TEICHMÜLLER SPACE

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Abstract. It is a well-known fact that every Riemann surface with negative Euler characteristic admits a hyperbolic metric. But this metric is by no means unique – indeed, there are uncountably many such metrics. In this paper, we study the space of all such hyperbolic structures on a Riemann surface, called the Teichmüller space of the surface. We will show that it is a complete metric space, and that it is homeomorphic to Euclidean space.

Contents

1. Introduction

Much of the theory of Riemann surfaces boils down to the following theorem, the two-dimensional equivalent of Thurston's geometrization conjecture:

Proposition 1.1 (Uniformization theorem)**.** *Every simply connected Riemann surface is isomorphic either to the complex plane* $\mathbb{C}$*, the Riemann sphere* $\mathbb{P}^1$*, or the unit disk,* $\boldsymbol{D}$*.*

By considering universal covers of Riemann surfaces, we can see that every surface admits a spherical, Euclidean, or (this is the case for all but a few surfaces) hyperbolic metric. Since almost all surfaces are hyperbolic, we will restrict our attention in the following material to them. The natural question to ask next is whether this metric is unique. We see almost immediately that the answer is no (almost any change of the fundamental region of a surface will give rise to a new metric), but this answer gives rise to a new question. Does the set of such hyperbolic structures of a given surface have any structure itself? It turns out that the answer is yes – in fact, in some ways, the structure of this set (called the Teichmüller space of the surface) is more interesting than that of the Riemann surface itself.

The culmination of our paper describes the Fenchel-Nielsen coordinatization of Teichmüller space, which give a very nice description of the space for certain Riemann surfaces:

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Definition 1.2 (Riemann surfaces of finite type)**.** A Riemann surface is of finite type if it is isomorphic to a compact Riemann surface with a finite (possibly zero) number of points removed.

The Fenchel-Nielsen coordinatization shows that for these Riemann surfaces of finite type, the corresponding Teichmüller space is homeomorphic to Euclidean space! However, before we can get to such topologically interesting results, we need to go through some analytic background.

Remark 1.3. There is a simpler way of defining Teichmüller space, but this way has the advantage of being generalizable to more types of Riemann surfaces, where Teichmüller spaces turn out to be infinite dimensional.

2. Quasiconformal maps

Conformal structures provide a valuable tool for understanding surfaces. However, sometimes the requirement that infinitesimal circles are mapped to infinitesimal circles can be too strong. For this reason, we define quasiconformal maps. The derivatives of quasiconformal maps map infinitesimal circles to infinitesimal ellipses of bounded eccentricity, though this is not *a priori* obvious from either of the definitions below. By measuring how eccentric these ellipses are, we can see how far from conformal a map is, which will give us a way to measure how "different" two surfaces are.

Definition 2.1 (Quasiconformal maps)**.** For any real number $K \geq 1$, we say that a homeomorphism f is K-quasiconformal if its partial derivatives (considered as distributions) are in L^2_{loc} and it satisfies the following equation almost everywhere:

$$\operatorname{Jac} f \geq \frac{1}{K} \parallel Df \parallel^2 . \tag{2.2}$$

If for some K, f is K-quasiconformal, then we say that f is quasiconformal.

Definition 2.3 (Quasiconformal maps (alternate definition))**.** Given $K \geq 1$, let $k \equiv (K-1)/(K+1)$. f is K-quasiconformal if it is a homeomorphism with distributional partial derivatives in L^2_{loc} which satisfy the following almost everywhere:

$$\left|\frac{\partial f}{\partial \bar{z}}\right| \leq k \left|\frac{\partial f}{\partial z}\right| \tag{2.4}$$

Note that a 1-quasiconformal map is conformal by this definition.

Proposition 2.5. *These two definitions are equivalent.*

Definition 2.6 (Quasiconformal constant)**.** f quasiconformal, the quasiconformal constant of f (denoted $K(f)$), is the infimum of all K such that f is K-quasiconformal.

Proposition 2.7 (Quasiconformal maps form a group)**.** . *If f is a K_1-quasiconformal map and g is a K_2-quasiconformal map, then f^{-1} is a K_1-quasiconformal map and $f \circ g$ is a (K_1K_2)-quasiconformal map.*

Note that we can topologize the set of quasiconformal functions by the topology of uniform convergence on compact sets. An important result about quasiconformal maps is that given this topology, the set of K-quasiconformal homeomorphisms homotopic to a given K-quasiconformal homeomorphism is in fact compact.

Proposition 2.8. *Let g be a quasiconformal homeomorphism between hyperbolic Riemann surfaces of finite type X and Y. Then $\forall K \geq 1$, the set of K-quasiconformal homeomorphisms $f : X \to Y$ homotopic to g is compact.*

Recall that if we have a conformal map between two cylinders of circumference one, then the two cylinders must have the same height. The following theorem generalizes this fact to quasiconformal maps.

Proposition 2.9 (Grötzsch's theorem)**.** *Let A_m and $A_{m'}$ be cylinders of circumference 1 and heights m and m' respectively. Let $f : A_m \to A_{m'}$ be K-quasiconformal. Then the following inequality holds:*

$$\frac{1}{K} \leq \frac{m}{m'} \leq K. \tag{2.10}$$

3. Beltrami Forms

Closely related to quasiconformal maps are Beltrami forms. They can be thought of as representing the infinitesimal ellipses in the tangent space of a Riemann surface which the derivative of the quasiconformal function acts on. They will prove useful in constructing "limits" of sequences of Riemann surfaces.

Definition 3.1 (Beltrami forms)**.** Given a Riemann surface, X, we denote by $L^\infty_*(TX, TX)$ the set of essentially bounded $\mathbb{C}$-antilinear bundle maps with the corresponding norm. The elements of this space with norm less than 1 are called Beltrami forms.

Proposition 3.2. *There is a unique pullback of Beltrami forms by $\mathbb{R}$-linear maps.*

The measurable Riemann mapping theorem ties together Beltrami forms and quasiconformal maps.

Theorem 3.3 (Measurable Riemann mapping theorem)**.** *Let $U \subset \mathbb{C}$ open, $\mu \in L^\infty(U)$ with $\| \mu \|_\infty < 1$. Then $\exists f : U \to \mathbb{C}$ quasiconformal which satisfies the Beltrami equation,*

$$\frac{\partial f}{\partial \bar{z}} = \mu \frac{\partial f}{\partial z}. \tag{3.4}$$

Proof. First, we consider the case μ real analytic. $\forall z \in U$, there is a neighborhood in which μ is an analytic function of two variables. We get

$$\frac{\partial f}{\partial x} = -i \frac{1+\mu}{1-\mu} \frac{\partial f}{\partial y} \tag{3.5}$$

directly from the Beltrami equation by simple algebraic manipulations. But we can then find the solution, because any function which is constant on the solutions to

$$\frac{dy}{dx} = -i \frac{1+\mu}{1-\mu} \tag{3.6}$$

will satisfy the Beltrami equation.

We can convolve arbitrary μ with a sequence of standard mollifiers (i.e. φ a non-negative real analytic function with integral 1, and φ_ϵ defined to be $\epsilon\varphi(\frac{x}{\epsilon})$ with ϵ

tending to 0). Note that the new sequence will consist of real analytic functions converging pointwise almost everywhere to μ. This sequence of real analytic functions will therefore converge to μ in the L^1 norm and will also be bounded in the L^∞ norm. Each such μ_ϵ defines a new Riemann surface on $\mathbb{C}$ by the solutions in each neighborhood of the differential equation for μ_ϵ. $\boldsymbol{D}$ in this Riemann surface is simply connected, relatively compact, and contained in a non-compact, simply connected surface, so it must by the uniformization theorem be isomorphic to $\boldsymbol{D}$. This gives us a sequence of K-quasiconformal maps by selecting such isomorphisms, but by 2.8 above, we can select a convergent subsequence, f_n. The derivatives must converge weakly in L^2 to the derivatives of the solution, f, of the Beltrami equation, so by a theorem of functional analysis, the f_n converge to the solution of the differential equation. □

The relationship between Beltrami forms and the measurable Riemann mapping theorem seems unclear until we recognize that the Beltrami equation can be rewritten as follows:

$$\bar{\partial} f = \partial f \circ \mu. \tag{3.7}$$

From the linearity and antilinearity of ∂ and $\bar{\partial}$ respectively, we see that for μ a Beltrami form, the conditions of the measurable Riemann mapping theorem are satisfied. Conversely, f a quasiconformal map, $\bar{\partial} f / \partial f$ will be a Beltrami form. There is thus a correspondence between quasiconformal maps and Beltrami forms.

We now need to know how to use Beltrami forms in the construction of Riemann surfaces. Given a base Riemann surface and a Beltrami form, we can construct a new Riemann surface as follows.

Definition 3.8. Given a Riemann surface X and a Beltrami form μ, we can construct a new Riemann surface, X_μ, as follows:

Given a collection of open covers of X, U_i, and corresponding atlas $\varphi_i : U_i \to V_i$, let μ_i be the Beltrami form such that $\mu|_{U_i}$ is the pullback by φ_i of $\mu_i \frac{d\bar{z}}{dz}$. We will now construct a new Riemann surface by giving a collection of open sets and an atlas. The open sets are the V_i, and the maps in the atlas are the quasiconformal maps which the measurable Riemann mapping theorem give us from the μ_i.

Proposition 3.9. *With X and μ defined as above, X_μ exists, is unique, and is a Riemann surface.*

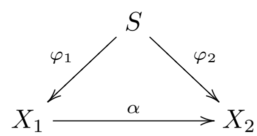
4. Teichmüller Space

We will study hyperbolic structures of a surface by looking at equivalence classes of pairs consisting of a Riemann surface together with a distinguished homeomorphism to this surface from S. The surface represents a hyperbolic structure, and the homeomorphism how we get from our surface to the hyperbolic structure. We can therefore think of the homeomorphism as a *marking* of the surface with respect to the hyperbolic structure. The equivalence relation tells us when we should consider two hyperbolic structures to be "the same."

Definition 4.1. A quasiconformal surface is a Riemann surface modulo the relation that two surfaces are equivalent if there is a quasiconformal map between them.

Definition 4.2 (Teichmüller Spaces)**.** Given a quasiconformal hyperbolic surface of finite type, S, the Teichmüller space modeled on S, denoted $\mathscr{T}_S$, is the set of equivalence classes of pairs (X, φ), where X is a Riemann surface, and $\varphi : S \to X$ is a quasiconformal homeomorphism, with two such pairs deemed to be equivalent if there exists an analytic isomorphism α such that the following diagram commutes up to homotopy:

(4.3)



We want to metrize Teichmüller space by finding out how non-conformal a map must be in order to go from one point to another. This is where the importance of quasiconformal maps becomes evident.

Definition 4.4. The Teichmüller metric, $d : \mathscr{T}_S \times \mathscr{T}_S \to \mathbb{R}^+$, is defined by

$$((X_1, \varphi_1), (X_2, \varphi_2)) \mapsto \inf_f \ln K(f), \tag{4.5}$$

where the infimum is taken over all quasiconformal homeomorphisms homotopic to $\varphi_2^{-1} \circ \varphi_1$.

Theorem 4.6. *The Teichmüller metric makes Teichmüller space into a complete metric space.*

Proof. Symmetry and positivity are immediate. The triangle inequality follows from 2.7 about the composition of quasiconformal maps. What remains are to demonstrate the identity of indiscernibles and completeness.

For the first part, suppose we have $\tau_1 = (X_1, \varphi_1)$ and $\tau_2 = (X_2, \varphi_2)$ such that $d(\tau_1, \tau_2) = 0$. Then we have a sequence K_i approaching 1 and a sequence of K_i-quasiconformal homeomorphisms from X_1 to X_2 homotopic to $\varphi_2 \circ \varphi_1^{-1}$. By 2.8, we can extract a subsequence which converges to an analytic mapping from X_1 to X_2 which is still homotopic to $\varphi_2 \circ \varphi_1^{-1}$, but by definition this means that τ_1 and τ_2 are equivalent.

Given a Cauchy sequence τ_j, we can look at a subsequence chosen such that the distance between the jth and $j+1$th terms is less than 2^{-j}.

Given ν_i such that

$$\ln \frac{1+ \parallel \nu_i \parallel}{1- \parallel \nu_i \parallel} \leq \frac{1}{2^i}, \tag{4.7}$$

we can find $f_i : X_i \to X_{i+1}$ which satisfy the Beltrami equation for ν_i. Now take $g_i : X_1 \to X_i$ to be the composition of the f_j for j going from 1 to $i-1$, and μ_i to be the corresponding Beltrami form. Note that τ_i is equivalent to $((X_1)_(\mu_i), id \circ \phi_i)$, so we need only show that the μ_i converge. We therefore have the following:

$$d(\mu_i, \mu_{i+1}) = d(g_i^*(0), g_{i+1}^*(\nu_i) = d(0, \nu_i) \leq \parallel \nu_i \parallel_\infty \leq \frac{1}{2^i}. \tag{4.8}$$

□

Interestingly enough, there is always a quasiconformal map which minimizes the quasiconformal constant.

Note that for S a Riemann surface of finite type, the mapping class group of S (denoted $\mathrm{Mod}(S)$) acts on $\mathscr{T}_S$ by homeomorphisms of S. This action will obviously preserve the Teichmüller metric. It turns out that for most important Teichmüller spaces, the mapping class group exhausts the isometries of the corresponding Teichmüller space. In addition, this action is well-behaved, so we can form a quotient space which turns out to be important too.

Proposition 4.9 (Fricke's theorem)**.** *The action described above is properly discontinuous (i.e. given K compact, all but a finite number of elements of* $\mathrm{Mod}(S)$ *map K to a set disjoint with K.*

Definition 4.10 (Moduli space)**.** The moduli space of a Riemann surface S is defined to be $\mathscr{T}_S/\mathrm{Mod}(S)$.

Since the "scrambling" of S by $\mathrm{Mod}(S)$ makes the notion of a distinguished homeomorphism moot, we can think of the moduli space as the Teichmüller space without the markings, so a point is determined solely by the intrinsic properties of the hyperbolic structure, not how it lies on S.

5. Fenchel-Nielsen Coordinates

Fenchel-Nielsen coordinates are a way of parametrizing one way of decomposing Riemann surfaces of finite type (i.e. each point in the parameter space corresponds to a different decomposition of this same type). Every such decomposition turns out to correspond to a point in the Teichmüller space of the surface, so this parameter space ends up telling us a lot about the Teichmüller spaces in question.

Proposition 5.1. *Every hyperbolic metric on a pair of pants is uniquely determined up to isometry by the lengths it assigns to the three boundary curves. Conversely, given three positive real numbers, there is a hyperbolic metric on the pair of pants which has each of these numbers as the lengths of the three boundary curves.*

Remark 5.2. Any surface of finite type (with genus g and n boundary components) has a family of $3g-3+n$ non-intersecting homotopy classes of simple closed curves. This is the most than can be achieved. The construction of such a family is fairly simple, and maximality can be seen by an Euler characteristic argument.

Proposition 5.3. *Any surface of finite type can be decomposed into pairs of pants by cutting along the geodesic representative of each of the homotopy classes described above.*

Fenchel-Nielsen coordinates are a parametrization of these pairs of pants decompositions. The Fenchel-Nielsen parameter space is $(\mathbb{R}^+)^{3g-3+n} \times \mathbb{R}^{3g-3+n}$. The first coordinates represent the length of the geodesics. By the above proposition, this uniquely determines each of the pairs of pants, so we are now only concerned with how the pants are glued together. These are called the length coordinates.

Definition 5.4. Let S be a quasiconformal surface of finite type. Let $FN_L : \mathscr{T}_S \to (\mathbb{R}^+)^{3g-3+n}$ such that $\tau = (X, \varphi) \mapsto l(\varphi(\gamma))\ \forall \gamma \in \Gamma$, where Γ is a maximal family of curves as described above, and l is the length of the geodesic representative of a homotopy class of curves.

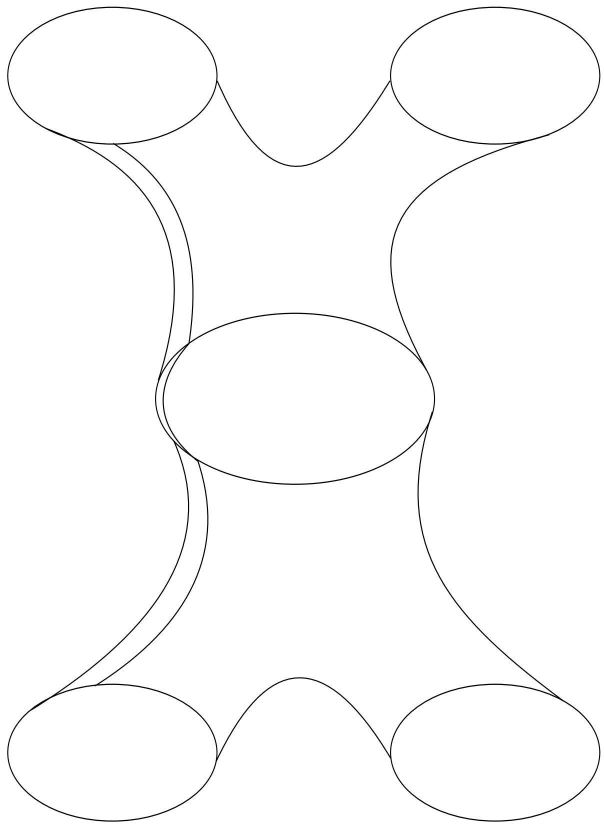


FIGURE 1. **We can see two curves from opposite boundary components, both perpendicular to the intervening boundary component, together with the curve along this component which connects them.**

Intuitively, the next parameter describes the relative "angle" of the pant legs at each curve. For this reason, they are called the twist coordinates. We now define for these twist coordinates the equivalent of FN_L.

Definition 5.5. Let S be a quasiconformal surface of finite type. Let $(X, \varphi) = \tau \in \mathscr{T}_S$. Given two boundary components, A and B, of adjacent pairs of pants in S on opposite sides of some boundary curve γ, we can consider the image of the homotopy classes of arcs directly connecting the two under φ. There is a unique representative of this homotopy class which consists of the minimal geodesics between each of A and B and the boundary on each side of a small annular neighborhood of $\varphi(\gamma)$ (denoted γ_A and γ_B respectively), along with a curve within this neighborhood (denoted γ_t). As this neighborhood is shrunk, γ_t will get arbitrarily close to a specific geodesic arc along γ. Let $t_\tau(\gamma)$ be the length of this geodesic. Then $FN_T : \mathscr{T}_S \to \mathbb{R}^{3g-3+n}$ is defined by $\tau \mapsto (t_\tau(\gamma))\ \forall \gamma \in \Gamma$.

Remark 5.6. Note that the sign of the twist coordinate is well-defined – no matter which way we traverse the arc from A to B, we turn in the same direction at $\varphi(\gamma)$.

We can combine these two functions to get a map from $\mathscr{T}_S$ to $(\mathbb{R}^+)^{3g-3+n} \times \mathbb{R}^{3g-3+n}$:

$$FN = (FN_L, FN_T) \tag{5.7}$$

Lemma 5.8. *FN_L is continuous.*

Proof. We will actually prove that given $\tau_1 = (X_1, \varphi_1)$ and $\tau_2 = (X_2, \varphi_2)$,

$$|\ln l(\varphi_1(\gamma)) - \ln l(\varphi_2(\gamma)) \leq d(\tau_1, \tau_2)| \tag{5.9}$$

holds.

By choosing a point s on γ, we see that γ generates a copy of $\mathbb{Z}$ in $\pi_1(S, s)$. The covering spaces of X_1 and X_2 determined by this subgroup will be annular, and it can be shown that their moduli will be

$$M_i = \frac{\pi}{l(\varphi_i(\gamma))} \tag{5.10}$$

Any continuous quasiconformal map $f : X_1 \to X_2$ with quasiconformal constant $K(f)$ which is homotopic to $\varphi_2^{-1} \circ \varphi_1$ (precisely those which determine the Teichmüller metric), lifts to a map $\tilde{f}_\gamma : \widetilde{(X_1)}_\gamma \to \widetilde{(X_2)}_\gamma$ with the same quasiconformal constant. Using Grötzsch's theorem, we get:

$$d(\tau_1, \tau_2) = \inf_{f \sim \varphi_2^{-1} \circ \varphi_1} \ln K(f) \tag{5.11}$$

$$= \inf_{f \sim \varphi_2^{-1} \circ \varphi_1} \ln K(\tilde{f}_\gamma) \tag{5.12}$$

$$\geq \left| \ln \frac{M_2}{M_1} \right| = \left| \ln l(\varphi_1(\gamma)) - \ln l(\varphi_2(\gamma)) \right|. \tag{5.13}$$

□

Lemma 5.14. *FN_T is continuous.*

Proof. If the distance between $\tau_1 = (X_1, \varphi_1), \tau_2 = (X_2, \varphi_2) \in \mathscr{T}_S$ is small, then we can write $X_2 = (X_1)_\mu$ and $\varphi_2 = id \circ \varphi_1$ with μ a Beltrami coefficient of small norm. We can lift μ to $\boldsymbol{D}$ and solve the corresponding Beltrami equation to find a quasiconformal map $f_{\widetilde{\mu}}$ which will "almost" be the identity. Let $X_1 = \boldsymbol{D}/\Gamma_1$. Let Γ_2 be the conjugation of Γ_1 by $f_{\widetilde{\mu}}$. Note that $(X_1)_\mu$ is isomorphic to $\boldsymbol{D}/\Gamma_2$. We can lift the three curves γ_A, γ_B, and γ_t which were used in the definition of FN_T to $\boldsymbol{D}$. Note that the endpoints are fixed points of Γ_1, so by the closeness of $f_{\tilde{\mu}}$ to the identity, they will be "almost" fixed by Γ_2. Therefore, the lengths of the curves will not be changed very much. □

Corollary 5.15. *FN is continuous.*

Theorem 5.16. *Let S be a quasiconformal surface of finite type; $\mathscr{T}_S$ is homeomorphic to $(\mathbb{R}^+)^{3g-3+n} \times \mathbb{R}^{3g-3+n}$.*

Proof. We know that *FN* is a continous map, so it suffices to construct its inverse and show that this inverse is continuous.

Given the lengths, we know by 5.1 that we can construct the corresponding pairs of pants. Suppose we want to glue two boundary components together along the curve γ. We can again pick geodesics on each of the pants (or two different geodesics on the same pair of pants, as the case may be), which both intersect the boundary curve corresponding to γ. Now consider t_γ/l_γ. This will have both an integral part and a fractional part. The fractional part corresponds to the distance between the endpoints of the two geodesics along γ after gluing the two boundary curves together.

We have now constructed the Riemann surface part of the point in Teichmüller space. The homeomorphism can be constructed as follows: in the interior of the pairs of pants, the homeomorphism simply corresponds to the map between the pair of pants in S and the hyperbolic pair of pants in X. In an annulus surrounding the boundary, we simply do a Dehn twist however many times the integral part of t_γ/l_γ tells us to. It can be verified that *FN* applied to this point in Teichmüller space gives us the identity, so we have constructed the inverse. □

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