = 0$ and the hypothesis states that $\sum a_n$ is convergent. By Lemma 3.3, we have to show that there is uniform convergence in every domain of the form $Re(z) \geq 0$, $\frac{|z|}{Re(z)} \leq k$. Given $\epsilon > 0$, there exists N such that $\forall j, j' \geq N, |A_{j,j'}| \leq \epsilon$. Applying Lemma 3.1 with $b_n = e^{-\lambda_n z}$, we obtain

$$S_{j,j'} = A_{j,j'} e^{-\lambda'_j z} + \sum_{n=j}^{j'-1} A_{j,n} (e^{-\lambda_n z} - e^{-\lambda_{n+1} z})$$

Applying Lemma 3.2, with $x = \text{Re}(z)$,

$$|S_{j,j'}| \leq \epsilon(1 + |\frac{z}{x}| \sum_{n=j}^{j'} (e^{-\lambda_n x} - e^{-\lambda_{n+1} x})) \leq \epsilon(1 + k)$$

Thus, we have uniform convergence. □

Corollary. *It follows that if f converges for $z = z_0$, then it converges for $Re(z) > Re(z_0)$ by Lemma 3.3. Hence, the function is holomorphic in an open half plane, and its analytic continuation is given by the same formula.*

Proposition 8. *If the Dirichlet series $f(z)$ has positive real coefficients $\{a_j\}$ and converges for $R(z) > \rho, \rho \in \mathbb{R}$, and the function f can be extended analytically to a function holomorphic in a neighborhood of the point $z = \rho$, then there exists $\epsilon > 0$ such that f converges for $R(z) > \rho - \epsilon$.*

Proof. Without loss of generality, assume $\rho = 0$, hence f is holomorphic in a disc $|z - 1| \leq 1 + \epsilon$ and has a convergent Taylor series. The derivatives can be calculated using lemma 3.3, obtaining

$$f^{(n)}(z) = \sum_{\ell} a_\ell(-\lambda_\ell)^n e^{-\lambda_\ell \cdot z} \qquad f^{(n)}(1) = \sum_{\ell} a_\ell(-1)^n(\lambda_\ell)^n e^{-\lambda_\ell \cdot 1}$$

The Taylor series expansion is

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{n!}(z-1)^n f^{(n)}(1), \text{ when } |z - 1| \leq 1 + \epsilon$$

Hence,

$$f(-\epsilon) = \sum_{n=0}^{\infty}[\frac{1}{n!}(1+\epsilon)^n(-1)^n \sum_{\ell} a_\ell(-1)^n(\lambda_\ell)^n e^{-\lambda_\ell \cdot 1}]$$

is a double series with positive coefficient. By rearranging terms,

$$\begin{aligned} f(-\epsilon) &= \sum_{\ell}[a_\ell e^{-\lambda_\ell} \sum_{n=0}^{\infty} \frac{1}{n!}(1+\epsilon)^n \lambda_\ell^n] \\ &= \sum_{\ell} a_\ell e^{-\lambda_\ell} e^{\lambda_\ell(1+\epsilon)} \\ &= \sum_{\ell} a_\ell e^{\lambda_\ell \epsilon} \end{aligned}$$

Hence, the Dirichlet series converges for $z = -\epsilon$ and thus for $Re(z) > \epsilon$ from Proposition 7. □

Corollary. *The set of convergence of a Dirichlet series is an open half plane bounded by a singularity.*

3.3 Ordinary Dirichlet Series

The ordinary Dirichlet series is obtained by setting $\lambda_n = \log n$, which gives us

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}, a_n \in \mathbb{C}, s \in \mathbb{C}$$

Corollary. *Since* $\sum_{n=1}^{\infty} \frac{1}{n^\alpha}$ *converges for* $\alpha > 1$*, if* $\{a_n\}$ *is bounded, then there is absolute convergence for* $\mathrm{Re}(s) > 1$*.*

Proposition 9. *If the absolute value of the partial sums* $A_{j,k} = \sum_{n=j}^{k} a_n$ *are bounded by* K*, then there is convergence for* $\mathrm{Re}(s) > 0$*.*

Proof. This is similar to the proof given for $\rho(s)$ in proposition 3. By applying Lemma 3.1,

$$|S_{j,j'}| \leq K \left(\sum_{n=j}^{j'-1} |\frac{1}{n^s} - \frac{1}{(n+1)^s} + |\frac{1}{j'^s}| \right)$$

By Proposition 7, we may assume that s is real, hence the inequality simplifies to $|S_{j,j'}| \leq \frac{K}{j^s}$, showing convergence. □

3.4 Dirichlet L-function

Recall that the strictly multiplicative function $\chi : \mathbb{N} \to \mathbb{C}$ was defined at the end of section 2.

$$\chi(n) = \begin{cases} 0 & (n, m) \neq 1 \\ \chi(a) & n = a \pmod{m} \end{cases}$$

The corresponding Dirichlet L-function is defined by

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$$

Example. *In the specific case where* $m = 1$*,* $\zeta(s) = L(s, \chi_1)$*.*

We now establish results for the L-functions that are analogous to Propositions 1, 2 and 3 given for the zeta function.

Corollary. *From proposition 2,* $L(s,\chi_1) = F(s)\zeta(s)$ *where* $F(s) = \prod_{p|m}(1 - r^{-s})$*. Thus* $L(s,\chi_1)$ *extends analytically for* $\mathrm{Re}(s) > 0$ *and has a simple pole at* $s = 1$*.*

Proposition 10. *For* $\chi \neq \chi_1$*,* $L(s,\chi)$ *converges for* $\mathrm{Re}(s) > 0$ *and converges absolutely for* $\mathrm{Re}(s) > 1$*. Moreover,*

$$L(s,\chi) = \prod_{p\in\mathbb{P}} \frac{1}{1 - \frac{\chi(p)}{p^s}} \text{ for } \mathrm{Re}(s) > 1$$

Proof. Since the absolute value of the coefficients $|\chi(n)|$ are bounded by 1, it follows that the series is absolutely convergent for $\mathrm{Re}(s) > 1$.
Let N be any natural number, let A_N be the set of natural numbers whose prime factors are not larger than N. Then we have

$$\sum_{n\in A_N} \frac{\chi(n)}{n^s} = \prod_{p\in\mathbb{P},p\leq N} \sum_{\ell=0}^{\infty} \frac{\chi(p)^\ell}{p^{\ell s}} = \prod_{p\in\mathbb{P},p\leq N} \frac{1}{1 - \frac{\chi(p)}{p^s}}$$

As N tends to infinity, we establish the product formula of $L(s,\chi)$ as above.

From Proposition 6, $\sum_{i=j}^{j+m-2} \chi(i) = 0$, hence $|A_{j,j'}|$ is bounded by $\phi(m)$. By Proposition 9, $L(1,\chi)$ converges in $\mathrm{Re}(s) > 0$. Thus, we have convergence in $\mathrm{Re}(s) > 0$ from Proposition 8. □

3.5 Product of Dirichlet L-functions for fixed m

For any prime $p \nmid m$, let $\overline{p}$ be the image of p in $(\mathbb{Z}/m\mathbb{Z})^\times$, $f(p)$ be the order of $\overline{p}$, and $g(p) = \phi(m)/f(p)$.

Proposition 11. *Let* $\zeta_m(s) = \prod_\chi L(s,\chi)$*. Then* $\zeta(s)$ *is a ordinary Dirichlet series, with positive integral coefficients, converging in the half plane* $\mathrm{Re}(s) > 1$*, and*

$$\zeta_m(s) = \prod_{p\nmid m} \frac{1}{\left(1 - \frac{1}{p^{f(p)s}}\right)^{g(p)}}$$

Proof. For a fixed p, let ω_p denote the $f(p)$-th roots of unity. From the construction of characters in Proposition 4, it follows that the multi-set $\{\chi(p)\}$ is $g(p)$ copies of ω_p.

Since $\prod_{\omega\in\omega_p}(1-\omega T) = (1-T^{f(p)})$, we have $\prod_\chi(1-\chi(p)T) = (1-T^{f(p)})^{g(p)}$. Convergence in $\mathrm{Re}(s) > 1$ is guaranteed as we are multiplying finitely many convergent functions. Replace each L-function by its product form and set $T = \frac{1}{p^s}$,

$$\begin{aligned}\zeta_m(s) &= \prod_\chi \prod_{p\nmid m} \frac{1}{1-\frac{\chi(p)}{p^s}}\\ &= \prod_{p\nmid m} \prod_\chi \frac{1}{1-\frac{\chi(p)}{p^s}}\\ &= \prod_{p\nmid m} \frac{1}{\left(1-\frac{1}{p^{f(p)s}}\right)^{g(p)}}\end{aligned}$$

Expansion of this product form shows that $\zeta_m(s)$ is a ordinary Dirichlet series with has positive coefficients.

$\square$

Theorem 2. $\forall\chi \neq 1, L(1,\chi) \neq 0$.

Proof. Proof by contradiction. Suppose that for some $\chi \neq 1, L(1,\chi) = 0$, then ζ_m would be holomorphic at $s = 1$, and hence in $\mathrm{Re}(s) > 0$ by Proposition 10. The product form given in Proposition 11 would thus be the analytic continuation of $\zeta_m(s)$. By Proposition 8, since the Dirichlet series has positive coefficients, the series would converge for all $\mathrm{Re}(s) > 0$. However,

$$\frac{1}{\left(1-\frac{1}{p^{f(p)s}}\right)^{g(p)}} \text{ is dominated by } \frac{1}{1-p^{-\phi(m)s}},$$

it follows that the coefficients of ζ_m are greater than $\prod_{p\nmid m} \frac{1}{1-p^{-\phi(m)s}}$ This sequence diverges for $s = \frac{1}{\phi(m)}$ as

$$\prod_{p\nmid m} \frac{1}{1-p^{-1}} = \prod_{p\mid m}(1-p^{-1}) \sum_n \frac{1}{n}$$

$\square$

Corollary. *$\zeta_m(s)$ has a simple pole at $s = 1$*

Comment. *Observe that $\zeta_1(s) = \zeta(s)$, so we have a proof of the corollary to Proposition 3, which is independent of Proposition 3.*

4 Proof of Dirichlet's Theorem

The result stated in Theorem 1 allows us to define the analytic density of a set of primes $A \subset \mathbb{P}$. We say that A has an analytic density k if the ratio

$$\sum_{p \in A} \frac{1}{p^s} / \sum_{p \in \mathbb{P}} \frac{1}{p^s}$$

tends to k as $s \to 1^+$ on the reals. It is clear that if A is finite, then it has an analytic density of 0. We can now formulate a stronger version of Dirichlet's Theorem.

Theorem 3. *The set $\mathbb{P}_{m,a}$ has an analytic density of $\frac{1}{\phi(m)}$.*

Comment. *In the spirit of this paper, this result is analogous to Theorem 1. To imitate the proof, we need to understand $\log \zeta_m(s)$. The 'principal determination' of the logarithm is $\log \frac{1}{1-\alpha} = \sum \alpha^n$ as given by the Taylor expansion. For $L(s, \chi)$, define*

$$\log L(s, \chi) = \log \prod_{p \in \mathbb{P}} \frac{1}{1 - \frac{\chi(p)}{p^s}} = \sum_{p \in \mathbb{P}} \log \frac{1}{1 - \frac{\chi(p)}{p^s}} = \sum_{p \in \mathbb{P}, l \in \mathbb{N}} \frac{\chi(p)^l}{l p^{ls}}$$

This series is clearly convergent in $\mathrm{Re}(s) > 1$.
Equivalently, we take the 'branch' of $\log L(s, \chi)$ in $\mathrm{Re}(s) > 1$ which becomes 0 as s tends to infinity on the real axis.

For $\chi \in \hat{G_m}$, define

$$f_\chi(s) = \sum_{p \nmid m, p \in \mathbb{P}} \frac{\chi(p)}{p^s} = \sum_{p \in \mathbb{P}} \frac{\chi(p)}{p^s}$$

This function is convergent for $s > 1$.

Proof. We require the following 2 lemmas.

Lemma 4.1. *For* $\chi = 1, f_\chi(s) \sim \log \frac{1}{s-1}$.

Proof. This follows from Theorem 1 as $f_\chi(s)$ differs from $\sum \frac{1}{p^s}$ by the finitely many terms $\sum_{p|m} \frac{1}{p^s}$. □

Lemma 4.2. *For* $\chi \neq 1$, $f_\chi(s)$ *is bounded as* $s \to 1$.

Proof. To prove this, we consider the logarithm of $L(s, \chi)$.

$$\log L(s, \chi) = f_\chi(s) + F_\chi(s) \quad \text{where } F_\chi(s) = \sum_{p \in \mathbb{P}} \sum_{l \geq 2} \frac{\chi(p^l)}{lp^{ls}}$$

$|F_\chi(s)| \leq \sum_{l \geq 2} \frac{|1|}{|lp^{ls}|} \leq 1$, and by Theorem 2, $\log L(s, \chi)$ is bounded, hence so is $f_\chi(s)$. □

Now, for the proof of Dirichlet's theorem, let $g(s) = \sum_{p \in \mathbb{P}_{m,a}} \frac{1}{p^s}$. Then,

$$\begin{aligned}
\frac{1}{\phi(m)} \sum_{\chi \in \hat{G}} \chi(a)^{-1} f_\chi(s) &= \frac{1}{\phi(m)} \sum_{\chi \in \hat{G}} \sum_{p \in \mathbb{P}} \frac{\chi(a)^{-1}\chi(p)}{p^s} \\
&= \frac{1}{\phi(m)} \sum_{p \in \mathbb{P}} \sum_{\chi \in \hat{G}} \frac{\chi(a^{-1}p)}{p^s} \\
&= \frac{1}{\phi(m)} \sum_{p \in \mathbb{P}, a^{-1}p \equiv 1} \frac{\phi(m)}{p^s} \\
&= g(s)
\end{aligned}$$

where the third equality follows from Proposition 6.
Thus,

$$g(s) = \frac{1}{\phi(m)} \sum_{\chi \in \hat{G}} \chi(a)^{-1} f_\chi(s) \sim \frac{1}{\phi(m)} 1(a)^{-1} f_1(s) \sim \frac{1}{\phi(m)} \cdot \log \frac{1}{s-1}$$

□

References

[1] Jean-Pierre Serre, *A course in Arithmetic*, 1973.