FUNDAMENTAL GROUPS AND COVERING SPACES

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Abstract. In this paper, I will briefly develop the theory of fundamental groups and covering spaces of topological spaces. Then I will point toward the manner in which covering spaces can be used prove some cool things in group theory. I presume a cursory knowledge of topological spaces—open and closed sets, continuous functions, and homeomorphisms should be enough background knowledge.

Contents

1. Fundamental Groups: An Intuitive Introduction

Given a topological space X and a point $x_0 \in X$, we want to know what the one dimensional structure of the space surrounding this point looks like. In particular, we wish to know how many "different" ways (up to homotopy equivalence) we can travel in continuous loops from that point and back to that point. This information is what the fundamental group tells us. Let us first have some definitions.

Definition 1.1. A **loop** at x_0 in a topological space X is a continuous function $f : [0, 1] \to X$ such that $f(0) = x_0 = f(1)$. That is, we start at x_0, wander around continuously for $t \in (0, 1)$, and return to x_0 at $t = 1$. Given two loops f and g, with the same base point, define $f \bullet g$ to be a loop that runs them, one after the other, in double speed: $(f \bullet g)(t) = f(2t)$ for $t \in [0, 1/2]$, and $(f \bullet g)(t) = g(2t - 1)$ for $t \in [1/2, 1]$.

Definition 1.2. Two loops f and g are **homotopy equivalent** if there is a continuous function $h : [0, 1] \times [0, 1] \to X$ such that, for all $t \in [0, 1]$, $h(t, 0) = f(t)$ and $h(t, 1) = g(t)$, and, for all s, $h(0, s) = x_0 = h(1, s)$. That is, h continuously deforms f to g through loops at x_0. It is easy to see that homotopies between loops at a given base point divide loops into equivalence classes, which we call homotopy equivalence classes.

Definition 1.3. Given a topological space X and a base point x_0, the **fundamental group** of X at x_0 is the set of all homotopy classes of loops at x_0, and is denoted $\pi_1(X, x_0)$. It is a fact (unproved here) that the operation $\bullet$ gives $\pi_1(X, x_0)$

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the structure of a group; the constant loop is the identity of the group, and, for a loop $f(t)$, the inverse $g(t) = f(1-t)$ runs the same loop backwards.

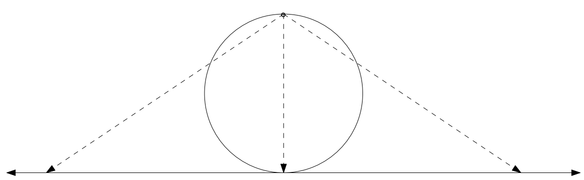
Remark 1.4. Generally $\pi_1(X, x_0)$ depends on the base point; but if our space X is path-connected, then all base points give isomorphic fundamental groups, and we can simply write (for all our purposes here) $\pi_1(X)$.

Definition 1.5. A space X is **simply connected** if it is path connected and if the fundamental group on that space is trivial (i.e., all loops are homotopic to the constant loop $f(t) = x_0$).

Proposition 1.6. *S^n is simply connected for $n \geq 2$.*

Proof. We know that S^n is path-connected, so let us show that its fundamental group is trivial. Let f be a loop in S^n at base point x_0. It suffices to show that the image of f in S^n misses one point x, because we can easily show that $S^n \backslash \{x\}$ is homeomorphic to $\mathbb{R}^n$, which is simply connected. Let us first sketch a proof of this:

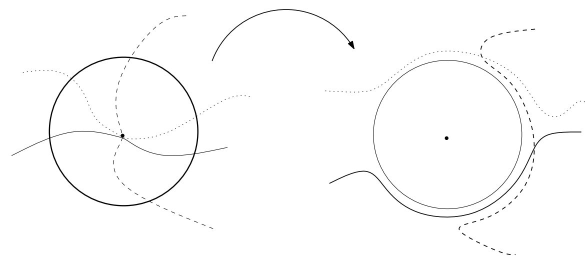
Embed both S^n and $\mathbb{R}^n$ in $\mathbb{R}^{n+1}$, putting S^n above $\mathbb{R}^n$ with the missing point x farthest away from $\mathbb{R}^n$. Construct the stereographic projection of S^n onto $\mathbb{R}^n$ by constructing rays ϕ starting at x, passing through $y \in S^n$ and landing on $\mathbb{R}^n$. Construct our homeomorphism such that each $y \in S^n$ goes to the place where this projection hits $\mathbb{R}^n$. This map will yield a homeomorphism onto $\mathbb{R}^n$. It is easy to visualize with $n = 1$:



We now have only to show that f is homotopic to a loop g that misses at least one point in S^n. Take any point $x \neq x_0$, and find a small open ball B around x. If the number of times f passes into the ball, hits x, and leaves the ball is finite, then we are done—just push each individual portion of f off of x. We must therefore show that this is in fact the case.

Now, the set $f^{-1}(B)$ is open in $[0, 1]$ since f is continuous, and thus is equal to the union of (possibly infinitely many) disjoint intervals. Recall that any open cover of a compact set has a finite subcover. Now consider $f^{-1}(x)$. The set $\{x\}$ is closed, which implies that $f^{-1}(x)$ is also closed (since with a continuous function, the preimage of a closed set is closed). But $f^{-1}(x) \subset f^{-1}(B)$, which is bounded—we therefore have $f^{-1}(x)$ closed and bounded, and hence compact. Now, the intervals of $f^{-1}(B)$ form a cover of $f^{-1}(x)$, and since $f^{-1}(x)$ is a compact set, we can find a finite subcollection of intervals $I_1, ..., I_n$ that cover $f^{-1}(x)$.

The rest is straightforward: homotope f to, for example, the loop g that avoids x on each I_i by going around the boundary δB. Then we have that f is homotopic to some loop g such that the image $g(I)$ is contained in $S^n \backslash \{x\}$, which is contractible. Hence our original loop f is nullhomotopic, and thus $\pi_1(S^n)$ is trivial. We can visualize this final homotopy thus:



□

Remark 1.7. This method fails for S^1 because when we remove a ball from S^1, we cannot homotope the path around, say, δB–there is only one way to get from one side of the "ball" to the other (while retaining the same direction). In fact, the fundamental group of the circle is isomorphic to $(\mathbb{Z}, +)$, but to prove this we need more advanced tools than we have at our disposal. For us, it will be an unjustified fact.

Definition 1.8. Let X and Y be topological spaces, and let $f : X \to Y$ be a continuous map. Define $f_* : \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$ by

$$[\alpha_{x_0}] \mapsto [f \circ \alpha_{x_0}]$$

for loops α_{x_0} based at x_0. We can check that f_* is a homomorphism, and satisfies **functoriality** conditions:

(1) $f_*([c_{x_0}]) = [c_{f(x_0)}]$
(2) $(f \circ g)_* = f_* \circ g_*$
(3) $(f^{-1})_* = f_*^{-1}$

Corollary 1.9. $\mathbb{R}^2$ *is not homeomorphic to* $\mathbb{R}^n$ *if* $n > 2$.

Proof. We will use the previous proposition, and proceed by contradiction. If there were a homeomorphism $\phi : \mathbb{R}^2 \to \mathbb{R}^n$, then there would also be a homeomorphism

$$\phi' : \mathbb{R}^2 \backslash \{0\} \longrightarrow \mathbb{R}^n \backslash \{\phi(0)\}.$$

Note that $\phi(0)$ must be a point in $\mathbb{R}^n$ since ϕ is a homeomorphism (and for continuous maps, the image of a point is a point). In particular, since ϕ is a bijection, ϕ must take nonzero points of $\mathbb{R}^2$ to points in $\mathbb{R}^n \neq \phi(0)$.

Now take the inclusion

$$i : S^1 \hookrightarrow \mathbb{R}^2 \backslash \{0\}$$

as the unit circle, and define

$$p : \mathbb{R}^2 \backslash \{0\} \to S^1$$

to be the function which collapses each ray from 0 to infinity onto the point where that ray hits the unit circle. It is easy to see that $p \circ i = Id_{S^1}$, and that $i \circ p$ is homotopic to $Id_{\mathbb{R}^2 \backslash \{0\}}$ via the straight line homotopy. This implies that $\mathbb{R}^2 \backslash \{0\}$ is homotopy equivalent to S^1, and hence $\pi_1(\mathbb{R}^2 \backslash \{0\}) \cong \pi_1(S^1) \cong (\mathbb{Z}, +)$.

But we can perform this same inclusion with S^n to show that $\pi_1(\mathbb{R}^n \backslash \{0\}) \cong \pi_1(S^n) \cong 0$ by Prop. 1.6. But since we have a homeomorphism ϕ', this would imply that

$$(\mathbb{Z},+) \cong \pi_1(S^1) \cong \pi_1(\mathbb{R}^2\backslash\{0\}) \cong \pi_1(\mathbb{R}^n\backslash\{0\}) \cong \pi_1(S^n) \cong 0.$$

Thus $\pi_1(\mathbb{R}^2\backslash\{0\})$ and $\pi_1(\mathbb{R}^n\backslash\{0\})$ have isomorphic fundamental groups. But this is absurd since the integers are not isomorphic to the trivial group. Thus there is no homeomorphism from $\mathbb{R}^2 \to \mathbb{R}^n$ for $n \neq 2$. □

We can also use facts about functoriality to prove a cool fact about continuous functions on the two dimensional unit disc, D^2:

Proposition 1.10. *Any continuous map $f : D^2 \to D^2$ has a fixed point—i.e., there exists $z \in D^2$ such that $f(z) = z$.*

Proof. We proceed by contradiction. Suppose there exists a continuous map f with no fixed point. Define $r : D^2 \to S^1$ thus: First, for each x, define a ray that passes from $f(x)$ through x, and then let $r(x)$ be the point on S^1 where this ray passes through it. Note that if we assume that there is no point such that $f(z) = z$, this is a well-defined, continuous function on S^1.

Let

$$i : S^1 \hookrightarrow D^2$$

be the inclusion of the boundary. Then

$$r \circ i : S^1 \to S^1 = Id_{S^1}$$

since r does not change anything on the boundary of the disc. Thus we have

$$r_* \circ i_* = (Id_{S^1})_* = Id_{\pi_1(S^1)}.$$

Thus r_* is onto, so we have a surjection

$$r_* : \pi_1(D^2) \to \pi_1(S^1).$$

But D^2 is just a convex subset of $\mathbb{R}^2$, and hence $\pi_1(D^2)$ is trivial; but $\pi_1(S^1)$ is $(\mathbb{Z},+)$, which is far from trivial. Hence r_* is not a surjection, thus $r(x)$ cannot be well-defined and continuous. But if f really does not fix a point, $r(x)$ *must* be well-defined and continuous. Thus there must be a fixed point z such that $f(z) = z$. □

2. Covering Spaces

Connected with the study of fundamental groups is the study of covering spaces: a space X is covered by another space Y, roughly, when there is a surjective local homeomorphism from Y onto X. As we will see in Section 3, the theory of covering spaces has some interesting applications to group theory.

Definition 2.1. Let X be a topological space. A **covering space** $\tilde{X}$ over X is a topological space together with a continuous and surjective map

$$p : \tilde{X} \to X$$

such that, for every $x \in X$, there exists an open neighborhood $U \subset X$ such that $x \in U$ and $p^{-1}(U)$ is a disjoint union of open sets $\coprod_{\alpha} C_\alpha \subset \tilde{X}$ such that, for each α, $p|_{C_\alpha}$ is a homeomorphism.

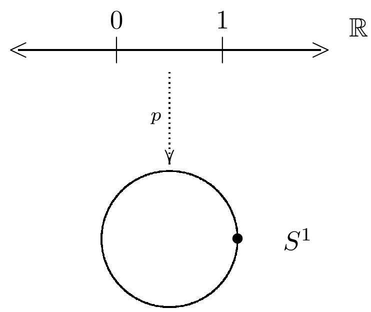
Remark 2.2. Here we call X the **base space**, $\tilde{X}$ the **covering space**, and p the **covering map**.

Example 2.3. Let our base space be the unit circle S^1 embedded in $\mathbb{R}^2$. Then $\mathbb{R}$ is a covering space of S^1 using the map

$$p : \mathbb{R} \to S^1$$
$$p : t \mapsto (\cos(t), \sin(t)).$$

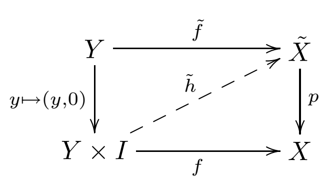
Note that for each point in $x \in S^1$, there are infinitely many $t \in \mathbb{R}$ such that $p(t) = x$ — namely, any number of the form $2n\pi + j$ where j is a number such that $p(j) = x$ and $n \in \mathbb{Z}$. It is a cover because it is continuous and surjective, and given a proper open set U on the circle, the preimage $p^{-1}(U)$ is a disjoint union of open sets in $\mathbb{R}$; it is easy to see that for any open interval in S^1 we get an open interval on $\mathbb{R}$, and then we remember that any open set can be decomposed into intervals. This map is a homeomorphism on each interval because each interval cannot map to a point in U more than once, since U is a proper open subset of S^1 and each interval must be less than distance 2π away from another interval that maps to the same point in S^1. If we had x_1 and x_2 with $p(x_1) = p(x_2)$, then the line connecting x_1 to x_2 would have as its image all of S_1, and U would not be a proper open subset of S^1.

We can visualize this thus:



Definition 2.4. Let $p : \tilde{X} \to X$ be a covering map. A **lift** of a map $f : Y \to X$ is a map $\tilde{f} : Y \to \tilde{X}$ such that $p\tilde{f} = f$.

Theorem 2.5. *Given a covering space $p : \tilde{X} \to X$, a map $f : Y \to X$, and a map $\tilde{f} : Y \to \tilde{X}$ lifting $h(t, 0)$, there exists a unique homotopy $\tilde{h} : Y \times I \to \tilde{X}$ of f that lifts $\tilde{f}$. We can visualize this in terms of a diagram: Given the homotopy on the bottom of the diagram, there is a unique homotopy lift $\tilde{h}$ represented by the middle arrow that makes the diagram commute (i.e. such that $\forall y, (p \circ \tilde{h})(y, y) = h(y, t)$):*



Sketch of proof. For a detailed proof, see Hatcher [2]. Since p is a covering map, it is a local homeomorphism, and so we can define $\tilde{h}$ locally. The difficult part of the proof, which we will not reproduce here, involves showing that these local lifts can be pieced together to give a well-defined homotopy $\tilde{h} : Y \times I \to \tilde{X}$. □

Theorem 2.6. *For any path connected covering space $\tilde{X}$ of a simply-connected space X, the covering map p is a global homeomorphism.*

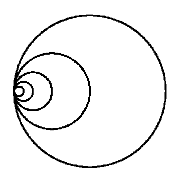
Proof. By the definition of a covering space, p is globally surjective and locally homeomorphic. Thus it suffices to show that p is one to one globally—that is, given any $x_1, x_2 \in \tilde{X}$ with $x_1 \neq x_2$, $p(x_1) \neq p(x_2)$. (Note that this fails for the previous example: S^1 is not simply connected since $\pi_1(S^1) = (\mathbb{Z}, +)$, and p is certainly not a global homeomorphism since each x in S^1 is covered infinitely many times by p.)

Suppose then that $p(x_1) = p(x_2)$. Consider any path f from x_1 to x_2 in $\tilde{X}$. Since p is continuous, the composition of p with f is a path in X. Now since $p(x_1) = p(x_2)$, the image of our path f, $p \circ f$, is a loop in X. But X is simply connected, so $\pi_1(X)$ is trivial, and thus the loop $p \circ f$ is homotopic to a constant loop. Now we are in a position to use homotopy lifting. We have a homotopy h from the loop $p \circ f$ to a constant loop g in X, and also, we have a lift of $p \circ f$, namely, f. By homotopy lifting, we get that there exists a unique homotopy $\tilde{h}$ from f to $\tilde{g}$ where $\tilde{g}$ is some lift (that we cannot choose) of the constant loop g into $\tilde{X}$. *A priori* we do not know what $\tilde{g}$ is. But since g is a constant loop in X, and since p is a local homeomorphism at x, we *must* have $\tilde{g}$ be a constant loop—otherwise, we would not have a local bijection. Then we have a homotopy $\tilde{h}$ from f to $\tilde{g}$, which must be constant. Now we have that $\tilde{h}$ is a map $[0,1] \times [0,1] \to X$, with $h(s,0) = f(s), h(s,1) = \tilde{g}(s)$. Since the original homotopy h fixes the endpoints, $t \mapsto \tilde{h}(0,t)$ and $t \mapsto \tilde{h}(1,t)$ give paths from, respectively, x_1 to a point in the image of g and from x_2 to the point in the image of g. But both of these paths are entirely contained within the preimage of the point $p(x_1) = p(x_2)$, so both paths must be constant. Thus $x_1 = x_2$, and p is a global homeomorphism.

□

It is tempting to think that the composition of covering spaces is a covering space. But this need not be the case, as the following two examples will illustrate.

Example 2.7. Construct a space as follows: consider the subspace $X \in \mathbb{R}^2$ that is the union of circles C_n of radius $1/n$, and center $(1/n, 0)$ for $n \in \mathbb{N}$. We can picture this thus:

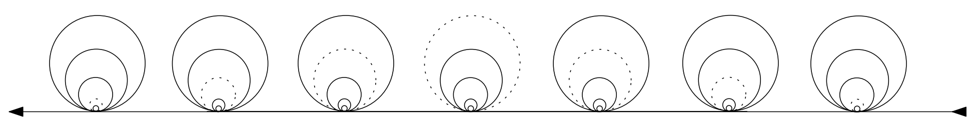


This space is interesting in many ways, and will eventually provide a counterexample to the compositions of covering spaces being a covering space. As an aside, here are some fun properties about this space:

Consider the surjection $r_n : X \to C_n$ collapsing all C_i's except C_n to the origin. Each r_n yields a surjection $\rho_n : \pi_1(X) \to \pi_1(C_n) \cong \mathbb{Z}$, with the origin as the base point. Taking the product of all the ρ_n's yields a homomorphism $\rho : \pi_1(X) \to \Pi_\infty \mathbb{Z}$ to the direct product of infinitely many copies of $\mathbb{Z}$. It is surjective because, for every sequence of integers, we can find a loop $f : I \to X$ that wraps around C_n that many times in the interval $[1 - 1/2^{n-1}, 1 - 1/2^n]$. This composition of loops is continuous at all $t < 1$, and it is continuous at $t = 1$ since every neighborhood of the base point contains all but finitely many of the circles C_n. $\pi_1(X)$ is uncountable since it maps onto the uncountable group $\Pi_\infty \mathbb{Z}$.

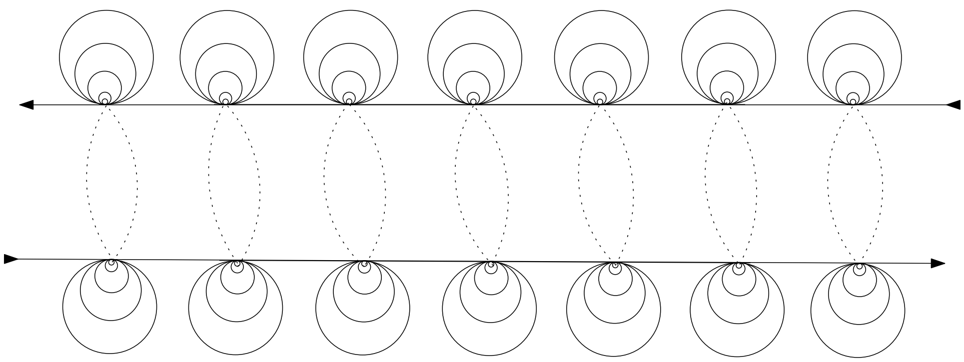
Now let us construct a covering of this space X. Pictures are the best way to get a sense of what types of covering spaces will give us our desired result, so I have included many.

Example 2.8. Let us start out with the following covering $\tilde{X}$ of X:

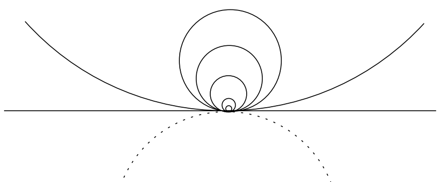


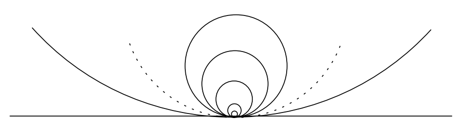
I have just placed a bunch of the inner loops $\bigcup_{i \geq 2} C_i$ of these spaces X on the real line $\mathbb{R}$, with the base points of X hitting $\mathbb{R}$ on, say, the integers. For each integer n, let X_n denote the copy of $\bigcup_{i \geq 2} C_i$ with basepoint at n on the real line. (Where the C_i notation is exactly that of the previous example.) Then define a covering map $p_1 : \tilde{X} \to X$ as follows: first, map each segment $[n, n+1]$ of $\mathbb{R}$ to C_1 in X via the usual covering map of $\mathbb{R}$ to the circle, and have p_1 take X_n via the identity to $\bigcup_{i \geq 2} C_i$. (Ignore for now the dotted loops: they will become important with the next cover.) I claim that this defines a covering map. Given any point x on an inner circle C_i of X and any $U \subset C_i$ such that $x \in U$ but the basepoint x_0 is not in U, $p_1^{-1}(U)$ consists of one copy of U in each X_n, and p_1 restricts to a homeomorphism on each of these. Similarly, given any point x on the outer loop C_1 of X, there is an open neighborhood U such that the preimage of U consists of open neighborhoods of $\mathbb{R}$ in $\tilde{X}$, each contained in $(n, n+1)$, and thus p_1 restricts to a homeomorphism on each of those components. Finally, at the base point x_0 of X, we can find a small open neighborhood U that misses at least one point of C_1 such that $p_1^{-1}(U)$ consists of disjoint open neighborhoods around the base points of $\tilde{X}$ such that each consists of a homeomorphic copy of $U \cap \bigcup_{i \geq 2} C_i$ (the intersection of U with the inner circles) together with an interval in $\mathbb{R}$ around the appropriate integer which is homeomorphic to $U \cap C_1$ (since the latter misses a point of C_1 and any open subset of $\mathbb{R}$ is homeomorphic to an open subset of S^1). Thus p_1 is a local homeomorphism, and hence $\tilde{X}$ covers X.

Now we will construct a double-sheeted covering of this space $\tilde{X}$ that will result in a non-cover of the initial space X. It looks like this:



Here, I have taken two copies of $\tilde{X}$, with the following change: for each dotted circle in $\tilde{X}$, I have made that circle a dotted line connecting the two sheets. (We get one dotted line from each sheet, making a dotted loop passing through the two vertically aligned base points.) Now, construct a covering map p_2 from $\tilde{\tilde{X}}$ to $\tilde{X}$ as follows. Map both copies of $\mathbb{R}$ in $\tilde{\tilde{X}}$ onto $\mathbb{R}$ in $\tilde{X}$ via the identity, and map each X_n in $\tilde{\tilde{X}}$ to the copy of X_n directly above it in $\tilde{X}$, skipping only the dotted loop in $\tilde{X}$. Map the dotted lines in $\tilde{\tilde{X}}$ to the dotted loop in the same X_n in $\tilde{X}$. Thus each point in $\tilde{X}$ has exactly two preimages in $\tilde{\tilde{X}}$. This is a perfectly acceptable covering of $\tilde{X}$; for any point $x \in \tilde{X}$ which is not an integer in $\mathbb{R}$, a small open set containing x but not any preimages of the basepoint has the required covering space property, and for each basepoint n, there is an open neighborhood U around it that does not contain any other integers and which misses at least one point of the dotted loop in X_n; then it will have a preimage of two disjoint open neighborhoods in $\tilde{\tilde{X}}$. It is easy to see that these two pictures are homeomorphic (where the left is an open neighborhood around a base point in $\tilde{\tilde{X}}$, and the right is the corresponding open neighborhood around the base point in $\tilde{X}$):

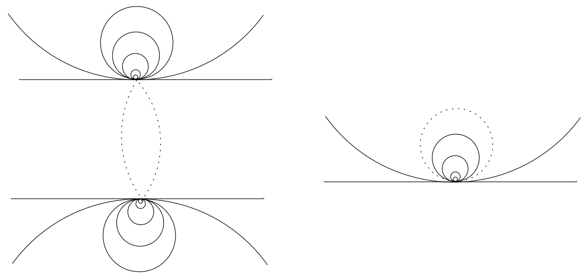




And hence, $\tilde{\tilde{X}}$ covers $\tilde{X}$.

Now look at the composition of these two covering maps, $p_1 \circ p_2$. I claim that there exists a point $x \in X$ such that for every open neighborhood U with $x \in U$, $(p_1 \circ p_2)^{-1}(U)$ not a disjoint union of open sets in $\tilde{\tilde{X}}$ such that $p_1 \circ p_2$ restricts to a local homeomorphism on each component. Unsurprisingly, the point x for which this fails will be the base point. Let U be any small open set including $x_0 \in X$. $p^{-1}(U)$ is a small open neighborhood around each of the base points in $\tilde{X}$. But given any collection of open neighborhoods including the base points in $\tilde{X}$, we can go far enough along $\mathbb{R}$ (i.e., take m large enough) to find a copy of X such that our open neighborhood $Y \subset \tilde{X}$ properly includes an entire dotted circle C_m (since the dotted circles become arbitrarily small as we go out to infinity in either direction).

But since the preimage of Y in $\tilde{X}$ contains all of the dotted loop in X_n, when we take the preimage $p_2^{-1}(Y)$ in $\tilde{\tilde{X}}$, we get a connected component, as shown, which includes two preimages of x_0 and therefore is not mapped homeomorphically to U by $(p_1 \circ p_2)$. In pictures, this says that the following two spaces are not homeomorphic:



Hence the composition $(p_1 \circ p_2)$ is not a covering map.

By the previous example, the composition of coverings is not necessarily a covering. But this example seems strange; are there naturality conditions that make the result hold? It turns out that there are, and to this end let us have a definition:

Definition 2.9. A space X is **semi-locally simply connected** if, for each $x \in X$, there is a neighborhood U such that the map $\pi_1(U, x) \to \pi_1(X, x)$ induced by the inclusion is trivial. That is to say, roughly, that every loop in U can be contracted in X.

Remark 2.10. It is easy to see that this fails miserably for the wedges in the previous example: any open neighborhood around the base point x_0 contains infinitely many little circles, none of which can be contracted in X.

Theorem 2.11. *Let X be a locally path-connected, semi-locally simply connected space, and let $p_1 : \tilde{X} \to X$ be a covering of X, and let $p_2 : \tilde{\tilde{X}} \to \tilde{X}$ be a covering of $\tilde{X}$. Then the composition $p_2 \circ p_1 : \tilde{\tilde{X}} \to X$ is a covering map.*

Proof. Take some $x \in X$, and take a neighborhood U such that $i_* : \pi_1(U, x) \to \pi_1(X, x)$ induces a trivial map. Now if $p_1 : \tilde{X} \to X$ is a covering map, we intend first to show that $p_1^{-1}(U)$ is a disjoint union of open sets C_α such that $p_1|_{C_\alpha}$ is a homeomorphism for all α.

To this end, consider the preimage $p_1^{-1}(U)$, and let C_α be any component of this preimage. I claim that $p|_{C_\alpha}$ is one to one: Let $x_1, x_2 \in C_\alpha$, and have $p_1(x_1) = p_1(x_2)$. Now, find some path f from x_1 to x_2 in C_α. The image $p_1 \circ f$ is some loop in U, which is in turn contractible in X. By lifting the contracting homotopy to $\tilde{X}$, we see (using, basically, the same method as in Theorem 2.6) that we must have $x_2 = x_1$. Now since p_1 is a covering map, it is a local homeomorphism; but since on C_α, p_1 is one-to-one, this implies that p_1 actually restricts to a homeomorphism everywhere on C_α. Therefore we must have that the open neighborhoods U around each x such that the inclusion $i_* : \pi_1(U, x) \to \pi_1(X, x)$ is trivial are also neighborhoods that satisfy the required properties of covering spaces.

Now we can proceed. Since $\tilde{X}$ covers X, any loop in any C_α must be contractible in $\tilde{X}$, because we can always just lift a contracting homotopy from its image in X.

Therefore the inclusion of C_α also induces the trivial map on homotopy groups. Thus if $p_2 : \tilde{\tilde{X}} \to \tilde{X}$ covers $\tilde{X}$, we can apply the same reasoning as above: the preimage under p_2 of each C_α must be a disjoint union of sets $C_{\alpha\beta}$ such that $p_2|_{C_{\alpha\beta}}$ is a homeomorphism to C_α. Then we just have a simple composition of homeomorphisms, $p_1 \circ p_2 : \tilde{\tilde{X}} \to X$. Then, given $x \in X$, there is an open neighborhood $U \subset X$ such that $x \in U$ and $(p_1 \circ p_2)^{-1}(x)$ is a disjoint collection of sets $C_{\alpha\beta} \subset \tilde{\tilde{X}}$ such that, for each α and β, $(p_1 \circ p_2)|_{C_{\alpha\beta}}$ is a homeomorphism. Therefore $p_1 \circ p_2$ is a covering map, and $\tilde{\tilde{X}}$ covers X. □

3. Cayley Complexes

A beautiful thing about covering space theory is that it gives us a very intuitive way to describe groups geometrically as graphs. First let us see that for every group G there is some cell complex that, roughly, corresponds to the group.

Definition 3.1. The **wedge sum** of two spaces X and Y, $X \bigvee Y$, is the quotient of the disjoint union $X \coprod Y$ obtained by identifying x_0 with y_0 to a single point.

Example 3.2. $S^1 \bigvee S^1$ is simply a figure eight—two circles, connected at one point.

Theorem 3.3. *For every group G there is a two-dimensional cell complex X_G with $\pi_1(X_G) \cong G$.*

Proof. Let us choose some presentation of $\langle g_\alpha | r_\beta \rangle$ of G. (This is just the group generators enumerated on one side, and all the group relations on the other. This presentation exists by the fact, unsubstantiated here, that every group is a quotient of a free group.) Now just take a wedge of circles $\bigvee_\alpha S^1_\alpha$, and attach two-cells e^2_β by the loops corresponding to words r_β. Thus for every generator $g \in G$, there is some S^1_α corresponding to that element with a fundamental group of $\mathbb{Z}$ corresponding to self-concatenating that basis element some n times—and then for each relation between the generators, we have a new two-cell e^2_β that connects basis elements. The fundamental group of this space, therefore, is just the basis elements S^1_α with connections at the relations e^2_β that bring us back to the identity. This fundamental group is by construction homeomorphic to G. □

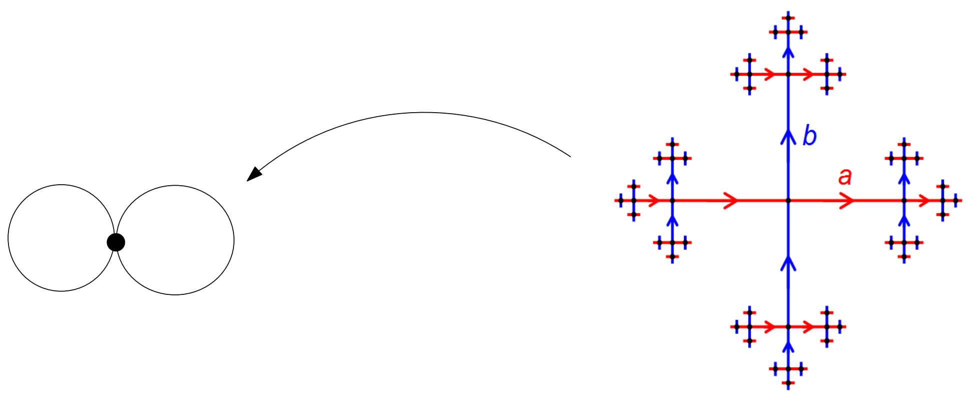
Definition 3.4. A **Cayley complex** is a cell complex $\tilde{X}_G$ covering X_G such that $\tilde{X}_G/G \cong X_G$, constructed as follows:

Let the vertices of $\tilde{X}_G$ be the elements of G themselves. Then, at each vertex $g \in G$, insert an edge joining g to gg_α for each of the chosen generators g_α. This graph is connected since every element of G is a product of g_α's, so there is a path in the graph joining each vertex to the identity vertex. Each r_α yields one loop in the graph, starting at any vertex g, and for each such loop, attach a two-cell.

For the covering map, take each vertex of $\tilde{X}_G$ to the basepoint of G, and have each edge corresponding to g_α wrap once around S^1_α in X_G. Take each two cell e^2_β in $\tilde{X_G}$ to the two cell in X_G whose attaching map is the image of the attaching map of e^2_β.

This is clearly a covering space of X_G, since at any vertex of $\tilde{X}_G$ we can find a small open neighborhood of that vertex which is homeomorphic to a small neighborhood of the basepoint of X_G.

Example 3.5. Let G be the free group on two generators, a and b. X_G is simply $S^1 \bigvee S^1$, and $\tilde{X}_G$ is the graph pictured below. It is a cover via the natural map which takes every vertex in the graph to the single vertex $S^1 \bigvee S^1$, and everything else to the edges coming out of the vertices. It is easy to see that this is a cover—locally, they look identical. For any path on $S^1 \bigvee S^1$, there is a path on the graph below that corresponds to it (going right or left on the graph corresponds to going around the right loop in $S^1 \bigvee S^1$, and going up or down on the graph corresponds to going around the left loop in $S^1 \bigvee S^1$).



This theory of covering spaces of graphs generated by groups comes in useful in proving things about groups themselves. For example, the next two results, which I shall state but not prove as a motivation for further study, give some idea of how the connections can be made:

Lemma 3.6. *Every covering space of a graph is also a graph, with the vertices and edges being all the lifts of the vertices and edges in the base graph.*

Theorem 3.7. *Every subgroup of a free group is free.*

References

[1] Farb, Benson. Class Notes. University of Chicago REU 2008.
[2] Hatcher, Allen. Algebraic Topology. Cambridge University Press 2002.