GROUP ACTIONS ON TREES

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1. Introduction

For this paper, we will define a (non-oriented) graph Γ to be a pair $\Gamma = (V, E)$, where $V = \text{vert}(\Gamma)$ is a set of vertices, and $E = \text{edge}(\Gamma) \subseteq V \times V/S_2$ is a set of unordered pairs, known as edges between them. Two vertices, $v, v' \in V$ are considered adjacent if $(v, v') \in E$, if there is an edge between them. An oriented graph has edge set $E = \text{edge}(\Gamma) \subseteq V \times V$, ordered pairs. For an edge $v = (v_1, v_2)$ in an oriented graph, $o(v) = v_1$ is the origin of v and $t(v) = v_2$ is the terminus of v. Unless noted, all graphs in this paper are non-oriented.

A group G acts on a set X if there is a map

$$G \times X \to X$$

$$(g, x) \mapsto gx$$

such that the following are true:
a) For e the identity of G, $ex = x$
b) For $h, g \in G$, $x \in X$, $h(gx) = (hg)x$.

A group G acts on a graph (V, E) if it acts on V and E in a compatible way; i.e. if $gv_1 = v_1'$ and $gv_2 = v_2'$ then $g(v_1, v_2) = (v_1, v_2)$. Of particular interest will be graphs that are trees. We will explore their properties, and the behavior of group actions on trees, looking especially at free groups and groups that act freely. All of the theorems and many of the proofs in this paper can be found in [1]. Our ultimate result will be to prove the following theorem:

Theorem 1.1. *A group is free if and only if there exists a tree on which it acts freely.*

2. Some Needed Definitions

First I will define a few terms used to describe graphs, and introduce a few basic propositions.

Definition 2.1. Let $Path_n$ be a graph with n+1 vertices, $\{v_0, v_1, ..., v_n\}$ such that for $0 \leq i \leq n$, vertices v_i and v_{i+1} are adjacent and $v_i \neq v_{i+1}$. A path of length n in a graph Γ is a morphism from $Path_n$ to Γ.

Definition 2.2. Let $Circuit_n$ be a graph with n vertices $\{v_0, v_1, ..., v_{n-1}\}$. Similarly to $Path_n$, for $0 \leq i \leq n$, vertices v_i and v_{i+1} are adjacent. Additionally, v_{n-1} is adjacent to v_0. A circuit of length n in a graph Γ is an injective morphism from $Circuit_n$ to Γ.

Definition 2.3. A tree is a connected non-empty graph with no circuits.

Definition 2.4. A geodesic is a path on a tree with edges $\{e_0, \ldots, e_n\}$ where for $0 \leq i \leq n$, $e_i \neq e_{i+1}$.

Proposition 2.5. *For v_1, v_2 vertices in a tree Γ, $v_1 \neq v_2$, there is exactly one geodesic from v_1 to v_2 and it is an injective path.*

Proof. Γ is connected, so there exists a geodesic between v_1 and v_2 (a path exists between v_1 and v_2, which trivially implies a geodesic). Let $f : Path_n \to \Gamma$ be a geodesic from v_1 to v_2 such that $f(0) = v_1$ and $f(n) = v_2$. If for $i \neq j$, $f(i) = f(j)$, then $\{f(i), ..., f(j)\}$ and the edges between them form a circuit, which is a contradiction because Γ is a tree. Therefore for $0 \leq i < j \leq n$, $f(i) \neq f(j)$, and so f is injective.

To show the geodesic is unique, assume there are two geodesics from $v_1$to v_2, f and g, with edges $\{x_1, ..., x_n\}$ and $\{y_1, ..., y_n\}$. If $y_n \neq x_n$, then there is a path from v_1 to v_1, namely f followed by g^{-1}. This contradicts the fact that Γ is a tree, so $x_n = y_n$, and inductively all the other edges are equal. This makes the two paths equal, so the geodesic is indeed unique. □

Definition 2.6. Let G be a group, and S be a subset of G. Then we will define $\Gamma(G, S)$, the Cayley graph of (G, S). The vertices of Γ are the elements of G, and two vertices g and h are adjacent if $\exists\ s \in S$ such that $gs = h$.

We have the following proposition.

Proposition 2.7. *Let Γ=$\Gamma(G, S)$ be the graph defined by a group G and a subset S of G. For Γ to be connected it is necessary and sufficient that S generate G.*

Proof. If S generates G, $\forall g, h \in G$, $\exists s_1, \ldots, s_n \in S \cup S^{-1}$ such that $gs_1 \ldots s_n = h$. This implies there is a path from g to h, and therefore Γ is connected.

If Γ is connected, $\forall g \in G$, $\exists$ a path from e to g for e the identity in G. This implies $\exists\ s_1, \ldots, s_n \in S \cup S^{-1}$ such that $es_1, \ldots, s_n = g$, but then $s_1, \ldots, s_n = g$, which implies S generates G. □

Definition 2.8. The distance on a tree from one vertex to another is the number of edges in the geodesic connecting them.

Definition 2.9. Let Γ be a tree. Let v be a vertex of Γ. Let X_n be the set of vertices at distance n from v. Each $x_n \in X_n$ is adjacent to exactly one $x_{n-1} \in X_{n-1}$. Define a map $f_n : X_n \to X_{n-1}$ such that x_n is adjacent to $f_n(x_n)$. Define an inverse system starting at v to be the set of these functions $\{f_n\}_{n\in\mathbb{N}}$.

It is important to note that each f_n is only well-defined if Γ is a tree, hence the following proposition.

Proposition 2.10. *If a graph can be represented by an inverse system, it is a tree.*

Proof. Take a graph Γ that can be represented by an inverse system $\{f_n\}_{n\in\mathbb{N}}$ starting at v. Assume Γ contains a cycle with vertices $\{v_1, \ldots, v_n = v_1\}$. Without loss of generality, we can assume $v = v_1$. Then $v_2 \in X_1$, $v_3 \in X_2$, $\ldots$, $v_n \in X_{n-1}$. But there is no function in $\{f_n\}_{n\in\mathbb{N}}$ that maps elements of X_{n-1} to elements of X_0 unless $n-1 = 1$, which implies the cycle has only two vertices v and v_2. Therefore $f_1(v_2) = v$, which means $v_2 \neq v$, but this contradicts the definition of a cycle. Therefore Γ is a tree. □

3. An Exploration of Subtrees

When examining a graph, it is often useful to look at subgraphs that are trees.

Proposition 3.1. *Every graph Γ has a maximal subtree.*

Proof. Consider the set of subtrees of Γ. This set has a partial ordering via inclusion. Let $\{T_i\}_{i\in I}$ be a chain of trees. If the $\bigcup_{i\in I} T_i$ is not a tree, then it contains a circuit. But a circuit is finite, so $\exists\ i \in I$ such that T_i contains a circuit, which is a contradiction. Therefore $\bigcup_{i\in I} T_i$ is a tree in $\{T_i\}_{I\in I}$, and so each chain is bounded by its union. By Zorn's lemma, the set of subtrees of Γ has a maximal element, which we will call a maximal tree of Γ. □

Proposition 3.2. *A maximal tree Λ of a connected non-empty graph Γ contains all the vertices of Γ.*

Proof. Assume for contradiction that Λ does not contain a vertex v of Γ. Then by connectedness, we can assume without loss of generality that there is an edge e from v to a vertex x of Λ. Because v is not a vertex of Λ, there exist no paths from vertices of Λ to v. Therefore, adjoining v and e to Λ creates a new graph that is still a tree, but that contains more vertices and edges than Λ. This contradicts the maximal quality of Λ. Therefore, Λ contains all the vertices of Γ. □

For any graph Γ, let $s = |\text{vert}(\Gamma)|$, and $a = |\text{edge}(\Gamma)|$.

Proposition 3.3. *For a connected graph Γ with a finite number of vertices, $a \geq s-1$, and $a = s-1 \iff \Gamma$ is a tree.*

Proof. Induction can be used to show equality holds for any tree Γ. If Γ has one vertex, then $a = 0$ and $s = 1$, so the claim holds. Assume for a tree Γ with n vertices that $a = s-1$. Consider a tree Ω with $n+1$ vertices, and assume all vertices of Ω are adjacent to at least two other vertices. Let $v_1 \in \text{vert}(\Omega)$, then v_1 is adjacent to a vertex v_2. But v_2 is adjacent to a vertex v_3 (which is not equal to v_1, since it is adjacent to two other vertices and Ω is a tree). By induction, it is clear there is a path of length n that touches all $n+1$ vertices once. But v_{n+1} is also adjacent to two vertices, so it must be adjacent to a vertex that already appears in the path, which implies the existence of a cycle. Therefore there is at least one vertex (call it a terminal vertex) that is only adjacent to one other vertex. By removing a terminal vertex and the edge that connects it to Ω we obtain a tree with n vertices, for which $a = s-1$. Since for Ω we increase a and s by one each, the claim holds for $n+1$ vertices. Therefore, if Γ is a tree, $a = s-1$.

Consider now any graph Γ. The claim is trivial for an empty graph, so assume Γ is not empty. Then by Proposition 3.1 Γ has a maximal subtree Γ', which by Proposition 3.2 contains all the vertices of Γ. So $s(\Gamma) = s(\Gamma')$, and $a(\Gamma) = a(\Gamma')$ only if $\Gamma = \Gamma'$. We have already shown that if Γ is a tree, $a = s-1$, so the second half of the proposition is complete.

Since Γ' is a tree, $a(\Gamma') = s(\Gamma') - 1 = s(\Gamma) - 1$, which implies $a(\Gamma) = s(\Gamma) - 1 + a(\Gamma) - a(\Gamma')$. And since $a(\Gamma) \geq a(\Gamma'), a(\Gamma) - a(\Gamma') \geq 0$, we have $a(\Gamma) \geq s(\Gamma) - 1$. □

We can define the contraction of a graph that takes each tree in a graph and sucks it into a single point, giving a new much simpler graph that can reveal properties of the original graph.

Definition 3.4. Given a connected graph Γ, take Λ to be a subgraph that is the union of disjoint trees $\Lambda_i (i \in I)$. Define Γ/Λ, a contraction of Γ, in the following way. The set of vertices of Γ/Λ is the quotient of the set of vertices of Γ under the equivalence relation $\sim$ so that $\forall x, x' \in \Lambda_i, x \sim x'$. The set $\text{edge}(\Gamma/\Lambda) = \text{edge}(\Gamma) \setminus \text{edge}(\Lambda)$. Two vertices are adjacent in Γ/Λ if they have representatives which are adjacent in Γ but not in Λ_i for any $i \in I$.

Proposition 3.5. *Γ is a tree $\iff \Gamma/\Lambda$ is one.*

Proof. Assume Γ is not a tree. Recall that $a(\Gamma) = |\text{edge}(\Gamma)|$ and $s(\Gamma) = |\text{vert}(\Gamma)|$. Then by Proposition 3.3 and the above arguments about contractions, the following holds:

$$a(\Gamma) > s(\Gamma) - 1$$

But $a(\Gamma) = a(\Gamma/\Lambda) + a(\Lambda)$ and $s(\Gamma) = s(\Gamma/\Lambda) + \sum_{i\in I}(s(\Lambda_i) - 1)$ and for each tree Λ_i, $a(\Lambda_i) = s(\Lambda_i) - 1$. So, the following holds:

$$a(\Gamma/\Lambda) + \sum_{i\in I} a(\Lambda_i) > s(\Gamma/\Lambda) + \sum_{i\in I}(s(\Lambda_i) - 1) - 1$$

$$a(\Gamma/\Lambda) > s(\Gamma/\Lambda) - 1$$

Which implies Γ/Λ is not a tree.

A similar logic can be applied to show that Γ/Λ is a tree if Γ is a tree. □

4. Trees of Representatives

If you have a graph that is acted on by a group, one way of understanding the group action is to take a tree of representatives. This section will explain this process, and an interesting result. First, if we have a group G which acts on a graph X, we will define the quotient graph $X' = X/G$ in the following way. The vertices of X' are the equivalence classes of vertices of X under the equivalence relation that for $x, x' \in vert(X), x \sim x'$ if $\exists\ g \in G$ such that $gx = x'$. Then the set $edgeX'$ is defined similarly. Furthermore, the equivalence class of (y, y') connects equivalence classes of y and y' by an edge. This gives us the following proposition.

Proposition 4.1. *Let X be a connected graph acted on by a group G. Then for every subtree T' of X/G $\exists$ a subtree of X that maps isomorphically to T' under the canonical projection map f.*

Proof. We assume for a contradiction that there is no subtree of X with the desired quality. Take Ω to be the set of subtrees of X that project injectively into T' under f. Then Ω has a partially ordering via the inclusion relation. Take a chain of subtrees $\{T_i\}_{i\in I}$ in Ω. As in the proof of Proposition 3.1, $\bigcup_{i\in I} T_i$ is a tree. All the vertices and edges of $\bigcup_{i\in I} T_i$ map under f to T', because they are all in T_i for some $i \in I$. If the map is not injective, then for some $i \in I$, T_i does not map injectively into T'. This implies each chain is bounded by its union. By Zorn's lemma, Ω has a maximal element T_x. Let T'_x be its image in T'.

If no subtree of X maps isomorphically to T' under f, then $T'_x \neq T'$. T_x projects injectively into T', so $|\text{vert}(T'_x)| \leq |\text{vert}(T')|$. Therefore, because both are connected, $\exists$ (y_1', y_2'), an edge in T' that is not an edge in T_x'. Because T' is connected, we can assume without loss of generality that y_1' is a vertex of T_x'. If y_2' is also a vertex of T_x', then because of connectedness, there is a path P_n from y_1' to y_2' in T_x' that does not include the edge (y_1', y_2'). But then P_n is also a path in T', which means if (y_1', y_2') is added to the beginning of P_n, we have a circuit in T', which is a contradiction. Therefore y_2' is not a vertex of T_x'. Because (y_1', y_2') is an edge of T', there is an edge (y_1, y_2) of X that maps to it in X/G. For any $g \in G$, (gy_1, gy_2) also maps to (y_1', y_2'). So, without loss of generality, we can assume y_1 is a vertex of T_x. Then y_2 cannot be a vertex of T_x, or y_2' would be a vertex of T_x'. Adjoin the vertex y_1 and the edge (y_1, y_2) to T_x to create T^*, which maps injectively into T'. We have created a subtree of X that maps injectively into T', but is strictly larger than T_x; this contradicts maximality of T_x. □

Definition 4.2. Suppose a group G acts on a graph X. Take T' to be a maximal tree in X/G. A tree of representatives of X/G is any subtree T of X that maps isomorphically to T' under the canonical projection map.

5. The Graph of a Free Group

Proposition 5.1. *Let G be a group, $S \subseteq G$, and $X = \Gamma(G, S)$ be the graph defined by G and S. Then the following are equivalent:*
a) X is a tree.
b) G is a free group with basis S.

Proof. First we will prove that if G is a free group with basis S then X is a tree.

If G is a free group, every $g \in G$ can be written uniquely in reduced form

$$g = s_1^{\epsilon_1} s_2^{\epsilon_2} \cdots s_n^{\epsilon_n} \;\; s_i \in S, \epsilon \in \{\pm 1\}$$

where for $1 \le i < n$ if $\epsilon_i = -\epsilon_{i+1}$, $s_i \neq s_{i+1}$. Trivially, assume $s_1 = e$ for e the identity in G. Define the length of the element g to be n, and then let X_n be the set of elements of G of length n. If $g \in X_n$, then g is adjacent to $g' = s_1^{\epsilon_1} s_2^{\epsilon_2} \cdots s_{n-1}^{\epsilon_{n-1}}$, which is a unique element of X_{n-1}. Because G is a free group, g' is the only element of X_{n-1} that g is adjacent to. Define functions

$$f_n : X_n \to X_{n-1}$$

so that $g \mapsto g'$ where $g' s_n^{\epsilon_n} = g$. $X_1 = S \cap S^{-1}$. $X_0 = \{e\}$ for e, the identity in G. Therefore the set $\{f_n\}$ is an inverse system, which by Proposition 2.10 implies $\Gamma(G, S)$ is a tree.

Now we will prove that if X is a tree, then G is a free group with basis S.

We know by Proposition 2.7 that because X is connected, S generates G. Consider $F(S)$, the free group generated by S. Assume G is not free. Every element of $F(S)$ corresponds canonically to an element of G since S generates both. Since G is not free, there is at least one element $g \in G$, $g = s_1^{\epsilon_1} s_2^{\epsilon_2} \cdots s_n^{\epsilon_n}$, such that $g = e$, the identity in G. Let g^* be one of these elements with minimal value of n. Then consider the set of elements of G, $H = \{e, s_1^{\epsilon_1}, s_1^{\epsilon_1} s_2^{\epsilon_2}, \ldots, s_1^{\epsilon_1} s_2^{\epsilon_2} \cdots s_{n-1}^{\epsilon_{n-1}}\}$. Assume $s_1^{\epsilon_1} s_2^{\epsilon_2} \cdots s_i^{\epsilon_i} = s_1^{\epsilon_1} s_2^{\epsilon_2} \cdots s_j^{\epsilon_j}$ for some $0 \leq i < j \leq n-1$. Then $s_i^{\epsilon_i} = s_i^{\epsilon_i} \cdots s_j^{\epsilon_j}$, but this implies $e = s_{i+1}^{\epsilon_{i+1}} \cdots s_j^{\epsilon_j}$. We know that $j - i < n$, but this contradicts the minimality of g^*. Therefore, all the elements of H are distinct except e and g. This implies that H projects injectively into X. Additionally, the vertices corresponding to $s_1^{\epsilon_1} s_2^{\epsilon_2} \cdots s_i^{\epsilon_i}$ and $s_1^{\epsilon_1} s_2^{\epsilon_2} \cdots s_{i+1}^{\epsilon_{i+1}}$ are adjacent, and the vertex corresponding to $s_1^{\epsilon_1} s_2^{\epsilon_2} \cdots s_{n-1}^{\epsilon_{n-1}}$ is adjacent to the vertex corresponding to e. But then H projects injectively into a cycle on X, which contradicts the fact that X is a tree. Therefore there are no such elements of G, and G is a free group. □

6. Proof of Theorem

We can now prove our theorem. As stated above, the theorem is:

Theorem 6.1. *A group is free $\iff$ there exists a tree on which it acts freely.*

Proof. Let G be a free group with basis S. Then by Proposition 5.1 G acts freely on the tree $\Gamma(G, S)$.

Proving the other direction of the theorem requires a little more work. Let G be a group that acts freely on a tree X. We will choose from G a particular set S based on its tree of representatives. We will take T to be a specific tree of representatives of X/G, and will prove that each subtree gT of X can be contracted to a single point simultaneously. Then we will prove S is a basis by creating an isomorphism between $\Gamma(G, S)$ and the contraction of X. Since any such contraction of a tree is a tree, this will prove that $\Gamma(G, S)$ is a tree. We can then invoke our previous work.

Choose an orientation for X by assigning each edge pair an origin and terminus so the orientation is preserved by the group action, in that if $g \in G$, and (y_1, y_2) is an oriented edge, then (gy_1, gy_2) is an ordered pair in the edge set of X. Let T be a tree of representatives of $X^* = X/G$ with the orientation induced by the orientation on X.

Consider the set of edges of X^* with origin in T, but terminus not in T. Then the terminus is in gT for some $g \in G$. Define S to be the set of such elements $g \in G, g \neq 1$. First we will prove that S is a basis for G by showing $\Gamma(G,S)$ is isomorphic to a tree, and then invoking Proposition 5.1.

Consider the map $g \mapsto gT$ that maps G onto the set of translates of T. Suppose $\exists\ p \in gT \cap g'T$, for $g, g' \in G$. Then $p = gt = g't'$ for $t, t' \in T$. So $t = g^{-1}g't'$. Since T is a lift of a tree of representatives, this implies $t = t'$. Since G acts freely, $e = g^{-1}g'$, so $g = g'$. Therefore $\forall\ g, g' \in G$ and $g \neq g'$, $gT \neq g'T$. This shows that the map $g \mapsto gT$ is a bijection. Because $g \mapsto gT$ is a bijection, it is possible to form the graph X' by contracting each tree gT to a single vertex (gT). X is a tree, so by Proposition 3.5 X' is a tree.

Now we will show that $\Gamma(G,S)$ is isomorphic to X', which will show that $\Gamma(G,S)$ is tree. The inverse of the bijection $g \mapsto gT$ is a bijection $f : vertX' \to G$. G is also the vertex set of $\Gamma(G,S)$. So $f : vertX' \to vert\Gamma(G,S)$ is also a bijection. We can give X' the orientation induced by X. Then the bijection can be extended to an isomorphism $f' : X' \to \Gamma(G,S)$. We need to define the isomorphism on the edges of X'. Since the trees gT are contracted to points, the only edges in X' are the ones in X that are not in gT for any $g \in G$. Take (y_1, y_2) to be an oriented edge in X'. Then $y_1 = (gT)$ and $y_2 = (g'T)$ for some $g, g' \in G$. Then let $s = g^{-1}g'$. There is an edge with origin in T and terminus in $g^{-1}g'T$, namely $(g^{-1}y_1, g^{-1}g'y_2)$, so we know that $s \in S$. Define $f'((y_1, y_2)) = (g, gs)$. As shown earlier, $f : vertX' \to vert\Gamma(G,S)$ is injective, so f' is injective. Let (g, gs) be an edge of $\Gamma(G,S)$, for $g \in G$ and $s \in S$. By the definition of S, there are vertices of T and sT that are adjacent. Therefore there are vertices in gT and gsT that are adjacent, which implies (gT) is adjacent to (gsT) in X'. Thus there is an edge of X' that f' maps to (g, gs), which implies surjectivity. So f' is bijective, and therefore is an isomorphism.

We now have an isomorphism between X' and $\Gamma(G,S)$, and we have shown X' is a tree. Therefore $\Gamma(G,S)$ is a tree. By Proposition 5.1, G is a free group. □

Here is a useful corollary.

Corollary 6.2. *If X/G has a finite number of vertices s, and if $|edgeX/G| = a$, then $|S| - 1 = a - s$.*

Proof. Let T be a tree of representatives of X/G. Let E be the set of oriented edges (y_1, y_2) in X such that y_1 is a vertex of T and y_2 is not. By the proof of Theorem 6.1, there is a bijection between

E and S, which implies $|E| = |S|$. Let T^* be the image of T in X/G. Define E^* similarly. Let (x_1, x_2) and (y_1, y_2) be distinct edges in E. Then x_1 and y_1 are distinct vertices of T, and since T maps isomorphically to T^*, there is no element of G that maps one to the other. Therefore, in the quotient graph X/G, (x_1, x_2) and (y_1, y_2) must map to two distinct edges. This implies $|E| = |E^*|$. By definition, T^* is a maximal tree, and $|vertX/G| = |vertT^*|$. Trivially, $edgeX/G$ is the disjoint union of $edgeX/G \cap edgeT^*$ and E^*. Also, by Proposition 3.3, $|edgeT^*| - |vertT^*| = -1$. If $|vertX/G|$ is finite, then we have the following:

$$\begin{aligned} |edgeX/G| - |vertX/G| &= (|edgeT^*| + |E^*|) - |edgeT^*| \\ &= |E^*| - 1 \\ &= |S| - 1 \end{aligned}$$

□

7. Applications of the Theorem

First we can prove a theorem that was originally published in a paper by Otto Schreier, who would probably have made many other significant contributions to combinatorial group theory, had he not died in 1929 at the age of 28.

Theorem 7.1. *Every subgroup of a free group is free.*

Proof. Let G be a free group with basis S, and let e be the identity for G. Then G acts freely on $X = \Gamma(G, S)$, which by Proposition 5.1 is a tree. Let H be a subgroup of G, then H acts freely on X. By Theorem 6.1, H is a free group. □

We will look at another of Schreier's results, the Schreier Index Formula. First we need two definitions:

Definition 7.2. The rank r_G of a free group G is the cardinality of its basis.

Remark 7.3. To make this well defined, it is necessary to show that all bases of G have the same cardinality. This is indeed true, and is not trivial to prove. I will not prove this here.

Definition 7.4. For a group G the index of a subgroup H, $[G : H]$, is the number of left cosets of H in G, where for $g \in G$, a left coset of H is $gH = \{gh | h \in H\}$.

Now we have:

Theorem 7.5 (Schreier Index Formula)**.** *Let G be a free group, and H be a subgroup of finite index n in G.*

$$r_H - 1 = n(r_G - 1).$$

Proof. G acts freely on $\Gamma(G, S)$ by Theorem 6.1. Take $\Gamma_G = \Gamma/G$ and $\Gamma_H = \Gamma/H$. Then let $s_G = |vert\Gamma_G|$ and $a_G = |edge\Gamma_G|$. Define s_H and a_H similarly. Because G acts freely on $\Gamma(G, S)$, for $x \in G$, $G \cong Gx$ as G-sets. Therefore there are the same number of H-orbits in Gx as there are in G, which implies $s_H = ns_G$ and $a_H = na_G$. Then by Corollary 6.2, the following is true:

$$r_H - 1 = a_H - s_H \text{ and } r_G - 1 = a_G - s_G$$

And since

$$r_H - 1 = a_H - s_H = na_G - ns_G = n(a_G - s_G) = n(r_G - 1)$$
$$r_H - 1 = n(r_G - 1)$$

□

References

[1] Serre, Jean-Pierre. *Trees*. Springer-Verlag, 1980.