A PROOF OF THE GAUSS-BONNET THEOREM

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Abstract. In this paper I will provide a proof of the Gauss-Bonnet Theorem. I will start by briefly explaining regular surfaces and move on to the first and second fundamental forms. I will then discuss Gaussian curvature and geodesics. Finally, I will move on to the theorem itself, giving both a local and a global version of the Gauss-Bonnet theorem. For this paper, I will assume that the reader has a knowledge of point-set topology, analysis in $\mathbb{R}^n$, and linear algebra.

Contents

1. Introduction

The Gauss-Bonnet theorem relates the sum of the interior angles of a triangle with the its Gaussian curvature, an intrinsic quantity of the geometry of the space that the triangle is drawn on. The theorem has numerous applications within and without its native field of differential geometry. In order to understand the Gauss-Bonnet theorem we must first understand some basic differential geometry. To this end, we start with the most basic idea in differential geometry, a regular surface.

2. Regular Surfaces

Definition 2.1. A subset $S \subset \mathbb{R}^3$ is a *regular surface* if for every point $p \in S$ there is a neighborhood $V \subset \mathbb{R}^3$ and a function f which maps an open set $U \subset \mathbb{R}^2$ onto $V \cap S \subset \mathbb{R}^3$ which has the following properties:

(1) f is differentiable.
(2) f is a homeomorphism.
(3) For every $q \in U$, the differential $df_q : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ is one-to-one.

Throughout this paper we denote the partial derivative of f with respect to u as $f_u = \frac{\partial f}{\partial u}$. Note that f is a vector-value function.

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We will discuss (3), often called the regularity condition, once we have the following definition.

Definition 2.2. The function f defined as above is called a *parametrization* in a neighborhood of p. The neighborhood $V \cap S$ of $p \in S$ is called a *coordinate neighborhood*.

Condition (3) is more familiar if we compute the matrix of the linear map df_q in the standard, or canonical bases $e_1 = (1,0), e_2 = (0,1)$ of $\mathbb{R}^2$ with coordinates (u,v) and $f_1 = (1,0,0), f_2 = (0,1,0), f_3 = (0,0,1)$ of $\mathbb{R}^3$, with coordinates (x,y,z).

Let $q = (u_0, v_0)$. Then e_1 is tangent to the curve $u \longrightarrow (u, v_0)$ where the image of this curve under f becomes the curve

$$u \longrightarrow (x(u,v_0), y(u,v_0), z(u,v_0)).$$

This image curve lies on our surface S and at $f(q)$ has the tangent vector

$$(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}) = \frac{\partial x}{\partial u}.$$

Here we compute the derivatives at (u_0, v_0) and a vector is indicated by its components in the basis f_1, f_2, f_3. By the definition of differential,

$$df_q(e_1) = (\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}) = \frac{\partial x}{\partial u}.$$

Similarly, we find

$$df_q(e_2) = (\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}) = \frac{\partial x}{\partial v}.$$

Combining this into a single matrix, we see that

$$df_q = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{pmatrix}$$

Condition (3) is equivalent to requiring that the two column vectors of the above matrix be linearly independent, or that the vector product

$$\frac{\partial x}{\partial u} \times \frac{\partial x}{\partial v} \neq 0,$$

or finally that one of the Jacobian determinants

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}, \frac{\partial(y,z)}{\partial(u,v)}, \frac{\partial(x,z)}{\partial(u,v)}.$$

is non-zero.

We defined surfaces as subsets of $\mathbb{R}^3$. We do this by covering a surface S in $\mathbb{R}^3$ with embeddings of open sets in $\mathbb{R}^2$ or *charts*. Condition (1) is very natural if we are to do differential geometry on S. Condition (2) makes sure charts only overlap in 2–dimensional subsets, so that the tangent plane at a point is unique. We will now show that condition (3) guarantees that the set of tangent vectors to parametrized curves of S at a point p makes up a plane.

Definition 2.3. A *tangent vector* to S at a point $p \in S$ is the tangent vector $\alpha'(0)$ of a differentiable parametrized curve $\alpha : (-\varepsilon, \varepsilon) \longrightarrow S$ with $\alpha(0) = p$.

Proposition 2.4. *Let $f : U \longrightarrow \boldsymbol{R}^3 \cap S$ be a parametrization of a regular surface S and let $q \in U$. The vector subspace of dimension 2,*

$$df_q(\mathbb{R}^2) \subset \mathbb{R}^3,$$

coincides with the set of tangent vectors to S at $f(q)$.

We will not prove this proposition, but we will note that the plane $df_q(\mathbb{R}^2)$, which passes through $f(q) = p$, does not depend on the parametrization f. We will denote this plane the *tangent plane* to S at p and write it as $T_p(S)$. It is easy to see that f_u and f_v span T_pS.

3. The First Fundamental Form

Besides differentiability, surfaces carry further geometric structures, the most important of which is called the first fundamental form.

Definition 3.1. By restricting The natural inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^3$ to each tangent plane $T_p(S)$ of a regular surface S, we get an inner product on $T_p(S)$. We call this inner product on T_pS *the first fundamental form* and denote it by I_p. So $I_p\langle v, w\rangle = \langle v, w\rangle$.

Thus the first fundamental form tells us how the surface S inherits the natural inner product of $\mathbb{R}^3$. We want to write it in terms of $\{f_u, f_v\}$, a basis associated to a parametrization $f(u, v)$ at p. To do this, we remember that a tangent vector $w \in T_p(s)$ is the tangent vector to a parametrized curve $\alpha(t) = f(u(t), v(t)), t \in (-\varepsilon, \varepsilon)$ with $p = \alpha(0) = f(u_0, v_0)$. Then, keeping in mind that u' and v' are the respective derivatives of u and v,

$$\begin{aligned} I\langle \alpha'(0), \alpha'(0)\rangle &= \langle f_u u' + f_v v', f_u u' + f_v v'\rangle \\ &= \langle f_u, f_u\rangle (u')^2 + 2\langle f_u, f_v\rangle u'v' + \langle f_v, f_v\rangle (v')2 \\ &= E(u')^2 + 2Fu'v' + G(v')^2 \end{aligned}$$

where we define

$$E(u_0, v_0) = \langle f_u, f_u\rangle \tag{3.2}$$

$$F(u_0, v_0) = \langle f_u, f_v\rangle \tag{3.3}$$

$$G(u_0, v_0) = \langle f_v, f_v\rangle. \tag{3.4}$$

To round off the section, we give the following

Definition 3.5. A parametrization is *orthogonal* if $F(u, v) = 0$.

4. Orientation

Definition 4.1. A regular surface S is *orientable* if it is possible to cover S with a family of coordinate neighborhoods so that if a point $p \in S$ is in two neighborhoods of this family, then the change of coordinates has positive Jacobian at p. The choice of family that satisfies this condition is called an *orientation* of S, and S is called *oriented*. If it is not possible to find such a family then S is called *nonorientable*.

Given a parametrization $f(u, v)$ at p, we have a definite choice of a unit normal vector N at p by the rule

$$N(p) = \frac{f_u \times f_v}{|f_u \times f_v|}(p).$$

(It may worth to say that the normal space is 1–dimensional.)
Taking a second parametrization $f'(u', v')$ at p, we see that

$$f'_u \times f'_v = (f_u \times f_v)\frac{\partial(u, v)}{\partial(u', v')},$$

where $\dfrac{\partial(u, v)}{\partial(u', v')}$ is the Jacobian of the coordinate change. From this we can see that N will not change its direction if the Jacobian is positive, and change its direction if the Jacobian is negative. So we can see from this that a surface is orientable if N keeps its direction no matter how it is moved around the surface.

Example 4.2. On the Möbius strip, we cannot find a differentiable field of unit normal vectors that are defined on the entire surface. Intuitively, we can see this by taking a vector field N around the middle circle of the figure and noticing that it would come back as $-N$, which contradicts the continuity of N. This is because we cannot decide which side of the surface we are on since we can go continuously to the other side without breaking the surface.

5. The Gauss Map

We denote S^2 as the unit sphere, i.e. $S^2 = \{(x, y, z) \in \mathbb{R}^3 |\ x^2 + y^2 + z^2 = 1\}$.

Definition 5.1. Let $S \subset \mathbb{R}^3$ be a surface with an orientation N. The *Gauss map* is defined to be $N : S \longrightarrow S^2 \subset \mathbb{R}^3$ is defined to be $p \longmapsto N(p)$.

It is easy to see that the Gauss map is differentiable. The differential dN_p is a linear map from $T_p(S)$ to $T_{N(p)}(S^2)$. We can identify T_pS and $T_{N(p)}S$ since they are parallel planes, hence dN_p is a linear map on $T_p(S)$. We now are ready for the following

Proposition 5.2. *The differential dN_p: $T_p(S) \longrightarrow T_pS$ of the Gauss-map is self-adjoint.*

Proof. We need to show that $\langle dN_p(w_1), w_2\rangle = \langle w_1, dN_p(w_2)\rangle$ for a basis $\{w_1, w_2\}$ of $T_p(S)$. To do this, let $f(u, v)$ be a parametrization of S at p and $\{f_u, f_v\}$ the associated basis of $T_p(S)$. If $\alpha(t) = f(u(t), v(t))$ is a parametrized curve in S, with $\alpha(0) = p$, we get

$$\begin{aligned} dN_p(\alpha'(0)) &= dN_p(f_u u'(0) + f_v v'(0)) \\ &= \frac{d}{dt} N(u(t), v(t)) \Big|_{t=0} \\ &= N_u u'(0) + N_v v'(0); \end{aligned}$$

in particular, $dN_p(f_u) = N_u$ and $dN_p(f_v) = N_u$. Thus, in order to prove that dN_p is self-adjoint, we only need to show that

$$\langle N_u, f_v \rangle = \langle f_u, N_v \rangle.$$

We can see this by taking the derivatives of $\langle N, f_u \rangle = 0$ and $\langle N, f_v \rangle = 0$, relative to v and u respectively, and get

$$\begin{aligned} \langle N_v, f_u \rangle + \langle N, f_{uv} \rangle &= 0 \\ \langle N_u, f_v \rangle + \langle N, f_{vu} \rangle &= 0. \end{aligned}$$

Hence,

$$\langle N_u, f_v \rangle = -\langle N, f_{uv} \rangle = \langle N_v, f_u \rangle.$$

□

For a parametrization $f(u, v)$ at a point $p \in S$ with $\alpha(t) = f(u(t), v(t))$ a parametrized curve on S, with $\alpha(0) = p$, the tangent vector to $\alpha(t)$ at p is $\alpha' = f_u u' + f_v v'$, and

$$dN(\alpha') = N'(u(t), v(t)) = N_u u' + N_v v'.$$

But N_u and N_v belong to $T_p(S)$, so we can write them in terms of our parameters

$$N_u = a_{11} f_u + a_{21} f_v, \tag{5.3}$$

$$N_v = a_{12} f_u + a_{22} f_v, \tag{5.4}$$

hence

$$dN(\alpha') = (a_{11} u' + a_{12} v') f_u + (a_{21} u' + a_{22} v') f_v,$$

which gives us

$$dN \begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} u' \\ v' \end{pmatrix}$$

Thus, in the basis $\{f_u, f_v\}, dN$ is given by the matrix $(a_{ij}); i, j = 1, 2$

Definition 5.5. The determinant of dN_p is the *Gaussian curvature*, K, of S at a point p.

6. The Second Fundamental Form

Now that we have the self-adjoint, linear map dN_p, we can associate with it a quadratic form which we imaginatively call the second fundamental form.

Definition 6.1. The quadratic form II_p is defined in $T_p(S)$ by $II_p(v) = -\langle dN_p(v), v\rangle$, and is called the *second fundamental form* of S at p.

Just like with the first fundamental form, we now proceed to write the second fundamental form in the basis $\{f_u, f_v\}$. Now

$$\begin{aligned} II_p(\alpha') &= -\langle dN(\alpha'), \alpha'\rangle \\ &= -\langle N_u u' + N_v v', f_u u' + f_v, v'\rangle \\ &= e(u')^2 + 2fu'v' + g(v')^2, \end{aligned}$$

where

$$e = -\langle N_u, f_u\rangle = \langle N, f_{uu}\rangle, \tag{6.2}$$

$$f = -\langle N_v, f_u\rangle = \langle N, f_{uv}\rangle = -\langle N_u, f_v\rangle, \tag{6.3}$$

$$g = -\langle N_v, f_v\rangle = \langle N, f_{vv}\rangle, \tag{6.4}$$

since $\langle N, f_u\rangle = \langle N, f_v\rangle = 0$. We can use the coefficients e, f, g to find the values of a_{ij}, giving us

$$-f = \langle N_u, f_v\rangle = a_{11}F + a_{21}G, \tag{6.5}$$

$$-f = \langle N_v, f_u\rangle = a_{12}E + a_{22}F, \tag{6.6}$$

$$-e = \langle N_u, f_u\rangle = a_{11}E + a_{21}F, \tag{6.7}$$

$$-g = \langle N_u, f_u\rangle = a_{12}F + a_{22}G. \tag{6.8}$$

Or in matrix form,

$$-\begin{pmatrix} e & f \\ f & g \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix}.$$

From this it is clear that

$$K = det(a_{ij}) = \frac{eg - f^2}{EG - F^2}.$$

7. geodesics

Definition 7.1. Given a differentiable vector field w in an open $U \subset S$ with $p \in U$, take a $y \in T_p(S)$. Consider the parametrized curve $\alpha : (-\varepsilon, \varepsilon) \longrightarrow U$, with $\alpha(0) = p$, $\alpha'(0) = y$, and let $g(t), t \in (-\varepsilon, \varepsilon)$, be the restriction of of the vector field g to the curve α. Then projecting $\frac{dw}{dt}(0)$ onto $T_p(S)$ forms the *covariant derivative* at p of the vector field w relative to the vector y, which we denote by $\frac{Dw}{dt}(0)$ or $(D_y w)(p)$.

The covariant derivative is the vector field analogue of the usual derivative in the plane. It is easy to verify that the covariant derivative does not depend on the choice of α.

Definition 7.2. A vector field along a parametrized curve $\alpha \longrightarrow S$ is *parallel* if $\frac{Dw}{dt} = 0$ for all $t \in I$.

Definition 7.3. A nonconstant, parametrized curve $\gamma : I \longrightarrow S$ is *geodesic* at $t \in I$ if the field of the tangent vectors $\gamma'(t)$ is parallel along γ at t, i.e. $\frac{D\gamma'(t)}{dt} = 0$. γ is called a *parametrized geodesic* if γ is geodesics for every $t \in I$.

Definition 7.4. Let w be a differentiable field of unit vectors along a parametrized curve $\alpha : I \longrightarrow S$ on an oriented surface S. Since $w(t), t \in I$, is a unit vector field, $\frac{dw}{dt}(t)$ is normal to $w(t)$, hence

$$\frac{Dw}{dt} = \lambda(N \times w(t)).$$

The real number $\lambda = \lambda(t)$, denoted by $\left[\frac{Dw}{dt}\right]$, is called the *algebraic value of the covariant derivative* of w at t.

Definition 7.5. Let C be an oriented regular curve contained on an oriented surface S, and let $\alpha(s)$ be a parametrization of C, in an neighborhood of $p \in S$, by the arc length s. The algebraic value of the covariant derivative

$$\left[\frac{D\alpha'(s)}{ds}\right] = k_g$$

of $\alpha'(s)$ at p is called the geodesic curvature of C at p.

Lemma 7.6. *Let a, b be differentiable functions in I with $a^2 + b^2 = 1$ and φ_0 be such that $a(t_0) = \cos\varphi_0$ and $b(t_0) = \sin\varphi_0$. Then the function φ defined by*

$$\varphi = \varphi_0 + \int_{t_0}^{t} (ab' - ba')\, dt$$

has the properties that $\cos\varphi(t) = a(t), \sin\varphi(t) = b(t)$ for $t \in I$, and $\varphi(t_0) = \varphi_0$.

Proof. To prove this we need only show that

$$(a - \cos\varphi)^2 + (b - \sin\varphi)^2 = 2 - 2(a\cos\varphi + b\sin\varphi)$$

is zero everywhere. Or in other words, that

$$A = a\cos\varphi + b\sin\varphi = 1$$

.

By using the fact that $aa' = -bb'$ and the definition of φ, we get

$$\begin{aligned} A' &= -a(\sin\varphi)\varphi' + b(\cos\varphi)\varphi' + a'\cos\varphi + b'\sin\varphi \\ &= -b'(\sin\varphi)(a^2 + b^2) - a'(\cos\varphi)(a^2 + b^2) + a'\cos\varphi + b'\sin\varphi \\ &= 0 \end{aligned}$$

This tells us that $A(t) =$ constant, and since $A(t_0) = 1$, we have proved the lemma. □

Lemma 7.7. *Let v, w be differentiable vector fields along the curve $\alpha : I \longrightarrow S$, with $|w(t)| = |v(t)| = 1, t \in I$. Then*

$$\left[\frac{Dw}{dt}\right] - \left[\frac{Dv}{dt}\right] = \left[\frac{d\varphi}{dt}\right]$$

where φ is the function given in the previous lemma.

Proof. Take the vectors $\bar{v} = N \times v$ and $\bar{w} = N \times w$. Then

$$\begin{aligned} (7.8)\quad w &= (\cos\varphi)v + (\sin\varphi)\bar{v}, \\ (7.9)\quad \bar{w} &= N \times w = (\cos\varphi)N \times v + (\sin\varphi)N \times \bar{v} = (\cos\varphi)\bar{v} - (\sin\varphi)v. \end{aligned}$$

Differentiating (7.8) we get

$$w' = -(\sin\varphi)\varphi' v + (\cos\varphi)v' + (\cos\varphi)\varphi'\bar{v} + (\sin\varphi)\bar{v}'$$

Taking the inner product of this with (7.9) and using the fact that $\langle v, v' \rangle = \langle v, \bar{v} \rangle = 0$ we get

$$\begin{aligned} \langle w', \bar{w} \rangle &= (\sin^2\varphi)\varphi' + (\cos^2\varphi)\langle v', \bar{v} \rangle + (\cos^2\varphi)\varphi - (\sin^2\varphi)\langle v', \bar{v} \rangle \\ &= \varphi' + (\cos^2\varphi)\langle v', \bar{v} \rangle - (\sin^2\varphi)\langle v', \bar{v} \rangle. \end{aligned}$$

but $\langle v, \bar{v} \rangle = -\langle v, \bar{v}' \rangle$, so

$$\langle w', \bar{w} \rangle = \varphi' + (\cos^2\varphi + \sin^2\varphi)\langle v, \bar{v}' = \varphi' + \langle v, \bar{v}' \rangle$$

.

Thus,

$$\left[\frac{Dw}{dt}\right] = \left[\frac{Dw}{dt}\right]\langle N \times w, \bar{w} \rangle = \langle \frac{dw}{dt}, \bar{w} \rangle = \langle w', \bar{w} \rangle = \varphi' + \langle v', \bar{v} \rangle = \frac{d\varphi}{dt} + \left[\frac{Dv}{dv}\right]$$

,

which proves the lemma. □

Proposition 7.10. *Let $f(u, v)$ be an orthogonal parametrization of a neighborhood, and w(t) a differentiable field of unit vectors along the curve $f(u(t), v(t))$. Then*

$$\left[\frac{Dw}{dt}\right] = \frac{1}{2\sqrt{EG}}\left(G_u \frac{dv}{dt} - E_v \frac{du}{dt}\right) + \frac{d\varphi}{dt}$$

where φ_0 is the angle from from f_u to w in the given orientation.

Proof. Let $e_1 = \frac{f_u}{\sqrt{E}}, e_2 = \frac{f_v}{\sqrt{G}}$ be the unit vectors tangent to the coordinate curves. Then $e_1 \times e_2 = N$, and by lemma 7.7 we have

$$\left[\frac{Dw}{dt}\right] = \left[\frac{d\varphi}{dt}\right] + \left[\frac{Dv}{dt}\right] \tag{7.11}$$

where $e_1(u(t), v(t))$ is the restriction of the field e_1 to the curve $f(u(t), v(t))$. We know that

$$\frac{De_1}{dt} = \left\langle \frac{de_1}{dt}, N \times e_1 \right\rangle = \left\langle \frac{de_1}{dt}, e_2 \right\rangle = \langle (e_1)_u, e_2 \rangle \frac{du}{dt} + \langle (e_1)_v, e_2 \rangle \frac{dv}{dt}$$

But since $F = 0$, we have $\langle f_{uu}, f_v \rangle = -\frac{1}{2}E_v$, giving us

$$\langle (e_1)_u, e_2 \rangle = \left\langle \left(\frac{f_u}{\sqrt{E}}\right)_u, \frac{f_v}{\sqrt{G}} \right\rangle = \frac{1}{2}\frac{E_v}{\sqrt{EG}}$$

.

Similarly,

$$\langle (e_1)_v, e_2 \rangle = \frac{1}{2}\frac{G_u}{\sqrt{EG}}$$

.

Putting all of this back into (7.11), we get

$$\left[\frac{Dw}{dt}\right] = \frac{1}{2\sqrt{EG}}\left(G_u\frac{dv}{dt} - E_v\frac{du}{dt}\right) + \frac{d\varphi}{dt},$$

exactly what we wanted.

□

8. The Local Gauss-Bonnet Theorem

Theorem 8.1 (Local Gauss-Bonnet)**.** *Given an orthogonal parametrization $f : U \longrightarrow S$ of an oriented surface S, where $U \subset \mathbb{R}^2$ is homeomorphic to an open disk and f is compatible with the orientation of S, let $R \subset f(U)$ be a simple region of S, and let $\alpha : I \longrightarrow S$ be so that $\partial R = \alpha(I)$. If α is positively oriented, parametrized by arc length s, and if $\alpha(s_0), ..., \alpha(s_k)$ and $\theta_0, ..., \theta_k$ are respectively the vertices and external angles of α, then*

$$\sum_{j=0}^{k}\int_{s_j}^{s_{j+1}} k_g(s)\ ds + \iint_R K\ d\sigma + \sum_{j=0}^{k}\theta_j = 2\pi$$

where k_g is the geodesic curvature of the regular arcs of α and K is the Gaussian curvature of S.

Proof. Let $u = u(s), v = v(s)$ be the expression of the parametrization of α in the parametrization f. By proposition 7.10 we have

$$k_g = \frac{1}{2\sqrt{EG}}\left(G_u\frac{dv}{ds} - E_v\frac{du}{ds}\right) + \frac{d\varphi_j}{ds}$$

where $\varphi_j(s)$ is the differentiable function which measures the positive angle from f_u to $\alpha'(s)$ in $[s_j, s_{j+1}]$. By integrating the above expression in every interval $[s_j, s_{j+1}]$, and adding the results we obtain

$$\sum_{j=0}^{k}\int_{s_j}^{s_{j+1}} k_g(s)\ ds = \sum_{j=0}^{k}\int_{s_j}^{s_{j+1}}\left(\frac{G_u}{2\sqrt{EG}}\frac{dv}{ds} - \frac{E_v}{2\sqrt{EG}}\frac{du}{ds}\right) ds + \sum_{j=0}^{k}\int_{s_j}^{s_{j+1}}\frac{d\varphi_j}{ds}\ ds$$

Now, the Gauss-Green theorem states the following: If P(u, v) and Q(u, v) are differentiable functions in a simple region $A \subset R^2$, whose boundary is given by $u = u(s), v = v(s)$, then

$$\sum_{j=0}^{k}\int_{s_j}^{s_{j+1}}\left(P\frac{du}{ds} + Q\frac{dv}{ds}\right)\ ds = \iint_A\left(\frac{\partial Q}{\partial u} - \frac{\partial P}{\partial v}\right)\ dudv.$$

Applying this theorem, we get

$$\sum_{j=0}^{k}\int_{s_j}^{s_{j+1}} k_g(s)\, ds = \iint_{f^{-1}(R)}\left[\left(\frac{E_v}{2\sqrt{EG}}\right)_v + \left(\frac{G_u}{2\sqrt{EG}}\right)_u\right] dudv + \sum_{j=0}^{k}\int_{s_j}^{s_{j+1}}\frac{d\varphi_j}{ds}\ ds$$

Now since we have an orthogonal parametrization, i.e. $F = 0$,

$$\iint_{f^{-1}(R)} \left[\left(\frac{E_v}{2\sqrt{EG}} \frac{du}{dt} \right)_v + \left(\frac{G_u}{2\sqrt{EG}} \right)_u \right] dudv = - \iint_{f^{-1}(R)} K\sqrt{EG}\, dudv = - \iint_R K\, d\sigma$$

And from topology we know the Theorem of Turning Tangents which tells us that

$$\sum_{j=0}^{k} \int_{s_j}^{s_{j+1}} \frac{d\varphi_j}{ds}\, ds = \sum_{j=0} k(\varphi_j(s_{j+1}) - \varphi_j(s_j)) = \pm 2\pi - \sum_{j=1}^{k} \theta_j$$

Since the curve is positively oriented, the sign should be plus. Putting all of this together gives us

$$\sum_{j=0}^{k} \int_{s_j}^{s_{j+1}} k_g(s)\ ds + \iint_R K\ d\sigma + \sum_{j=0}^{k} \theta_j = 2\pi$$

□

Theorem 8.2 (Global Gauss-Bonnet). *Let $R \subset S$ be a regular region of an oriented surface and let $C_1, ..., C_n$ be the closed, simple, piecewise regular curves which form the boundary ∂R of R. Suppose that each C_j is positively oriented and let $\theta_1, ..., \theta_p$ be the set of all external angles of the curves $C_1, ..., C_n$. Then*

$$\sum_{j=1}^{n} \int_{C_j} k_g(s)\ ds + \iint_R K\ d\sigma + \sum_{j=1}^{p} \theta_j = 2\pi\chi(R)$$

where s denotes the arc length of C_j, and the integral over C_j means the sum of integrals in every regular arc of C_j. Also, $\chi = F - E + V$ is the Euler-Poincaré Characterization where for a given triangulation, F denotes the number of faces, E denotes the number of edges, and V denotes the number of vertices of the triangulation.

Proof. By a theorem in Topology, we know that we can take a triangulation $\mathfrak{J}$ of the region R with the property that every triangle T_j is contained in a coordinate neighborhood of a family of orthogonal parametrizations compatible with the orientation of S. By making the boundary of every triangle of $mathfrakJ$ is oppositely oriented, we get opposite orientations in the edges that adjacent triangles share. To every triangle we apply the Local Gauss-Bonnet Theorem, add them up, remembering that each "interior" side is described twice in opposite orientations,

$$\sum_j \int_{C_j} k_g(s)\ ds + \iint_R K\ d\sigma + \sum_{j,k=1}^{F,\mathfrak{J}} \theta_{j,k} = 2\pi F$$

4 where F denotes the number of triangles of $\mathfrak{J}$, and $\theta_{j1}, \theta_{j2}, \theta_{j3}$ are the external angles of the triangle T_j.

The *interior* angles of the triangle T_j we shall denote $\varphi_{jk} = \pi - \theta_{jk}$. From this, we see that

$$\sum_{j,k} \theta_{j,k} = \sum_{j,k} \pi - \sum_{j,k} \varphi_{j,k} = 3\pi F - \sum_{j,k} \varphi_{j,k}.$$

I will now introduce the following notation:

$$\begin{aligned} E_e &= \text{number of external edges of } \mathfrak{J}, \\ E_i &= \text{number of internal edges of } \mathfrak{J}, \\ V_e &= \text{number of external vertices of } \mathfrak{J}, \\ V_i &= \text{number of exteranl vertices of } \mathfrak{J}. \end{aligned}$$

Since the curves C_i are closed, $E_e = V_e$. It is also clear that

$$3F = 3E_i + E_e$$

hence

$$\sum_{j,k} \theta_{j,k} = 2\pi E_i + \pi E_e - \sum_{j,k} \varphi_{j,k}.$$

Now either the external vertices are vertices of some curve C_i, which we'll call V_{ec}, or they are vertices introduced by the triangulation, which we'll call V_{ej}. So we can write $V_e = V_{ec} + V_{ej}$. And the sum of the angles around each internal vertex is 2π, so we can write

$$\sum_{j,k} \theta_{j,k} = 2\pi E_i + \pi E_e - 2\pi V_i - 2\pi V_{ei} - \sum_j (\pi - \theta_j)$$

Adding and subtracting πE_e to the right hand side of this equation, and using the fact that $E_e = V_e$, we have

$$\begin{aligned} \sum_{j,k} \theta_{j,k} &= 2\pi E_i + 2\pi E_e - 2\pi V_i - \pi V_{ei} - \pi V_{ec} + \sum_j \theta_j \\ &= 2\pi E - 2\pi V + \sum_j \theta_j. \end{aligned}$$

Finally, we can put it all back together again to get

$$\begin{aligned} \sum_{j=1}^{n} \int_{C_j} k_g(s)\ ds + \iint_R K\ d\sigma + \sum_{j=1}^{p} \theta_j &= 2\pi(F - E + V) \\ &= 2\pi\chi(R), \end{aligned}$$

which is exactly what we wanted to prove. □