PLANE SYMMETRY GROUPS

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Abstract. This paper discusses plane symmetry groups, also known as planar crystallographic groups or wallpaper groups. The seventeen unique plane symmetry groups describe the symmetries found in two-dimensional patterns such as those found on weaving patterns, the work of the artist M.C. Escher, and of course wallpaper. We shall discuss the fundamental components and properties of plane symmetry groups.

Contents

1. What are Plane Symmetry Groups?

Since plane symmetry groups basically describe two-dimensional *images*, it is necessary to make sense of how such images can have isometries. An isometry is commonly understood as a distance- and shape-preserving map, but we must define isometries for this new context.

Definition 1.1. A *planar image* is a function $\Phi\colon \mathbb{R}^2 \longrightarrow \{c_1, \ldots, c_n\}$ where $c_1 \cdots c_n$ are colors.

Note that we must make a distinction between a planar image and an image of a function.

Definition 1.2. An *isometry* of a planar image Φ is an isometry $f\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ such that $\forall \vec{x} \in \mathbb{R}^2, \Phi(f(\vec{x})) = \Phi(\vec{x})$.

Definition 1.3. A *translation* in the plane is a function $T\colon \vec{x} \longrightarrow \vec{x} + \vec{v}$ where $\vec{v}$ is a vector in the plane.

Two translations $T_1\colon \vec{x} \longrightarrow \vec{x} + \vec{v_1}$ and $T_2\colon \vec{x} \longrightarrow \vec{x} + \vec{v_2}$ are linearly independent if $\vec{v_1}$ and $\vec{v_2}$ are linearly independent. Note also that $T^n(\vec{x}) = \vec{x} + n\vec{v}$.

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Definition 1.4. A *pattern* is a planar image that is invariant under two linearly independent *and* isometric translations in the plane. Furthermore, in both directions there is a translation of minimal length preserving the pattern.

The second part of the definition exists to exclude patterns of stripes. Thus, the patterns that we discuss are those that repeat in two different directions [6]. The pattern we define can also be referred to as a two dimensional repeating pattern or an ornament.

Example: Consider an infinite square grid made up of squares of side length s that share sides. In this case, the minimal linearly independent translations are the maps $T_x(\vec{x}) = \vec{x} + (s, 0)$ and $T_y(\vec{x}) = \vec{x} + (0, s)$.

Example: Imagine that the plane is tiled by black and white equilateral triangles with side length s and with one side parallel to the x-axis. Each triangle exclusively borders triangles of the opposite color. Suppose every point on the plane is translated to the right by s with $T_1(\vec{x}) = \vec{x} + (s, 0)$ and translated diagonally up and right by $T_2(\vec{x}) = \vec{x} + (s/2, s)$. White triangles map onto white triangles and black triangles map onto black triangles. Hence, we have a pattern. Note that the two translations are linearly independent but *not* orthogonal.

Definition 1.5. A *plane symmetry group* is a group of isometries that acts on a two-dimensional repeating pattern.

Thus, the plane symmetry group of a *particular* pattern is the group that includes *all* of the isometries that can act on that particular pattern (and the isometries can be considered equivalent up to affine transformation) . Hence, the set of patterns with symmetries described by a particular plane symmetry group form an equivalence class. The fact that there are only seventeen such equivalence classes becomes more interesting because all patterns can be catalogued so precisely.

Proposition 1.6. *The plane symmetry groups are, in fact, groups.*

Proof. Consider the set of isometries. First note that for an isometry *f*, the fact that $\Phi \circ f(x) = \Phi(x) \Rightarrow \Phi \circ f^{-1} \circ f(x) = \Phi \circ f^{-1}(x) \Rightarrow \Phi(x) = \Phi \circ f^{-1}(x)$shows us that the inverse of an isometry is an isometry. The group operation is composition of isometries. The identity is the identity function. The inverse of a composition of isometries $f_n \circ \cdots \circ f_1$ is $f_1^{-1} \circ \cdots \circ f_n^{-1}$. Since the image of every isometry is the same, it is clear that this set is closed under the composition of operations. □

2. The Lattice

Definition 2.1. A *lattice* is a set of points that is generated by the two translations in the plane symmetry group. In other words, the lattice is the set $\mathcal{L}$ built upon a point $\vec{x}$ in $\mathbb{R}^2$ such that $\mathcal{L} = \{(T_1^n + T_2^m)(\vec{x}) | n, m \in \mathbb{Z}\}$.

Some texts define a *lattice group* to be a group $\langle X \rangle$ generated using T_1 and T_2 [3]. Note that a lattice can be built upon any point $x \in \mathbb{R}^2$.

Definition 2.2. The *fundamental domain* of the underlying pair of translations T_1 and T_2 is the smallest parallelogram whose corners are points of the lattice [6].

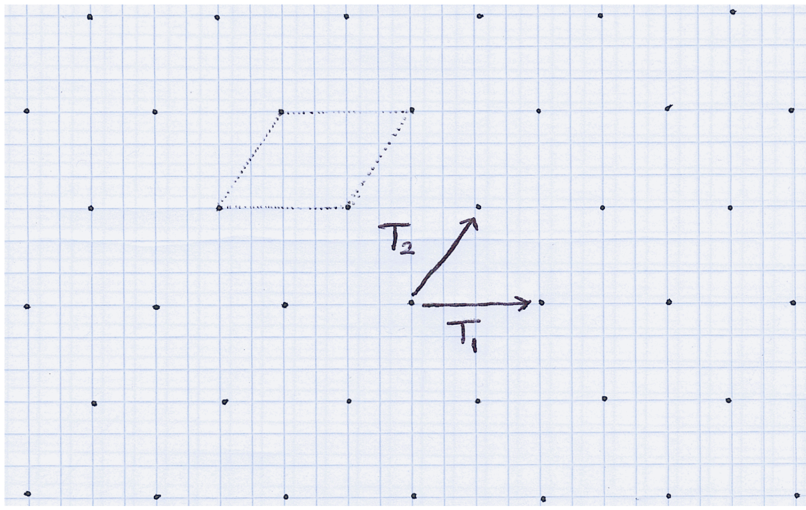

FIGURE 1. The Lattice

Consider the lattice $\mathcal{L}_0$ which is built upon the $\vec{0}$ vector. Since $\mathcal{L}_0$ is a normal subgroup of $(\mathbb{R}^2, +)$, we can define cosets of $\mathcal{L}_0$ for $\vec{x} \in \mathbb{R}^2$ as the sets $\vec{x}\mathcal{L}_0 = \{x + \vec{l} | \vec{l} \in \mathcal{L}_0\}$. It follows by the first isomorphism theorem there exists a group $\mathbb{R}^2/\mathcal{L}$ [2]. Thus, the cosets of $\mathcal{L}$ in $\mathbb{R}^2$ are equivalent points in the parallelograms that cover the plane.

We choose the lattice centered at the zero vector because it divides conveniently, but when we consider an infinite two-dimensional pattern, we must keep in mind that a fundamental domain can be chosen starting from any point as long as it has the proper parallelogram outline. Thus, a given fundamental domain is just a suitable shape to conceive as a building block of the pattern.

Example: Recall the infinite grid. A fundamental domain of this pattern is a single $s \times s$ square.

Example: A fundamental domain of the black and white triangle pattern mentioned in the first section is the parallelogram formed by one black triangle and one adjacent white triangle.

It appears that the fundamental domain is thus invariant under the translations T_1 and T_2. When we consider applying any transformation to a pattern, we can imagine what that transformation would do to the fundamental domain in that pattern. We understand the isometries by their operations on the various points within the fundamental domain. Thus, what we prove about lattices allows us to better understand planar symmetry groups.

3. The Components of Plane Symmetry Groups

The first kind of planar isometry is a *translation*, which is defined above. All patterns have translation isometries acting on them by definition, whereas that is not necessarily the case with the other types of isometries. With regard to translations, it is clear that the only possible translations of a planar symmetry group are the translations T_1 and T_2 that define the lattice of the pattern. This is because all of the isometric translations are multiples of each minimal translation. Since the function $\phi((nT_1, mT_2)(\vec{x})) = (n, m)$ is bijective, we know that $T_\Phi \cong (\mathbb{Z} \times \mathbb{Z}, +)$.

A *rotation* is a turns points by some angle around a central invariant point. Since each rotation should divide a complete rotation, the angle of rotation should always be $2\pi/n$ for some $n \in \mathbb{N}$. If the rotation in question is isometric, then multiples of that rotation are also isometric. In fact, the set of rotations that are multiples of $2\pi/n$ is isomorphic to $\mathbb{Z}$ mod $n \cong C_n$. Thus, we can refer to a rotation by the order of its cyclic group.

Note that a center of rotation of order n is also a center of rotation for all of the divisors of n. This is because $n|m \Rightarrow C_n \leq C_m$ [2]. Furthermore, a planar symmetry group may have multiple centers of rotation of different orders, but they must be divisible by either $\pi/3$ or $\pi/4$ for reasons discussed in section 6.

Example: Imagine an infinite chess board with black and white squares of side length s that cover the entire plane. The board has rotational symmetry of order 4 in the middle of each square. It also has rotational symmetry of order 2 on the corners where the squares meet.

Example: The black and white triangle pattern has symmetry or order 3 in the centers of the triangles and at the corners where the triangles meet.

A *reflection* flips points over a central invariant axis or line, and a *glide reflection* is a reflection composed with a translation. If a reflection is composed with a translation that is a multiple of one of the translations composing the lattice, then the resulting glide reflection is trivial, since it is equivalent to a reflection. Thus, the difference lies in the fact that a glide reflection is a staggered reflection.

Example: The infinite grid has reflective symmetry along the sides of its squares.

Example: Picture the fundamental domain of the pattern of black and white triangles, and consider the line of height s (the same measure as the sides of the triangles) that cuts the parallelogram in half (this line would be a $3/4 \times s$ distance from either of the far corners). This is an axis of glide reflection. An isometry is created by reflecting along this axis and composing it with the upward translation $T(\vec{x}) = \vec{x} + (0, s)$.

Note that the four above types of transformations are not isometries in *all* cases. Whether or not they are isometries depends on the crystallographic group in question and how they are applied (i.e., depending on the length of the translation, the center of rotation, and the axis of reflection or glide reflection).

Proposition 3.1. *There are exactly four isometries in the plane: translation, rotation, reflection, and glide reflection.*

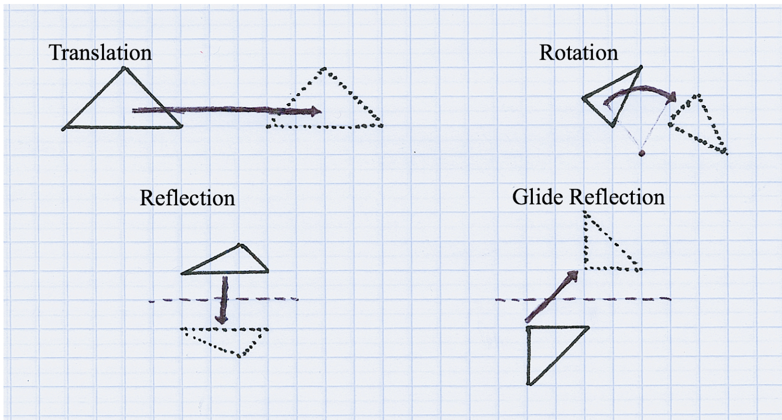

FIGURE 2. The Four Isometries

Proofs of this proposition are found in [1] and [3]. They argue that an isometry that preserves orientation is a rotation if it leaves an invariant point and a translation if it leaves no invariant point. An isometry that does not preserve orientation is a reflection if it leaves invariant points and a glide reflection if it does not. This proposition allows us to understand that these transformations completely describe plane symmetry groups.

There is an important restriction with regard to reflections:

Proposition 3.2. *Planar symmetry groups must exemplify one of the following:*
I) Alternating axes of reflection and glide reflection in one or both translational directions.
II) An axis of reflection in one direction an axis glide reflection in the other.
III) Only axes of glide reflections, in one or both directions.

The proof of this proposition is found in [3]. It is mainly a result of the fact that reflections and non-trivial glide reflections are mutually exclusive.

4. Generating Regions

Because of the multitude of members of some plane symmetry groups, it is possible for most (except the trivial one) to have a subregion that is smaller than the unit cell that, under repeating application of the isometries of the group, can be used to generate the entire pattern.

Definition 4.1. Let Φ be a planar image and let G be the planar symmetry group of that image. A *generating region* for a planar image Φ is a convex set $\Gamma \subset \mathbb{R}^2$ such that $\forall x \in \mathbb{R}^2, \exists y \in \Gamma$ and a composition of isometries $f_1 \circ \cdots \circ f_n$ in G mapping y to x.

Note that the concept of a generating region is *not* the same as that of a generating set of a group.

The lattice unit is clearly always a generating region. And interesting property of some patterns is that same have generating regions that are in fact smaller than their lattice units. Observe figure 3, in which we begin with a generating region (1) and build it up through a series of reflections (2,3, and 4). We thus create our

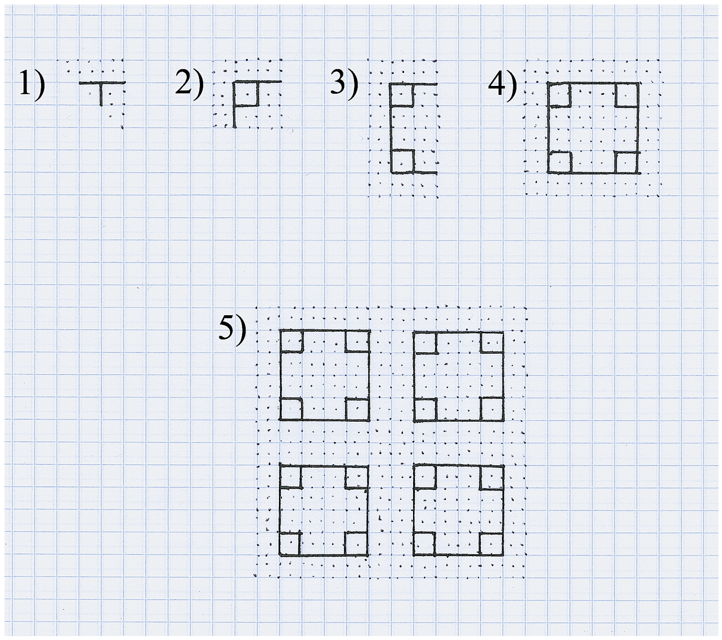

Figure 3. Generating Region

fundamental domain (4) which in turn can be used to create the pattern as a whole through translations (5, etc.).

5. The Crystallographic Restriction

The crystallographic restriction is the simplest and most important theorem that describes the limitations of planar symmetry groups.

Theorem 5.1. *The only possible orders of rotation for a lattice of points are 2,3,4, and 6.*

In other words, the only possible rotations are multiples of 60 or 90 degrees [3]. The following proof is found in [1]. We use simple plane geometry for the proof.

Proof. It is easy to conceive of lattices with orders of rotations of 2,3,4, or 6. Thus, it is only necessary to prove that a rotation of order of 5 or of an order greater than 6 is impossible.

We choose an arbitrary center of rotation A of order n. Since we have some sort of lattice, let B be another center of rotation on the lattice of minimal distance from A. Since all of the points on the lattice are identical with respect to rotational symmetry, we know that B has rotational symmetry of order n.

Hence, we can find a point A' by a $2\pi/n$ rotation around B and another point B' by a $2\pi/n$ rotation around A'. Because these transformations are isometries, we know that $AB = A'B = A'B' = L$. The question that remains is the relation between AB, AB' and AA', which we will denote as X and Y respectively. But since B was chosen to be the minimal lattice distance from A, it follows that $AB \leq AB'$ and $AB \leq AA'$ or rather that $L \leq X$ and $L \leq Y$.

If $n = 5$, then, A, B, A', and B' form a trapezoid $ABA'B'$. Since $2\pi/5 < 2\pi/4$,

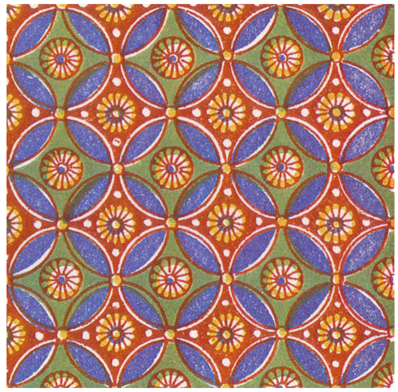

FIGURE 4. Egyptian Mummy Case

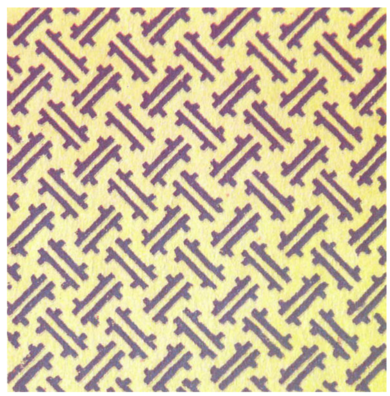

FIGURE 5. Painting, China

we have: $X = A'B(1-2cos2\pi/5) < A'B(1-2cos2\pi/4) = A'B = L$. Thus, we have $L > X$, a contradiction of minimality, and the contradiction proves that rotational symmetry of order 5 is impossible.

Now if $n > 6$, then we employ the law of cosines: $c^2 = a^2 + b^2 - 2ab\cos 2\pi/n$. Since ABA' is an isosceles triangle, we rewrite it as $Y = \sqrt{2L^2(1-\cos 2\pi/n)} < \sqrt{2L^2(1-\cos \pi/3)} = L$. Again we have a contradiction, and it is clear that it is impossible for a lattice to have rotational symmetry of order $n > 6$.

$\square$

Although we do not include the full enumeration of the 17 unique planar symmetry groups, it is worth noting that we can conceive this limitation through the restrictions that we have encountered.

6. Some Illustrated Examples

Example: The pattern displayed in figure 4 has a square fundamental domain. We can picture the square's corners on the blue dots in the wheels that rest on the green background. The symmetries of this pattern are exactly the same as those of the infinite grid. In other words, both patterns have the same plane symmetry

FIGURE 6. Persian Ornament

group, even though they do not appear similar at first.

Example: The pattern that appears in figure 5 is a good example of a glide reflection isometry. The pattern is essentially made up of one repeated figure that is aligned in two different ways. If the image is reflected across a horizontal line that cuts a row of the figures in half, then the result is that each figure is aligned in the opposite manner (this effectively rotates all of the figures by $\pi/2$). The pattern can then be transposed so that it is identical to its original form. Note that the same symmetry occurs across a vertical axis.

Example: The pattern from figure 6 is interesting because it has no reflective or glide-reflective symmetry. It does, however, have rotational symmetry of order 6 at the centers of the hexagon-encased flowers, of order 3 at the centers of the grey triangles, and of order 2 at the midpoints between the hexagons.

The examples in this section come from [4].

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