Irreducible Representations of the Symmetric Group

Brian Taylor

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1 Abstract

The intent of this paper is to give the reader, in a general sense, how to go about finding irreducible representations of the Symmetric Group $\mathbf{S}_n$. While I would like to be thorough toward this end, I fear we must assume some results from Wedderburn Theory that will be given without proof because although they are important, proving and discussing these results is not in the scope of this paper. On that note, I'd like to mention that I make no claim to originality in the paper that follows as my work was heavily guided by two very excellent books:

- "Groups and Representations" by J.L. Alperin and R.B. Bell
- "Representation Theory" by W. Fulton and J. Harris

I would also like to thank my Graduate Mentors Michael Broshi and Sundeep Balaji for recommending texts and providing helpful suggestions.

2 Preliminaries

To begin, we shall need some preliminary facts and a brief discussion of notation. We will use $\mathbf{S}_n$ where possible in the examples, but where any group G will do we assume that the order of G is finite and that all $\mathbb{C}G$-modules are finitely generated, $\mathbb{C}$ being the field of complex numbers. We will also briefly use $\mathcal{M}_n(D)$ as the set of $n \times n$ matrices with entries in the ring D.

Proposition 2.1. *Suppose that A is a semisimple Algebra, and let $S_1 \ldots S_r$ be a collection of simple A-modules such that every simple A-module is isomorphic with exactly one S_i. Let M be an A-module, then we can write $M \cong n_1 S_1 \oplus \ldots \oplus n_r S_r$ for some non-negative integers n_i. Then the n_i are uniquely determined.*

The proof of this proposition follows from the Jordan-Hölder theorem for modules.

Definition 2.1. An algebra D is said to be a *division algebra* if the non-zero elements of D form a group under multiplication.

Theorem 2.2. *Let D be a division algebra, and let $n \in \mathbb{N}$. Then any simple $\mathcal{M}_n(D)$-module is isomorphic with D^n, and $\mathcal{M}_n(D)$ is isomorphic as $\mathcal{M}_n(D)$-modules with the direct sum of n copies of D^n. Specifically, $\mathcal{M}_n(D)$ is a division algebra.*

Theorem 2.3. *Let $r \in \mathbb{N}$. For each $1 \leqslant i \leqslant r$, let D_i be a division algebra over F, let $n_i \in \mathbb{N}$, and let $B_i = \mathcal{M}_{n_i}(D_i)$. Let B be the external direct sum of the B_i. Then B is a semisimple algebra having exactly R isomorphism classes of simple modules and exactly 2^r two-sided ideals. Specifically, these are all the sums of the form $\bigoplus_{j \in J} B_j$, where J is a subset of $\{1, \ldots, n\}$.*

Theorem 2.4. *Suppose that the field F is algebraically closed. Then any semisimple algebra is isomorphic with a direct sum of matrix algebras over F.*

These preceding theorems are powerful in their own right, and it is easy to see how useful they are regarding $\mathbb{C}$-modules. These four theorems entirely comprise the background we will need for this paper. Thus, without further ado, let's get started on some real mathematics!

3 Basics of Representation Theory

In this section, I aim to introduce Representation Theory and discuss a few specific theorems that will be helpful towards the final goal. In addition, I will prove a very useful theorem of Frobenius regarding the arithmetic function p(n) and the number of irreducible representations of $\mathbf{S}_n$.

Theorem 3.1. *1. There is some $r \in \mathbb{N}$ and some $f_1, \ldots, f_r \in \mathbb{N}$ such that $\mathbb{C}G \cong \mathcal{M}_{f_1}(\mathbb{C}) \oplus \ldots \oplus \mathcal{M}_{f_r}(\mathbb{C})$ as $\mathbb{C}$ algebras.*

2. There are exactly r isomorphism classes of simple $\mathbb{C}G$-modules, and if we let $S_1, \ldots, S_r$ be representatives of these r classes, then we can order the S_i so that $\mathbb{C}G \cong f_1S_1 \oplus \ldots \oplus f_rS_r$ as $\mathbb{C}G$ modules, where $\dim_{\mathbb{C}} S_i = f_i$ for each i.

3. Any $\mathbb{C}G$-module can be written uniquely in the form $a_1S_1 \oplus \ldots \oplus a_rS_r$, where the a_i are non-negative integers.

Proof. 1.: We first make use of the fact that since $\mathbb{C}$ has characteristic zero, by a consequence of Maschke's Theorem, every non-zero $\mathbb{C}G$-module is semisimple. This statement then follows directly from Theorem 2.4. 2.: Let S_i be the space of column vectors of length f_i with canonical module structure over $\mathcal{M}_{f_i}(\mathbb{C})$. This statement now follows from Theorems 2.2 and 2.3. 3.: This is a direct consequence of Proposition 2.1. □

Now we have the number of non-isomorphic simple $\mathbb{C}G$-modules for some group G, which is r. But how is this number r determined, and how is it related to G? These questions have very satisfying answers, and they are the subjects of the next theorem.

Theorem 3.2. *The number r of simple $\mathbb{C}G$-modules is equal to the number of conjugacy classes of G.*

Proof. Let Z be the center of $\mathbb{C}G$. By Theorem 3.1, we can see that $Z \cong Z(\mathcal{M}_{f_i}(\mathbb{C}) \oplus \ldots \oplus \mathcal{M}_{f_r}(\mathbb{C}))$ and is therefore also isomorphic with $\bigoplus Z(\mathcal{M}_{f_i}(\mathbb{C}))$. The center of any $\mathcal{M}_{f_i}(\mathbb{C})$ consists only of scalar matrices, thus it is also isomorphic with $\mathbb{C}$. In other words, $Z \cong \mathbb{C}^r$ and $dim_{\mathbb{C}} Z =$ r.

Now consider an element $\sum_{g \in G} \lambda_g g$ of Z. For any $h \in G$, we have $\left\{\sum_g \lambda_g g\right\} h = h \left\{\sum_g \lambda_g g\right\}$, giving $\sum_g \lambda_g g = \sum_g \lambda_g h^{-1} g h = \sum_g \lambda_{hgh^{-1}} g$ since it commutes in $\mathbb{C}G$. Therefore, if we look at the first and last of the above equalities, we see that $\lambda_g = \lambda_{hgh^{-1}}$ for every $g, h \in G$, and so we can conclude that the coefficients λ_g of the elements of Z are constant on conjugacy classes. Then we have a basis for Z with the set of class sums, which are of the form $\sum_{g \in K} g$ where K is a conjugacy class of G. Thus, $dim_{\mathbb{C}} Z$ is equal to the number of conjugacy classes of G. Since we've already established that $dim_{\mathbb{C}} Z$ is equal to the number of simple $\mathbb{C}G$-modules, the proof is complete.

□

For a moment, we will switch gears to talk about some of the fundamental components of representation theory and some of their properties which will be useful later on.

Definition 3.1. Let U be a $\mathbb{C}G$-module, where G is finite group. We let each g define an invertible linear transformation of U that sends $u \in U$ to gu. The *character* of U is the function denoted χ_U from $G \to \mathbb{C}$, where $\chi_U g$ is just the trace of this linear map defined by g. In other words, if $\rho\ G \to GL(U)$ is the *representation* corresponding to U, then $\chi_U(g)$ is just the trace of the map $\rho(g)$.

As we observed above in the proof of the theorem, the coefficients λ are constant on conjugacy classes. Similarly, we can see that the linear transformations of any $g, h \in G$ defined by g and hgh^{-1} are similar, thus having the same trace. Like above, therefore, characters of G are constant within conjugacy classes. Since we've shown that there are r simple $\mathbb{C}G$-modules and that each module has a character, we now assign the names of these characters in a useful way.

Definition 3.2. The characters $\chi_1, \ldots, \chi_r$ are known as *irreducible characters* of G. When we say that $S_1, \ldots, S_r$ are the r distinct simple $\mathbb{C}G$-modules, we order them such that $\chi_i = \chi_{S_i}$ for each i. If we let f_1 denote the degree of the trivial representation, then we also let χ_1 be its character and we call it the *principal character.* $\chi_1(g) = 1$ for all $g \in G$.

The following proposition outlining some properties of characters and representations I give without proof, though one can be found on p.140 in "Groups and Representations."

Proposition 3.3. *Let U be a $\mathbb{C}G$-module, let $\rho : G \to GL(U)$ be the representation corresponding to U, and let $g \in G$ be of order n. Then:*

1. *$\rho(g)$ is diagonalizable*
2. *$\chi_U(g)$ equals the sum, including multiplicities, of the eigenvalues of $\rho(g)$.*
3. *$\chi_U(g)$ is a sum of $\chi_U(1)$ roots of unity.*
4. *$\chi_U(g^{-1}) = \overline{\chi_U(g)}$.*
5. *$|\chi_U(g)| \leqslant \chi_U(1)$.*
6. *$\{x \in G | \chi_U(x) = \chi_U(1)\}$ is a normal subgroup of G.*

Suppose we have two characters χ and ψ of G. Define new functions $\chi + \psi$ and $\chi\psi$ from G to $\mathbb{C}$ by $(\chi + \psi)(G) = \chi(g) + \psi(g)$ and $(\chi\psi)(g) = \chi(g)\psi(g)$ for $g \in G$. Similarly, given a $\lambda \in \mathbb{C}$, we can define a new function $\lambda\chi : G \to \mathbb{C}$ by $(\lambda\chi)(g) = \lambda\chi(g)$, and consequently we can look at the characters of G as elements of a $\mathbb{C}$-vector space. An important implication of this is the subject of the following theorem, which will allow us to prove our first result concerning S_n.

Proposition 3.4. *The irreducible characters of G are, as functions from G to $\mathbb{C}$, linearly independent (over $\mathbb{C}$).*

Proof. From Theorem 3.1, we have that $\mathbb{C}G \cong \mathcal{M}_{f_1}(\mathbb{C}) \oplus \ldots \oplus \mathcal{M}_{f_r}(\mathbb{C})$. Let $S_1, \ldots, S_r$ be the distinct simple $\mathbb{C}G$-modules, and for each i let e_i be the identity of $\mathcal{M}_{f_1}(\mathbb{C})$. Fix an i. Since for any $g \in G$, $\chi_i(g)$ is the trace of a linear transformation on S_i defined by g, we can linearly extend χ_i to a linear map from $\mathbb{C}G$ to $\mathbb{C}$ so that $\chi_i(a)$ for $a \in G$ is the trace of the linear transformation on S_i defined by a. Observe that this linear transformation on S_i given by e_i is the identity map, and therefore that $\chi_i(e_i) = dim_{\mathbb{C}} S_i = f_i$. Further, if $j \neq i$, then this linear transformation on S_j is the zero map, and hence $\chi_j(e_i) = 0$.

Now assume we have some $\lambda_1 \ldots \lambda_r \in \mathbb{C}$ such that $\sum_{j=1}^{r} \lambda_j \chi_j(e_i) = \lambda_i f_i$ for each i; therefore $\lambda_j = 0$ for all j and the χ_i's are linearly independent. □

Now we just need a few more facts before we are ready to proceed with the major result of this section.

- Consider a natural number n. The *partition function* $p(n)$ is an arithmetic function that counts the number of different ways to partition the number n into other natural numbers. For example, if $n = 2$, then there are two partitions: 2 and 1+1. Thus $p(2) = 2$. $p(3) = 3$ corresponding to 1+1+1, 2+1, and 3 (note that 2+1 and 1+2 are considered the same partition). A fair amount of research has been devoted to this function and finding its exact value, but this does not concern us here.

- We take it for granted that the reader is familiar with the symmetric group S_n, and the concept of a conjugacy class. The conjugacy classes for S_n are the sets of permutations with similar cycle notation. For example, the permutations (12)(345) and (123)(45) would be in the same conjugacy class despite the difference in cycle order (this is due to the fact that disjoint cycles commute). However, the permutations (12)(34) and (1234) are not in the same conjugacy class.

- Finally, we note that conjugacy classes of S_n are bijective correspondence with partitions of n. This is accomplished by sending a partition $a_1 + \ldots + a_i$ to a permutation whose cycle lengths are $a_1, \ldots, a_i$. For a quick proof, consider if two non-isomorphic partitions map to the same conjugacy class. Then the two partitions induce the same cycle lengths, which means that one must be a rearrangement of the other, so they are the same partition. Now consider some conjugacy class with cycle lengths $b_1, \ldots, b_j$. The partition $b_1 + \ldots + b_j$ maps to it. Throughout this process we assume that where a partition has a +1, this corresponds to an element that is unpermuted by an element of its corresponding conjugacy class. For example, in S_3, the cycle (12) does not permute the number 3, so it corresponds with 2 + 1 as a partition, whereas it corresponds with 2 in S_2.

Theorem 3.5. *Fix some $n \in \mathbb{N}$. The number of irreducible characters of S_n is equal to $p(n)$.*

Proof. We first have to note that since $\mathbb{C}G$ is a semi-simple algebra, any $\mathbb{C}G$-module is a direct sum of the r distinct simple $\mathbb{C}G$-modules. Therefore, the list of irreducible characters we get from the r simple modules is complete; there are no more irreducible characters. From Prop 3.4 we know that there are no less, and from Theorem 3.2 we know that, since each irreducible character corresponds to a simple $\mathbb{C}G$-module, the number of irreducible characters is equal to the number of conjugacy classes, which in turn is equal to $p(n)$. □

4 Techniques for Finding and Evaluating Characters

Before we begin our investigation of the characters of S_n, we must introduce a few more properties of characters and the way in which this information is commonly displayed, the character table. In addition, we will begin to focus our attention more towards S_n and less on the ambiguous group G. Since for any group there are the same number of irreducible characters as conjugacy classes, and each character is constant on a conjugacy class, the values of each character on each conjugacy class are stored in an $r \times r$ array known as a *character table.* It is called $\mathcal{X}$, and looks something like the following:

	1	g_2	$\ldots$	g_r
χ_1	1	1	1	1
χ_2	f_2	$\chi_2(g_2)$	$\ldots$	$\chi_2(g_r)$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
χ_r	f_r	$\chi_r(g_2)$	$\ldots$	$\chi_r(g_r)$

Here we're using f_i to represent the degrees of G and χ_1 is, of course, the trivial representation.

We now return to some properties of characters that will help us to calculate explicit values in a character table.

Definition 4.1. We noted earlier that characters are constant on conjugacy classes, and they therefore belong to a set of functions known as *class functions.* If we let α and β be two class functions, then we define their *inner product* to be the complex number

$(\alpha, \beta) = \frac{1}{|G|} \sum_{g \in G} \alpha(g)\overline{\beta(g)}$
(1)

This has the following properties that one might expect:

- $(\alpha, \alpha) \geq 0$ for all α, and $(\alpha, \alpha) = 0$ iff $\alpha = 0$.
- $(\alpha, \beta) = \overline{(\beta, \alpha)}$ for all α, β.
- $(\lambda\alpha, \beta) = \lambda(\alpha, \beta)$ for all α, β and for all $\lambda \in \mathbb{C}$
- $(\alpha_1 + \alpha_2, \beta) = (\alpha_1, \beta) + (\alpha_2, \beta)$ for all $\alpha_1, \alpha_2, \beta$.

We state the following Lemma without proof to use in the next theorem.

Lemma 4.1. *If U is a $\mathbb{C}G$-module, then $dim_{\mathbb{C}} U^G = \frac{1}{|G|} \sum_{g \in G} \chi_U(g)$.*

Theorem 4.2. *For any $\mathbb{C}G$-modules U and V, we have $(\chi_U, \chi_V) = dim_{\mathbb{C}} Hom_{\mathbb{C}G}(U, V)$.*

Proof. First of all, we observe that $Hom_{\mathbb{C}G}(U, V)$ is a subspaces of the $\mathbb{C}G$-module $Hom(U, V)$. If $\varphi \in Hom_{\mathbb{C}G}(U, V)$ and $g \in G$, then $(g\varphi)(u) = g\varphi(g^{-1}u) = gg^{-1}\varphi(u) = \varphi(u)$ for any $u \in U$. Thus we have $g\varphi = \varphi$ for all $g \in G$, which shows that $\varphi \in Hom(U, V)^G$, where $Hom(U, V)^G = \{\varphi \in Hom(U, V) | g\varphi = \varphi$ for all $g \in G\}$. If we assume the reverse above, we can see that $Hom_{\mathbb{C}G}(U, V) = Hom(U, V)^G$. Therefore

$$\dim_{\mathbb{C}} Hom_{\mathbb{C}G}(U, V) = dim_{\mathbb{C}} Hom(U, V)^G = \frac{1}{|G|} \sum_{g \in G} \chi_{Hom(U,V)}(g) = \frac{1}{|G|} \sum_{g \in G} \overline{\chi_U(g)} \chi_V(g) = (\chi_V, \chi_U) \quad (2)$$

by the Lemma above. The significance of this theorem is that now we know the inner product of two characters is real-valued and it allows us to prove the following important theorem. □

Row Orthogonality Theorem 1. $(\chi_i, \chi_j) = \delta_{ij}$ for any i and j.

Proof. Let $S_1, \ldots, S_r$ be the distinct simple $\mathbb{C}G$-modules. By Theorem 4.2, we know that $(\chi_i, \chi_j) = dim_{\mathbb{C}} Hom_{\mathbb{C}G}(S_i, S_j)$ for any i and j. For each i, we have $Hom_{\mathbb{C}G}(S_i, S_i) = End_{\mathbb{C}G}(S_i) \cong \mathbb{C}$ since S_i is a simple $\mathbb{C}G$-module. If $i \neq j$ then $Hom_{\mathbb{C}G}(S_i, S_j) = 0$ because non-zero homomorphisms between simple $\mathbb{C}G$-modules are isomorphisms. □

Remark 4.1. In other words, we have the following useful equalities:

$$\delta_{ij} = \frac{1}{|G|} \sum_{g \in G} \chi_i(g) \overline{\chi_j(g)} = \frac{1}{|G|} \sum_{t=1}^{r} k_t \chi_i(g_t) \overline{\chi_t(g_t)}$$

where the g_t represent the conjugacy classes and the k_t are the orders of the corresponding conjugacy classes. If we interpret the rows of the character table as vectors, then, we can see that they are orthonormal with respect to the above inner product when considered as vectors in $\mathbb{C}^r$. We will not prove it here, but it is in fact true that row orthogonality and column orthogonality are equivalent in this case.

Proposition 4.3. *If α is a linear character of G and χ is an irreducible character of G, then $\alpha\chi$ is an irreducible character of G.*

Proof. Since α is linear, it follows from Proposition 3.3 that $\alpha(g)$ is a root of unity for any $g \in G$, and in particular that $1 = |\alpha(g)| = \alpha(g)\overline{\alpha(g)}$ for every $g \in G$. We now have,

$$(\alpha\chi, \alpha\chi) = \frac{1}{|G|}\sum_{g\in G}\alpha(g)\chi(g)\overline{\alpha(g)\chi(g)} = \frac{1}{|G|}\sum_{g\in G}\chi(g)\overline{\chi(g)}\alpha(g)\overline{\alpha(g)} = \frac{1}{|G|}\sum_{g\in G}\chi(g)\overline{\chi(g)} = (\chi, \chi).$$

And since χ is irreducible, $(\chi, \chi) = 1$ which means that $\alpha\chi$ is also irreducible. □

Remark 4.2. We already know about the trivial representation χ_1, but now we will briefly mention a second linear (that is, $f_i = 1$) character belonging to all S_n. Not surprisingly, perhaps, it is known as the alternating character and it's values are 1 or -1, depending on whether a permutation is odd or even. To be more precise, it is the result of modding out by A_n for the same n, but computing based on $sgn(\sigma)$ is just as easy. Now we are finally ready for an example.

Example 4.1. Consider the group S_3. Extracting from above, we see that the conjugacy classes are the identity, the transpositions, and the 3-cycles. Since every group has the two linear characters mentioned above, we begin by inserting their values into the table

	1	3	2
	1	(12)	(123)
χ_1	1	1	1
χ_2	1	-1	1
χ_3			

Now we invoke a corollary of Theorem 3.1 which states that the sum of the squares of the degrees of G is the order of G. Therefore, since $|S_3| = 6$, we must have $1 + 1 + f_3^2 = 6$, so it must be that f_3 is 2. We make use of column orthogonality to get:

$$0 = \sum_{i=1}^{3} f_i\chi_i((12)) = 1\cdot 1 + 1\cdot(-1) + 2\chi_3((12))$$

so we see from this that $\chi_3((12)) = 0$, and

$$0 = \sum_{i=1}^{3} f_i\chi_i((123)) = 1\cdot 1 + 1\cdot 1 + 2\chi_3((12))$$

from which we see that $\chi_3((123)) = -1$. Thus the full character table of S_3 is

	1	3	2
	1	(12)	(123)
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	-1

Remark 4.3. For a more satisfactory explanation of the third irreducible character above, we'll briefly discuss what is sometimes known as the "original fixed-point formula." It states that if V is the permutation representation associated to the action of the group G on some finite set X (in our case the set $\{1, \ldots, n\}$, then $\chi_V(g)$ is the number of elements of X fixed by g. This makes sense if we remember that a character is defined to be the trace of a linear transformation on V, so its value is actually determined by the number of unpermuted elements of X. To see how this applies, we first note that $\mathbb{C}^3 = U \oplus V$, where V is the character we're trying to calculate and U is the trivial character's module. Since the permutation representation as presented has values (3,1,0), and χ_U is (1,1,1), we can subtract these to find χ_V = (2,0,-1). Thus we now have 3 easily calculable characters for each S_n.

We note a quick corollary to row orthogonality before moving on to the next example.

Corollary 4.4. *A representation (therefore its character) V is irreducible if and only if $(\chi_V, \chi_V) = 1$.*

Example 4.2. Based on the available information, here's what we would expect the table of S_4 to look like with the first 3 characters:

	1	6	8	6	3
S_4	1	(12)	(123)	(1234)	(12)(34)
χ_1	1	1	1	1	1
χ_2	1	-1	1	-1	1
χ_3	3	1	0	-1	-1
χ_4					
χ_5					

We can check that χ_3 is irreducible using the inner product, and indeed this is so. To proceed, we once again look at the sum of the squares of the degrees. Since $|S_4| = 24$, we want the sum of the degrees of the five representations to be 24. The sum of the degrees of the known characters is 11, so we need another 13 for the last 2 characters. Thus, one has degree 2 and the other degree 3. To find the character of degree 3, we can simply tensor χ_2 and χ_3 to get another irreducible representation, χ_4 which equals (3,-1,0,1,-1). As for the last character, we know its degree (2), so we can get the rest of the information from row and column orthogonality. Thus, the completed table for S_4 looks like:

	1	6	8	6	3
S_4	1	(12)	(123)	(1234)	(12)(34)
χ_1	1	1	1	1	1
χ_2	1	-1	1	-1	1
χ_3	3	1	0	-1	-1
χ_4	3	-1	0	1	-1
χ_5	2	0	-1	0	2

The character values for S_5 can be found in essentially the same way as above. In fact, knowing S_n can give you all the values for S_{n+1}. Unfortunately, this is a very inefficient method for computing values. However, there is a useful way of finding them for the symmetric group using what are known as Young diagrams. It is an ingenious blend of algebra and graph theory that represents the final step in the completion of these tables. Its downside is the depth and complexity required for its formulation, which unfortunately is beyond the scope of this paper. I would recommend to interested readers that they look at this subject in more detail in "Representation Theory" where a full treatment is presented.