Cohomology and Vector Bundles

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Abstract

Vector bundles are a generalization of the cross product of a topological space with a vector space. Characteristic classes assign to the each vector bundle a cohomology class of the base space. The Euler class, the Thom class and the Chern classes are a few characteristic classes of vector bundles.

The primary source for this paper is Bott and Tu's Differential Forms in Algebraic Topology [1].

1 Vector bundles

1.1 Definition and examples

Definition 1.1 (Vector bundle)**.** For a field $F = \mathbb{R}$ or $\mathbb{C}$ and a positive integer k, a rank k *F-vector bundle* ξ consists of a triple $(E,\ p,\ B)$ where E and B are topological spaces and $p : E \to B$ is a surjective continuous map. The triple must satisfy the condition that there be a *local trivialization* $\{\phi_\alpha, U_\alpha\}$ where $\{U_\alpha\}$ is an open cover of B and the ϕ_α are homeomorphisms satisfying the following commutative diagram:

$$\begin{array}{ccc} U_\alpha \times F^k & \xrightarrow{\phi_\alpha} & E|_{U_\alpha} \\ \downarrow{\scriptstyle \pi} & \swarrow{\scriptstyle p} & \\ B & & \end{array}$$

Here π is the natural projection onto the first factor and $E|_{U_\alpha} = p^{-1}(U_\alpha)$. Each fiber $p^{-1}(b)$ is a vector space and the restrictions $\phi_\alpha|_{b\times F^k} : b \times F^k \to p^{-1}(b)$ are vector space isomorphisms. The final property satisfied by $\{\phi_\alpha, U_\alpha\}$ is that the *transition functions* $g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL_k(F)$ given by $g_{\alpha\beta}(b) = \phi_\beta^{-1}\phi_\alpha|_{\{b\}\times F^k}$ be continuous.

E is called the *total space*, B the *base space* and F^k the *fiber space*. The preimage $p^{-1}(b) \subset E$ of a point $b \in B$ is called the *fiber* over b, and is often denoted E_b. The dimension k of the fiber space is called the *rank* of the vector bundle. Rank 1 vector bundles are commonly referred to as *line bundles*.

Definition 1.2 (Bundle morphism)**.** A *bundle morphism* between two vector bundles (E, p, B) and (E', p', B') is composed of a pair of functions (h, f) where f maps between the base spaces and h maps between the total spaces. We require that the restriction of h to any fiber be linear and that the fiber over $b \in B$ be mapped to the fiber over $f(b)$. In other words the following commutative diagram is satisfied:

$$\begin{CD} E @>h>> E' \\ @VpVV @VVp'V \\ B @>f>> B' \end{CD}$$

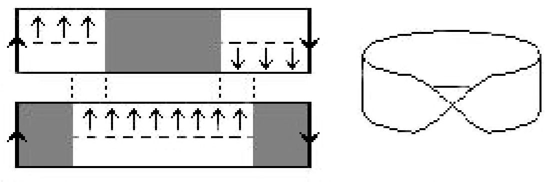
Example 1.3 (Trivial bundle)**.** For any topological space B and vector space V, the cross product $B \times V$ is a vector bundle call the trivial bundle. The projection map is simply the natural projection onto the first factor, and the local trivialization is given by the identity map. Thus the open cylinder is a trivial real line bundle over S^1.

Example 1.4 (Möbius band)**.** The Möbius band

$$[0, 1] \times \mathbb{R}/(0, t) \sim (1, -t)$$

is a nontrivial line bundle on S^1. Thinking of the circle as the unit interval with zero and one associated, the projection map comes from the natural projection onto the first factor. We take the open cover consisting of two overlapping semicircles. The trivializations amount to choosing an orientation of $\mathbb{R}$ for each of the semicircles (see figure 1). The resulting transition maps are both constant and therefore continuous.

Figure 1: Möbius band



Example 1.5 (Grassmannian)**.** One important vector bundle is the canonical bundle on the Grassmannian. The *Grassmannian*, $G_k(F^n)$, is the space of k-dimensional subspaces of F^n with the quotient topology defined by the map: $(v_1, \ldots, v_k) \mapsto \mathrm{Span}(v_1, \ldots, v_k)$. The infinite Grassmannian $G_k(F^\infty) = \bigcup_{n \geq k} G_k(F^n)$ is given the topology induced from the natural inclusions

$$G_k(F^n) \hookrightarrow G_k(F^{n+1}).$$

The *canonical vector bundle* γ_k^n on the Grassmannian $G_k(F^n)$ $(0 < n \leq \infty)$ is a subbundle of the trivial bundle $G_k(F^n) \times F^n$ composed of elements of the form (V, x) where $x \in V$. Thus the fiber over a subspace $V \in G_k(F^n)$ consists of the elements of V.

1.2 Transition functions

The transition functions of a vector bundle are essentially glueing maps that describe how the trivial neighborhoods stick together to make the total space.

Theorem 1.6. *Let B be a topological space and $\{U_\alpha\}$ an open cover of B. Given a set of functions $\{g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL_k(F)\}$ such that $g_{\alpha\beta} g_{\beta\gamma} = g_{\alpha\gamma}$ on the triple intersection $U_\alpha \cap U_\beta \cap U_\gamma$. There is a rank k F-vector bundle on B with transition functions $\{g_{\alpha\beta}\}$.*

Proof. Consider the disjoint union

$$\mathcal{E} = \bigcup_\alpha U_\alpha \times F^k$$

Use the functions $\{g_{\alpha\beta}\}$ to define the relation $(a, x)_\alpha \sim (b, y)_\beta$ if and only if $a = b$ and $y = g_{\alpha\beta}(a)x$. The cocycle condition $g_{\alpha\beta} g_{\beta\gamma} = g_{\alpha\gamma}$ implies that $g_{\beta\alpha} = (g_{\alpha\beta})^{-1}$. Therefore if $a = b$ and $y = g_{\alpha\beta}(a)x$, then $b = a$ and $x = (g_{\alpha\beta}(a))^{-1}y = g_{\beta\alpha}(b)y$. Thus $\sim$ is an equivalence relation. The quotient $\mathcal{E}/\sim$ is a vector bundle on B. Its trivialization functions result from the inclusion maps $U_\alpha \times F^k \hookrightarrow \mathcal{E}$ and its transition functions are exactly the $\{g_{\alpha\beta}\}$. □

Definition 1.7. Two sets of transition functions $\{g_{\alpha\beta}\}$ and $\{g'_{\alpha\beta}\}$ with respect to the same cover $\{U_\alpha\}$ are said to be equivalent if there exists a collection of continuous maps $\{\lambda_\alpha : U_\alpha \to GL_k(F)\}$ such that $g_{\alpha\beta} = \lambda_\alpha \cdot g'_{\alpha\beta} \cdot (\lambda_\beta)^{-1}$. Here $\cdot$ is multiplication in $GL_k(F)$.

Theorem 1.8. *Two rank k F-vector bundles ξ and η on a space B with local trivializations relative to some cover $\{U_\alpha\}$ are isomorphic if and only if their transition functions are equivalent.*

Remark 1.9. If the transitions functions of a vector bundle are equivalent to a set of transition functions living in some subgroup $H < GL_n(F)$, then we say that the structure group can be *reduced* to H.

Definition 1.10 (Orientable)**.** A rank k real vector bundle is call *orientable* if its transition functions are equivalent to a set of transition functions mapping to $GL_k^+(F)$, the linear isomorphisms with positive determinant.

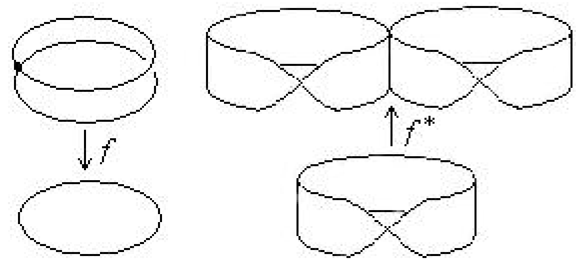
Proposition 1.11. *The structure group of a real vector bundle of rank k can be reduced to $SO(k)$ if and only if it is orientable.*

1.3 Induced bundles

Definition 1.12 (Bundle morphism)**.** A *bundle morphism* between two vector bundles (E, p, B) and (E', p', B') is composed of a pair of functions (h, f) where f maps between the base spaces and h maps between the total spaces. We require that the restriction of h to any fiber be linear and that the fiber over $b \in B$ be mapped to the fiber over $f(b)$. In other words the following commutative diagram is satisfied:

$$\begin{array}{ccc} E & \xrightarrow{h} & E' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ B & \xrightarrow{f} & B' \end{array}$$

Figure 2: Induced bundle



Definition 1.13 (Induced bundle)**.** Given a vector bundle $\xi = (E, p, B)$, a continuous map $f : A \to B$ induces a bundle $f^*(\xi)$ on A that has total space $\{(a, x) \in A \times E \mid f(a) = p(x)\}$ and projection map $(a, x) \mapsto a$. The fiber $f^*(\xi)_a$ is canonically identified with $p^{-1}(b)$.

Example 1.14 (Figure eight)**.** Consider the Möbius band μ as a real line bundle over the circle and the continuous map f from the figure eight to the circle obtained by viewing the figure eight as the wedge of two circles. The induced bundle $f^*(\mu)$ is then two Möbius bands glued together at a single fiber (see figure 2).

Example 1.15 (Projectivization)**.** The projectivization $P(V)$ of a vector space $V = F^n$ is simply the first Grassmannian $G_1(F^n)$. In example 1.5, two tautological bundles over this space were discussed, namely the trivial bundle $P(V) \times F^n$ and the canonical bundle γ_1^n. The canonical bundle on $P(V)$ is also know as the *universal subbundle* and is often denoted by S.

Given a vector bundle $\xi = (E, p, B)$ with fiber space V, we can take the projectivization $P(E)$ by taking the projectivization of each fiber. The resulting space is a fiber bundle over B with fiber space $P(V)$. A fiber bundle is similar to a vector bundle; the primary difference is that the fiber space need not be a vector space. The structure of this fiber bundle can be understood in terms of its transition functions $\{\bar{g}_{\alpha\beta}\}$ which are simply transition functions $\{g_{\alpha\beta}\}$ of ξ modulo scalar matrices. Thus the $\bar{g}_{\alpha\beta}$ map into the projective general linear group rather than the general linear group.

The tautological vector bundles on $P(E)$ are similar to those on the projectivization of a vector space. The analog of the trivial bundle is the induced bundle $\pi^*(\xi)$ where $\pi : P(E) \to B$ is the projection map derived from p. In this bundle, the fiber over a line l_b in E_b is exactly E_b.

When restricted to any fiber $\pi^{-1}(b) = P(E_b)$, this bundle is trivial i.e. $\pi^*(\xi)|_{P(E)_b} = P(E_b) \times E_b$. The analogue of the universal subbundle is the subbundle $S = \{(l_b, x) \in \pi^*(\xi) \mid x \in l_b\}$. The fiber in S over a line $l_b \in P(E)$ consists of the points in l_b, and the restriction of S to any fiber $\pi^{-1}(b) = P(E_b)$ is the universal subbundle of $P(E_b)$.

Proposition 1.16 (Functoriality)**.** *Given a vector bundle ξ over a space B and two continuous maps $X \xrightarrow{g} Y \xrightarrow{f} B$, the bundle induced by the composition fg is isomorphic to that induced from $g^*(\xi)$ by f. In other words, the following relation is satisfied: $g^*(f^*(\xi)) \cong (fg)^*(\xi)$.*

Theorem 1.17 (Classification of vector bundles)**.** *Every rank k, F-vector bundle on a paracompact space B is isomorphic to a bundle induced by a map from the base space to the Grassmannian $G_k(F^\infty)$.*

Proof. Let $\xi = (E, p, B)$ be a rank k, F-vector bundle on a paracompact space B. Also let $\{(\phi_\alpha, U_\alpha)\}$ be a local trivialization of ξ. As B is paracompact we can assume without loss of generality that $\{U_\alpha\}$ is locally finite and countable. Let $\{\varepsilon_\alpha\}$ be a partion of unity subordinate to $\{U_\alpha\}$.

Now consider the map $g : E \to \sum_\alpha F^k$ defined by:

$$g = \sum_\alpha g_\alpha \text{ such that } g_\alpha|_{p^{-1}(U_\alpha)} = (\varepsilon_\alpha p) \cdot (\pi_2 \phi_\alpha^{-1}) \text{ and } g_\alpha = 0 \text{ everywhere else.}$$

Here $\pi_2 : U_\alpha \times F^k \to F^k$ is the natural projection onto the second factor. The composition $\varepsilon_\alpha p$ assigns a scalar from the interval [0,1] to each point in $E|U_\alpha$. The function g_α then multiplies this scalar by the vector given by $\pi_2\phi_\alpha^{-1}$. The scalar given by the partition of unity ensures that the extension of g_α by zero is continuous. The sum of the g_α makes sense, because $\{U_\alpha\}$ is locally finite and countable. The continuity of g follows from that of the g_α.

The $g_\alpha : E \to F^k$ restrict to linear injections on each fiber $p^{-1}(b)$ where $\varepsilon_\alpha(b)$ is nonzero. This follows from the fact that both π_2 and ϕ_α^{-1} restrict to linear injections on fibers. This implies g restricts to linear injections on fibers as the g_α map to linearly independant subspaces of $\sum_\alpha F^k$.

Thus the map g associates each fiber of E to a k-dimensional subspace of F^∞ in a continuous way. This association can be used to define a map from B to $G_k(F^\infty)$ in the following way:

$$f : B \to G_k(F^\infty) \text{ such that } f(b) = g(p^{-1}(b)).$$

In order to see that f is continuous, consider p^{-1} as a function from B to the fibers of E. Locally $^{-1}$ maps U_α to the fibers of $E|U_\alpha$. The trivializing map ϕ_α is a fiber preserving homeomorphism between $E|U_\alpha$ and $U_\alpha \times F^k$. The projection $\pi_1 : U_\alpha \times F^k \to U_\alpha$ gives a homeomorphism between U_α and the space of fibers of $U_\alpha \times F^k$. Thus $p = \phi_\alpha^{-1}\pi_1$ can be viewed as a homeomorphism between the space of fibers of $E|U_\alpha$ and U_α. This implies that p^{-1} is a continuous map. Therefore $f = gp^{-1}$ must also be continuous.

All that remains to be shown is that $f^*(\gamma_k^\infty)$ is isomorphic to ξ.

From the definition of γ_k^∞ and the properties of induced bundles, we have that $f^*(B)$ is a subbundle of the trivial bundle $B \times F^\infty$. The map $(p.g) : E \to B \times F^\infty$ is clearly continuous. Furthermore its image is exactly $f^*(\gamma_k^\infty)$ and the restriction of g to any fiber E_b is a linear isomorphism to the fiber $f(b) = g(p^{-1}(b))$ in $f^*(\gamma_k^\infty)$. This implies that $f^*(\gamma_k^\infty)$ is isomorphic to ξ. □

Theorem 1.18. *Let B be a paracompact space. Two continuous maps $f, g : B \to G_k(F^\infty)$ induce isomorphic bundles on B if and only if they are homotopic.*

The proof of this theorem is too long to be included here, but can be found in Husemöller's Fibre Bundles [2].

By the previous two theorems, there is a correspondence between isomorphism classes of vector bundles on B and homotopy classes of maps from B to $G_k(F^\infty)$.

2 Cohomology

2.1 Differential forms

A more thorough explanation of this subject can be found in Munkres's Analysis on Manifolds [3].

This section focuses on real manifolds, but the analogous theorems and definitions hold for complex manifolds.

Definition 2.1 (Differential form)**.** A differential form ω on a smooth k-dimensional manifold M is a collection of forms ω_α on $\mathbb{R}^k$ corresponding to an atlas $\{(\phi_\alpha, U_\alpha)\}$ such that each ω_α is a form on $\phi_\alpha(U_\alpha)$. The ω_α's agree on the intersections $U_\alpha \cap U_\beta$ in the following sense. The maps induced from the inclusions $U_\alpha \cap U_\beta \overset{i}{\hookrightarrow} U_\alpha$ and $U_\alpha \cap U_\beta \overset{j}{\hookrightarrow} U_\beta$ take ω_α and ω_β to the same form i.e. $i^*\omega_\alpha = j^*\omega_\beta$

The wedge product of forms on M is derived from that of forms on $\mathbb{R}^k$ in the following way: $(\omega \wedge \nu)_\alpha = \omega_\alpha \wedge \nu_\alpha$. Similarly the differential operator d is obtained from that on forms on $\mathbb{R}^k$ by defining $(d\omega)_\alpha = d\omega_\alpha$. The familiar properties of the differential operator still hold.

Proposition 2.2. *Let ω and ν be differential forms on a manifold M. The following relations hold:*

$$d(d(\omega)) = 0$$
$$d(\omega + \nu) = d\omega + d\nu$$
$$d(\omega \wedge \nu) = d\omega \wedge d\nu$$

2.2 de Rham cohomology

Definition 2.3 (Closed and exact forms)**.** A differential form ω is called closed if $d\omega = 0$. It is called exact if there exists a form η such that $d\eta = \omega$.

Definition 2.4 (de Rham cohomology)**.** The q^{th} de Rham cohomology of a manifold $H^q(M)$ is obtained by taking the quotient of the closed q-forms on M by the exact q-forms.

$$H^q(M) = \{\text{closed } q\text{-forms}\}/\{\text{exact } q\text{-forms}\}$$

This construction results in the same cohomology as the standard singular cohomology with $\mathbb{R}$ (or $\mathbb{C}$) coefficients used in algebraic topology and thus also results in a contravariant functor. The key difference between these two contsructions is that the de Rham cohomology only works for smooth spaces and smooth maps. Here are a few important theorems regarding the basic properties of singular cohomology.

2.3 Properties of cohomology

Theorem 2.5. *Continuous maps $f : A \to B$ between spaces induce homomorphisms of the cohomologies $f^* : H^*(B) \to H^*(A)$. Furthermore the homomorphism induced by a composition fg is exactly the composition g^*f^*.*

Theorem 2.6. *Homotopic maps between spaces induce the same homomorphism on their cohomology groups.*

Definition 2.7 (Compact vertical support)**.** A vector bundle (E, p, M) over a manifold M has a collection of forms $\Omega^*_{cv}(E)$ that have compact support on each fiber of E i.e. the restriction of a form $\omega \in \Omega^*_{cv}(E)$ to a fiber $E_p = p^{-1}$ has compact support.

Definition 2.8 (Thom map)**.** Let (E, p, M) be an orientable rank k real vector bundle on a manifold M and $\{(\phi_\alpha, U_\alpha)\}$ be a local trivialization. The q-forms on E are locally of the form $p^*\omega_\alpha \wedge (\pi_2\phi_\alpha^{-1})^*\nu_\alpha$ where $\pi_2 : U_\alpha \times \mathbb{R}^k \to \mathbb{R}^k$ is the natural projection onto the second factor, the ω_α are m-forms on the U_α and the ν_α are n-forms on $\mathbb{R}^k$ such that $m + n = q$. Formally we write $\omega_\alpha \otimes \nu_\alpha$ rather than $p^*\omega_\alpha \wedge (\pi_2\phi_\alpha^{-1})^*\nu_\alpha$.

The *Thom map* takes forms on (E, p, M) to forms on M. More precisely it maps from $\Omega^*_{cv}(E)$ to $\Omega^*(M)$. The map is given locally by integration over fibers and is defined as follows:

$$\omega_\alpha \otimes \nu_\alpha \mapsto 0 \text{ if } \deg(\nu_\alpha) < k \text{ and } \omega_\alpha \otimes \nu_\alpha \mapsto \omega_\alpha \cdot \int_{\mathbb{R}^k} \nu_\alpha \text{ if } \deg(\nu_\alpha) = k$$

Here integration is done on each fiber giving a map from M to $\mathbb{R}$.

Theorem 2.9. *The local definition of the Thom map extends to a global definition.*

Proof. Let $\eta = \{\omega_\alpha \otimes \nu_\alpha\}$ be a global q-form on (E, p, M), and π_* denote the Thom map. If the ν_α have degree less than k, then $\pi_*(\eta) = 0$ as each of its components maps to zero. In the case where the ν_α have degree k, they can be written as $\nu_\alpha = f_\alpha(x, v)dv$ where $f_\alpha : U_\alpha \times \mathbb{R}^k \to \mathbb{R}$ is a smooth map. Restricting to $U_\alpha \cap U_\beta \times \mathbb{R}^k$ we have that

$$\begin{aligned}
\pi_*(\omega_\alpha \otimes \nu_\alpha) &= \omega_\alpha \cdot \textstyle\int_{\mathbb{R}^k} \nu_\alpha && \text{by definition}\\
&= \omega_\alpha \cdot \textstyle\int_{\mathbb{R}^k} f_\alpha(x, v)dv && \text{by definition}\\
&= \omega_\alpha \cdot \textstyle\int_{\mathbb{R}^k} f_\alpha(x, g_{\beta\alpha}(x)w)\det g_{\beta\alpha}(x)dw && \text{change of variables } \phi_\alpha^{-1}\phi_\beta\\
&= \omega_\alpha \cdot \textstyle\int_{\mathbb{R}^k} f_\alpha(x, g_{\beta\alpha}(x)w)dw && \text{because WLOG } \det g_{\beta\alpha}(x) = 1\\
& && \text{due to orientability}\\
&= \omega_\beta \cdot \textstyle\int_{\mathbb{R}^k} f_\beta(x, w)dw && \text{because } \eta \text{ is a global form}\\
&= \pi_*(\omega_\beta \otimes \nu_\beta) && \text{by definition}
\end{aligned}$$

Thus the components $\pi_*(\omega_\beta \otimes \nu_\beta)$ piece together into a global form $\pi_*(\eta)$. □

Theorem 2.10 (Thom isomorphism)**.** *The Thom map induces an isomorphism of cohomology rings:* $H^{*+n}_{cv}(E) \xrightarrow{\cong} H^*(M)$.

Definition 2.11 (Thom class)**.** Let (E, p, M) be a rank k real vector bundle on a manifold M. The Thom class of E is the cohomology class $\omega \in H^{*+n}_{cv}(E)$ that is mapped to $1 \in H^0(M) \cong \mathbb{R}$ by the Thom isomorphism.

Lemma 2.12 (Projection formula)**.** *Let (E, p, M) be a rank n real vector bundle over a manifold M. Also let τ be a form on M and ω be a form on E with compact vertical support. The following relation is satisfied by π_* the Thom map described in definition 2.8.*

$$\pi_*(p^*\tau \wedge \omega) = \tau \wedge \pi_*\omega$$

Proof. ω is locally $p^*\mu \wedge j^*\nu$ where μ is a form on M, ν a form on $\mathbb{R}^n$ and j is the composition of the trivialization map with the natural projection on to $\mathbb{R}^n$. There are two cases to consider: $\deg(\nu) < n$ and $\deg(\nu) = n$. In the first case, we have that

$$\begin{aligned}
\pi_*(p^*\tau \wedge \omega) &= \pi_*(p^*\tau \wedge p^*\mu \wedge j^*\nu) \\
&= \pi_*(p^*(\tau \wedge \mu) \wedge j^*\nu) && \text{because } p^* \text{ is a homomorphism} \\
&= 0 && \text{by the definition of the Thom map} \\
&= \tau \wedge \pi_*\omega && \text{by the definition of the Thom map}
\end{aligned}$$

In the second case, we have that

$$\begin{aligned}
\pi_*(p^*\tau \wedge \omega) &= \pi_*(p^*\tau \wedge p^*\mu \wedge j^*\nu) \\
&= \pi_*(p^*(\tau \wedge \mu) \wedge j^*\nu) && \text{because } p^* \text{ is a homomorphism} \\
&= (\tau \wedge \mu) \cdot \textstyle\int_{\mathbb{R}^n} \nu && \text{by the definition of the Thom map} \\
&= \tau \wedge (\mu \cdot \textstyle\int_{\mathbb{R}^n} \nu) && \text{property of differential forms} \\
&= \tau \wedge \pi_*\omega && \text{by the definition of the Thom map}
\end{aligned}$$

Thus the projection formula holds locally. Because the Thom map is derived from a local definition the formula also holds globally. □

Definition 2.13 (Poincaré dual)**.** Let M be an oriented m-dimensional manifold and $S \subseteq M$ a closed oriented submanifold of dimension k. The Poincaré dual of S in M is the unique cohomology class $\bar{\eta}_S \in H^{n-k}(M)$ such that

$$\int_S i^*\omega = \int_M \omega \wedge \eta_S \text{ for all closed } k\text{-forms } \omega \text{ with compact support.}$$

Here $i : S \hookrightarrow M$ is the inclusion map.

This definition only makes sense if such a form exists. The following theorem proves the existance of such a form and relates the the Poincaré dual of a submanifold to the Thom class of its normal bundle.

Theorem 2.14. *The Poincaré dual of a closed oriented submanifold S in an oriented manifold M and the Thom class of the normal bundle of S can be represented by the same form.*

Proof. Let M be an n dimensional oriented real manifold, $S \subseteq M$ be a k dimensional closed oriented submanifold and $i : S \hookrightarrow M$ be the inclusion map. Also let $T \subseteq M$ be a tubular neighborhood of S. There exists a map $p : T \to S$ such that (T, p, S) is isomorphic to the normal bundle on S. Let $\alpha : S \hookrightarrow T$ be the inclusion of S into T. Note that this is map is equivalent to taking the zero section of the normal bundle. Forms with compact vertical support on T (viewed as the normal bundle) can be extended by zero to forms on M as their support lies in the interior of T. Let $j_* : H^*_{cv}(T) \to H^*(M)$ be the map obtained by extending by zero. Finally let Φ be the Thom class of T and ω be a k-form on M with compact support.

$\int_M \omega \wedge j_*\Phi$	$=$	$int_T \bar{\omega} \wedge \overline{j_*\Phi}$	because the support of $j^*\Phi$ lies in T Here the bar denotes the restriction to T
	$=$	$\int_T (p^*\alpha^*\bar{\omega} + d\tau) \wedge \Phi$	because p^* and α^* are inverse isomorphisms in cohomology
	$=$	$\int_T (p^*\alpha^*\bar{\omega}) \wedge \Phi$	by Stokes theorem and the fact that Supp(Φ) lies in the interior of T
	$=$	$\int_S \alpha^*\bar{\omega} \wedge \pi_*\Phi$	by the projection formula lemma 2.12. Here π_* is the Thom map.
	$=$	$\int_S \alpha^*\bar{\omega}$	because $\pi_*\Phi = 1$ by definition
	$=$	$\int_S i^*\omega$	

Thus the Thom class of T satisfies the condition for the Poincaré dual. □

3 Characteristic Classes

For the entirety of this section let F be $\mathbb{R}$ or $\mathbb{C}$ and k be a positive integer. Also let B be paracompact space and $BG = G_k(F^\infty)$ be the Grassmann variety of k-dimensional subspaces of F^∞. Finally let γ be the canonical vector bundle on the Grassmannian BG. All vector bundles will be rank k F-vector bundles.

3.1 The definition

The classification of vector bundles provides a correspondence between isomorphism classes of vector bundles on B and homotopy classes of maps from B to BG. The cohomology functor associates a homomorphism of cohomology groups, $H^*(BG) \to H^*(B)$, to each homotopy class of maps from B to BG. Thus the classification of vector bundles in combination with cohomology theory can be used to assign to each vector bundle on B a subgroup of the cohomology of B, namely the image of the induced map between the cohomology groups $H^*(BG)$ and $H^*(B)$ (see figure 3).

By looking at the image of a specific class in $H^*(BG)$ rather than that of the entire group, this same technique can also be used to assign a cohomology class in $H^*(B)$ to each vector bundle on B.

Remark 3.1. The following notation provides a concise way of talking about this assignment: given a cohomology class $\alpha \in H^*(BG)$ and a vector bundle ξ induced by a map $f : B \to BG$, let $\alpha(\xi) = f^*(\alpha)$ be the image of α under the induced cohomology homomorphism $f^* : H^*(BG) \to H^*(B)$.

Definition 3.2 (Characteristic class)**.** A characteristic class α assigns to each vector bundle η on A a cohomology class $\alpha(\eta) \in H^*(A)$. This assignment commutes with induced maps in the following way: for all continuous maps $f : B \to A$ we have that

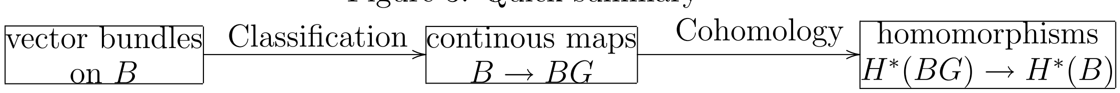
$$f^*(\alpha(\eta)) = \alpha(f^*(\eta)).$$

Here $f^*(\eta)$ is the induced bundle on B and f^* is the induced cohomology homomorphism.

Theorem 3.3. *Each cohomology class of the Grassmannian corresponds to a unique characteristic class and all characteristic classes correspond to a cohomology class of the Grassmannian.*

Proof. Let A and B be paracompact spaces and ξ be a vector bundle on B. Also let $g : A \to B$ be a continuous map and $\alpha \in H^*(BG)$ be a cohomology

Figure 3: Quick summary



class of the Grassmannian. By the classification theorem there exists a map $f : B \to BG$ that induces ξ from the canonical bundle on BG i.e. $\xi = f^*(\gamma)$. The assignment $\alpha()$ defined in remark 3.1 is a characteristic class because we have that:

$$\begin{aligned}
\alpha(g^*(\xi)) &= \alpha((fg)^*(\gamma)) && \text{by the functoriality of induced bundles} \\
&= (fg)^*(\alpha) && \text{by the definition of } \alpha() \\
&= g^*(f^*(\alpha)) && \text{because Cohomology is a functor} \\
&= g^*(\alpha(f^*(\gamma))) && \text{by definition of } \alpha() \\
&= g^*(\alpha(\xi)) && \text{by definition of } f
\end{aligned}$$

Clearly distinct cohomology class of the Grassmannian must correspond to different characteristic classes as they assign different classes to the canonical bundle on the Grassmannian.

All that remains is to show that all characteristic classes correspond to a cohomology class of the Grassmannian. Let α be a characteristic class. This implies $\alpha(\gamma) = \beta$ for some $\beta \in H^*(BG)$. Now let ξ be a vector bundle on B induced by the map $f : B \to BG$. The following equation follows directly from the definitions:

$$\beta(\xi) = f^*(\beta) = f^*(\alpha(\xi)) = \alpha(f^*(\gamma)) = \alpha(\xi)$$

Thus $\beta(\xi) = \alpha(\xi)$ for all vector bundles ξ. Therefore all characteristic classes correspond to a cohomology class of the Grassmannian. □

Example 3.4 (Euler class)**.** Let M be an n-dimensional manifold and $\xi = (E, p, M)$ be an oriented rank 2 real vector bundle on M with atlas $\{(\phi_\alpha, U_\alpha)\}$. Also let $\{\varepsilon_\alpha\}$ be a partition of unity on M subordinate to $\{U_\alpha\}$.

By proposition 1.11 we can assume without loss of generality that the transition functions $\{g_{\alpha\beta}\}$ of ξ map to $SO_2(\mathbb{R})$, thus $g_{\alpha\beta}$ assigns an angle of rotation to each point in the intersection $U_\alpha \cap U_\beta$. Because ξ is orientable, counterclockwise is globally defined. Let $\psi_{\alpha\beta} : U_\alpha \cap U_\beta \to [0, 2\pi)$ such that $\psi_{\alpha\beta}(x)$ is the angle of rotation assigned by $g_{\alpha\beta}(x)$ measured in the counterclockwise direction. Because $g_{\alpha\beta} g_{\beta\gamma} = g_{\alpha\gamma}$, we have that $\psi_{\alpha\gamma} - \psi_{\beta\gamma} = \psi_{\alpha\beta} + 2n\pi$ For some integer n.

On each U_α, define the 1-form $\eta_\alpha = \sum_\gamma \varepsilon_\alpha d\psi_{\alpha\gamma}$. Here d is the differential operator and the functions $d\psi_{\alpha\gamma}$ are extended by zero outside of $U_\alpha \cap U_\gamma$. On the intersection $U_\alpha \cup U_\beta$ we have that:

$$\begin{aligned} \eta_\alpha - \eta_\beta &= \sum_\gamma \varepsilon_\gamma d\psi_{\alpha\gamma} - \varepsilon_\gamma d\psi_{\beta\gamma} \\ &= \sum_\gamma \varepsilon_\gamma d(\psi_{\alpha\gamma} - \psi_{\beta\gamma}) \\ &= \sum_\gamma \varepsilon_\gamma d\psi_{\alpha\beta} \\ &= d\psi_{\alpha\beta} \end{aligned}$$

This implies that $d\eta_\alpha - d\eta_\beta = d(\eta_\alpha - \eta_\beta) = d(d\psi_{\alpha\beta}) = 0$ on $U_\alpha \cup U_\beta$ Thus the 2-forms $\{d\eta_\alpha\}$ piece together to form a global 2-form on M. This form is clearly closed and thus corresponds to a cohomology class. This class is called the Euler class and is a characteristic class.

Proposition 3.5. *The pullback of the Thom class to M by the zero section is the Euler class.*

3.2 Chern classes

Chern classes are some of the most important characteristic classes on complex vector bundles. The following two theorems provide a means of defining the Chern classes.

Theorem 3.6 (Leray-Hirsch)**.** *Let E be a fiber bundle on B. If there are global cohomology classes $\{x_1, \ldots, x_n\}$ on E that when restricted to each fiber freely generate the cohomology of that fiber, then $H^*(E)$ is a free module over $H^*(B)$ with basis $\{x_1, \ldots, x_n\}$.*

Lemma 3.7. *Let $P(\mathbb{C}^n)$ be the projectivization of $\mathbb{C}^n$ and S be the universal subbundle of $P(\mathbb{C}^n) \times \mathbb{C}^n$. We have that*

$$H^*(P(\mathbb{C}^n)) = \mathbb{R}[x]/(x^n) \text{ where } -x = e(S) \text{ the Euler class of } S.$$

Example 3.8 (Chern classes)**.** Let $\xi = (E, p, B)$ be a complex vector bundle on B. Also let $P(E)$ be the projectivization of E and S be the universal subbundle on $P(E)$ (see example 1.15). Finally let $x = -e(S)$ be the negative of the Euler class of S viewed as a rank 2 real vector space. As the restriction of S to any fiber of $P(E)$ is the universal subbundle $\widetilde{S}$ on the fiber and $e()$ is a characteristic class, x restricts to $e(\widetilde{S})$ on each fiber of $P(E)$.

This implies that the global classes $\{1, x, \ldots, x^{n-1}\}$ restrict to generators of cohomology of each fiber by theorem 3.7. Therefore the cohomology $H^*(P(E))$ of $P(E)$ is a free module over $H^*(B)$ with basis $\{1, x, \ldots, x^{n-1}\}$ by Leray-Hirsch theorem 3.6.

This implies that x^n can be written as a linear combination of $1, x, \ldots, x^{n-1}$ with coefficients in $H^*(B)$ i.e. there exists $\{c_1(\xi), \ldots, c_n(\xi)\} \subset H^*(B)$ such that

$$x^n + c_1(\xi)x^{n-1} + \cdots + c_n(\xi) = 0$$

The $c_i()$ are called the Chern classes and are characteristic classes on complex vector bundles. Their sum

$$c(\xi) = 1 + c_1(\xi) + \cdots + c_n(\xi)$$

is also a characteristic class and is called the total Chern class. By definition the higher Chern classes of a rank n vector bundle are zero i.e. $c_i(\xi) = 0$ for $i > n$.

Theorem 3.9. *The first Chern class $c_1(\xi)$ of a complex line bundle $\xi = (E, p, B)$ is the same as the Euler class $e(\xi_{\mathbb{R}})$ of the corresponding rank 2 real bundle on B.*

Proof. The projectivization $P(E)$ of a line bundle is isomorphic to the base space B, because there is exactly one line in each fiber. Similarly the universal subbundle S of $P(E)$ is isomorphic to the original line bundle ξ. Therefore $x = -e(S) = -e(\xi_{\mathbb{R}})$, and $x + c_1(\xi) = 0$ implies that $e(\xi) = c_1(\xi)$. □

Study of the cohomology of complex Grassmannians yields the following important result.

Theorem 3.10. *Every characteristic class of complex vector bundles over a paracompact manifold can be written as a polynomial in the Chern classes.*

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