Maximum Knotted Switching Sequences

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Abstract

In this paper, we define the maximum knotted switching sequence length of a knot projection and investigate some of its properties. In particular, we examine the effects of Reidemeister and flype moves on maximum knotted switching sequences, show that every knot has projections having a full sequence, and show that all minimal alternating projections of a knot have the same length maximum knotted switching sequence.

Contents

1 Introduction

1.1 Basic definitions

A *knot* is a smooth embedding of S^1 into $\mathbb{R}^3$ or S^3. A knot is said to be an *unknot* if it is isotopic to $S^1 \subset \mathbb{R}^3$. Otherwise it is *knotted.* Knots are often studied by looking at projections of the knot onto $\mathbb{R}^2$. No more than two strands of a knot may be projected onto any point, and wherever there is such a crossing, it is represented in the following manner, with the break in the understrand:

Figure 1: A crossing. The overstrand is solid, the understrand is broken.

A *link* is a collection of disjoint smooth embeddings of S^1 into $\mathbb{R}^3$ or S^3. The number of embeddings in a link is the number of *components* of the link. If some components of a link can be separated by a plane from the others after isotopy, it is called a *split link.* A knot is a one component link.

Knots are frequently considered with *orientation.* That is, the strand of the knot is given one of two directions, indicated by arrows along the strand. Crossings can be either right-handed crossings, or left-handed crossings with respect to the current orientation of a projection. The difference is demonstrated below. In a knot, with one component, reversing the orientation does not alter the handedness of its crossings. However, with a link, each component receives its own independent orientation, so changes to the orientation of components can change the handedness of crossings.

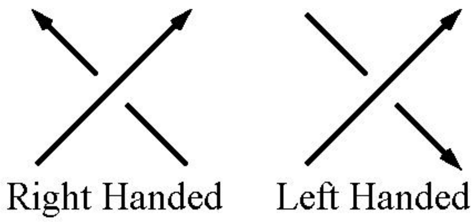


Figure 2: A right handed crossing and a left handed crossing.

Another structure frequently considered is a *tangle.* A n–tangle is n disjoint embeddings of the unit interval into $\mathbb{R}^3$, so that they are contained in a 3–ball, each embedding begins and ends on the boundary of that 3–ball, and except for those beginning and ending points, the embeddings are contained in the interior of the 3–ball. Also, any number of disjoint embeddings of S^1 may be in the interior of the 3-ball. In practice, we allow deformed boundaries.

When projecting a tangle, it is standard to isotop all the endpoints of the embeddings to the 2–sphere parallel to the plane being projected to, so that the projection of that 2–sphere acts as a boundary in the plane in the same way the

3–sphere did in $\mathbb{R}^3$. Again, it is equivalent to use any Jordan closed curve as the boundary.

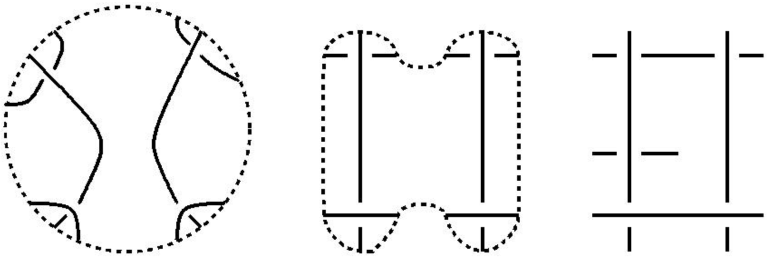


Figure 3: A projection of a tangle; The same tangle with non-circle boundary; Not a tangle, no boundary can exist without intersecting with a non-endpoint — note that it is not a tangle even though it is isotopic to one.

1.2 Changing knot projections

When studying knot projections it is important to know what changes can be made without changing the knot itself, considered up to isotopy in $\mathbb{R}^3$. An important and constantly used result is that the three Reidemeister moves, shown below in Figure 2, along with isotopies of the projection in $\mathbb{R}^2$ never change a knot, and also allow all representations of a knot.

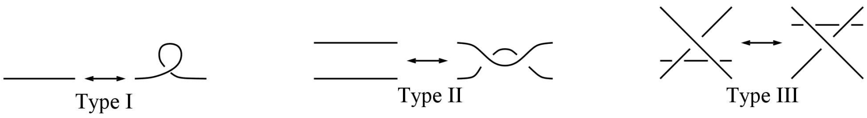


Figure 4: The three Reidemeister moves.

If we do not care to preserve the knot, there are other changes we may perform on the knot projection. One is to switch the understrand and overstrand of a crossing. This frequently results in a different knot, although not always, as shown below. From here, I will refer to this as *switching* the crossing.

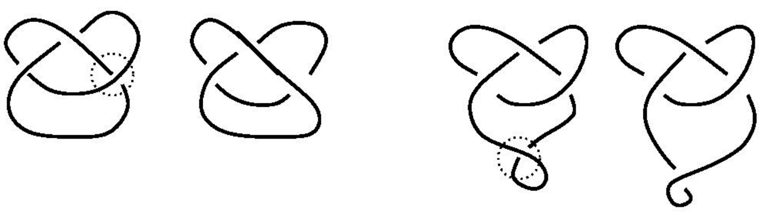


Figure 5: Switching the marked crossing of a trefoil; switching the new crossing after a Type I move.

The effects of switching moves will be the focus of this paper, specifically, how many such crossing switches can be made while still remaining knotted. Let us define this question more precisely. Given any n–crossing projection of a knotted knot, switching every crossing will obviously produce a knotted knot, the mirror of the original knot. However, this will not be true if we require crossing switches to be made one at a time in sequence. For example, any one crossing switch made to the standard projection of the trefoil will result in the unknot.

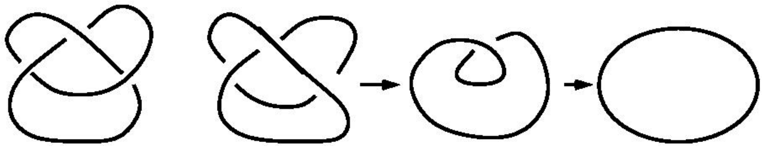


Figure 6: A trefoil; trefoil with switched crossing.

We consider sequences of crossing switches on a knotted knot projection, requiring that after each switch the resulting projection be knotted. Also, we will restrict the same crossing from being switched more than once in this sequence of crossing switches. [1] We call such a sequence a *knotted switching sequence*. We are specifically looking for a projection's *maximum knotted switching sequences* and their length, where a maximum sequence is defined to be one of maximal length. We call a knotted sequence that switches every crossing of a knot projection a *full sequence*.

2 Maximum knotted switching sequences

This section contains a number of quick results, the most significant of which are that all knots have a projection with a full sequence, and that there are an infinite number of knots with projections not having a full sequence.

2.1 Immediate bounds

We can immediately give simple bounds on this. For any n–crossing projection of a knot, the length of any maximum knotted switching sequence will be less than or equal to n.

The *unknotting number* of a knot is the minimum number of crossings that need to be switched to produce the unknot, considered over all projections. So the length of a maximum knotted switching sequence of any projection of a knot must be at least one less than the knot's unknotting number.

Also, if a projection of a knot with n crossings has a knotted switching sequence of length $n-1$, it has a full sequence. After $n-1$ switches, exactly one crossing will be unswitched, and as the original knot must have been knotted, its mirror will be as well, so that last crossing may also be switched, creating a full sequence.

[1]If we do not, then once any crossing can be switched an infinite sequence can be created by switching that crossing back and forth indefinitely.

2.2 Not all maximal knotted switching sequences are maximum

We are looking for the maximum knotted switching sequences of a knot projection. A maximal knotted switching sequence, one which cannot be increased in length by additional crossing switches, is much easier to find. We just switch legal crossings arbitrarily until no further crossing can be switched without unknotting the projection. However, as the example below demonstrates, this simple process need not create a maximum sequence.

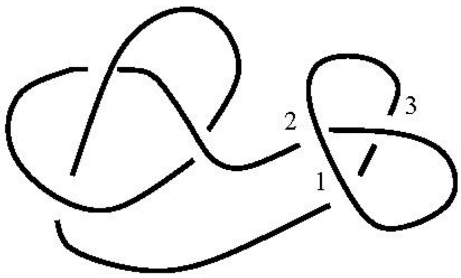


Figure 7: A maximal knotted switching sequence on a projection of the trefoil. A full sequence exists, as demonstrated in Section 2.4.

2.3 Composite knots

Given two oriented knots one can compose them by slicing each and attaching the open ends together, in the manner shown below.

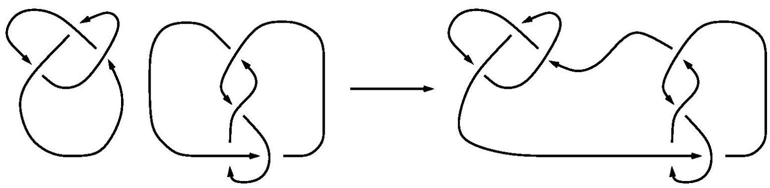


Figure 8: A trefoil being composed with a figure eight knot.

Composing any projection of the unknot with a knot is the identity operation. Also, if two knots may be composed to create the unknot, neither is knotted. This allows us to immediately show a full sequence for at least one projection of any composite knot.

Given a composite knot made by composing knotted knots K_1 and K_2, the knot can be projected in the manner below. Then we can switch every crossing in the K_1 component in any order. As K_2 is knotted, any composite knot formed by composing a knot (here a partially switched projection of K_1 which may be unknotted) with K_2 is knotted. When we have switched every crossing in K_1, we have the mirror of K_1, which is knotted. Then we can switch every crossing in K_2 in the same manner. Therefore the projection has a full sequence.

Conjecture 2.1. *Every projection of a composite knot has a full sequence.*

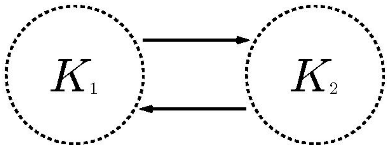


Figure 9: A composite made of knots K_1 and K_2.

2.4 All knots have a projection with a full sequence

Given any knot K, perform the sequence of Reidemeister moves shown below. The final projection has a full sequence. After switching crossing A, the knot will be a composite knot of the trefoil and K. So as above, we can switch every crossing of K while staying knotted. Then, as we have reached the mirror of K we can unknot the trefoil by switching B and C, while staying knotted. This is a full sequence.

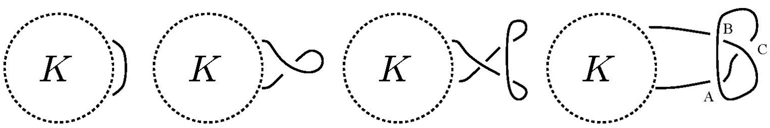


Figure 10: A construction of a projection with full sequence for any knot K.

2.5 An infinite family of knots with projections not having full sequences

So far we have seen many example of projections with full sequences. In fact they seem to occur whenever a knot has two separate "knottable structures." For example, in Figure 10 we add a knottable structure to the knot K — the crossings A B and C can be knotted by a crossing switch.

Conjecture 2.2. *Among prime knots, the frequency of knots with a projection without a full sequence approaches zero as the crossing number of the knots considered increases.*

However, there are knots with projections not having a full sequence at arbitrary crossing number. Figure 11 shows an infinite family of such knots. m is the number of crossing in the bottom twisting section, so this projection of the knot always has $m + 2$ crossings. Clearly, switching either of the top two crossings will unknot the projection, so the maximum knotted switching sequence length for each must be less than or equal to m.

We can do better than that. The effects of switching a twist crossing is equivalent for all of them: you remove two twist crossings from the reduced figure. If m is even, then the $m/2$th switch unknots the projection, so the maximum knotted switching sequence length of the projection is $m/2 - 1$. For m odd, the $\lfloor m/2 \rfloor - 1$th crossing switch creates a trefoil, one twist crossing left at the bottom. The next crossing switch reverses the handedness of the crossing left after Type II moves remove the rest, creating the unknot. So there again, the maximum sequence length is $\lfloor m/2 \rfloor - 1$.

Conjecture 2.3. *For all n,* $\lfloor (n-2)/2 \rfloor - 1$ *is the lowest possible maximum knotted switching sequence length of a knotted knot projection with n crossings.*

Remark 2.4. Note that the below family of knot projections is an example of this minimum. The conjecture implies, for example, that all knot projections with at least six crossings have at least one crossing that may be switched without unknotting the projection.

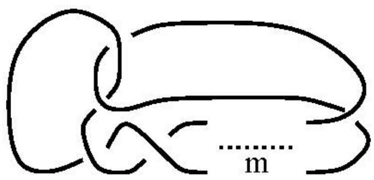


Figure 11: An infinite family of knots created by two hooked loops separated by a section of twists containing m crossings. Note that neither of the top two crossings can be switched without unknotting the projection.

3 The effects of Reidemeister moves on the maximum knotted switching sequence length of a knot projection

In this section, we will completely describe the action of a Type I Reidemeister move on the length of the maximum knotted switching sequences of a knot projection. We then conjecture the effects of a Type II Reidemeister move, and restrict the possible counterexamples. Last we give examples of the disparate effects of Type III Reidemeister moves.

3.1 Type I Reidemeister moves

As we are investigating a property of knot projections, the effects of the three Reidemeister moves on the maximum knotted sequence length must be investigated.

The first Reidemeister move acts predictably. If you add a loop to a knot projection (assuming that it is knotted) the knot's maximum sequence will increase by one move: whatever the sequence was, you can now first switch the new crossing, since that produces the same knot (by two Type I moves) and so at every stage of the original sequence the new knot will still be equivalent to the original knot after the sequence of crossing changes. Similarly, when you remove a loop with a Type I move, you must be reducing the length of all maximum knotted sequences of the knot by one.

Thus, for any knot we can create a projection of that knot with an arbitrarily large knotted switching sequence, through the use of Type I Reidemeister moves.

Remark 3.1. If we choose a knot projection without a full sequence for K, we can modify it as in Figure 12 to have an arbitrarily large number of crossings without having a full sequence.

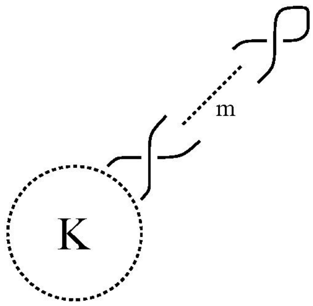


Figure 12: A knot with at least m switchable crossings.

3.2 Type II Reidemeister moves

The second Reidemeister move is less obvious. We consider the question of whether, if the original knot projection is knotted, at least one of the two new crossings may be switched to produce a knotted projection. If this is always possible then, as the original projection was knotted, we may switch the other new crossing and continue on with whatever the original projection's maximum sequence was. Thus such a Type II Reidemeister move would add at least two to the length of a projection's maximum knotted switching sequence. Similarly, removing crossings with a Type II move would subtract at least two from the length of a projection's maximum sequence.

There is some difficulty here, as it is possible to unknot a projection with a Type II move and the wrong crossing switch, as shown below.

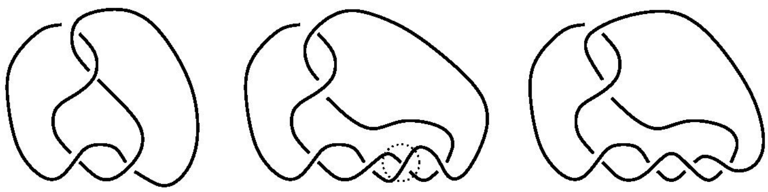


Figure 13: A knot being unknotted by a Type II Reidemeister move and a crossing switch.

It is also possible that by adding the two crossings we might allow a new sequence, one in which a crossing that could not be switched before may be switched. So the second Reidemeister move does not act as predictably as the first.

For example, no crossings can be switched in standard projection of the trefoil. After performing the Type II move below, two crossings can be switched (the intermediate knot is a presentation of 5_2). However, if a Type I move is added first, creating a one switch projection, the Type II move creates a projection with a full sequence, as indicated. So in this case, the Type II move increases the maximum knotted switching sequence length by five.

Conjecture 3.2. *A Type II Reidemeister move will add or subtract at least two from the length of a projection's maximum knotted switching sequence.*

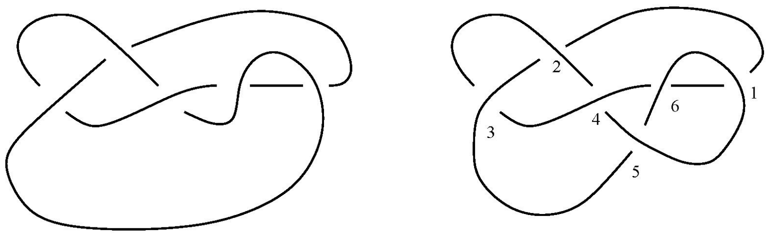


Figure 14: A trefoil after a Type II move; A trefoil after a Type I and two move, with full sequence.

We can only fully prove this for *alternating* knots, which we shall do in Section 3.3. In Section 3.4 we will further winnow the set of possible counterexamples.

Furthermore, if a Type II move has increased the length of a projection's maximum knotted switching sequences by more than two, it seems likely that it has added a new knottable structure to the projection, a situation analogous to that of Figure 10 in section 2.4.

Conjecture 3.3. *A Type II Reidemeister move will either add or subtract two from the length of a projection's maximum knotted switching sequence, or it will create or remove a full sequence.*

3.3 Alternating knots and Type II Reidemeister moves

It will be easier to discuss the effects of Type II Reidemeister moves on alternating knots, so we will begin there.

Definition 3.4. An alternating projection is one in which no strand has two overcrossings or undercrossings consecutively.

Definition 3.5. An alternating knot is one which has an alternating projection.

There is a powerful result about alternating knots that we will use: if an alternating projection is *reduced*, by which we mean that there is no obvious crossing that may be removed, then the projection is minimal, i.e. that knot will have no projections with a fewer number of crossings [3]. This also directly implies that the projection is knotted.

More formally, a *reduced* knot is one without any *nugatory* crossings. A *nugatory* crossing is one through which one can draw a Jordan closed curve which splits the knot projection so that some of it is in the interior and some in the exterior, and the curve only passes through the projection at the nugatory crossing, as shown in Figure 15.

Theorem 3.6. *Given any alternating knot projection, and any possible Type II Reidemeister move on that projection, one of the two new crossings may be switched so that the resulting knot projection is alternating.*

Proof. Consider the two strands to be crossed, and the previous crossings along each strand, specifically whether the strand we are following was the overstrand or understrand in that crossing. There are three cases to consider, up to rotation

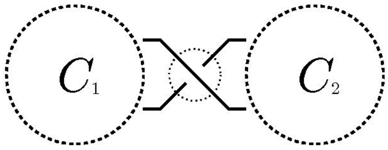


Figure 15: A *nugatory* crossing.

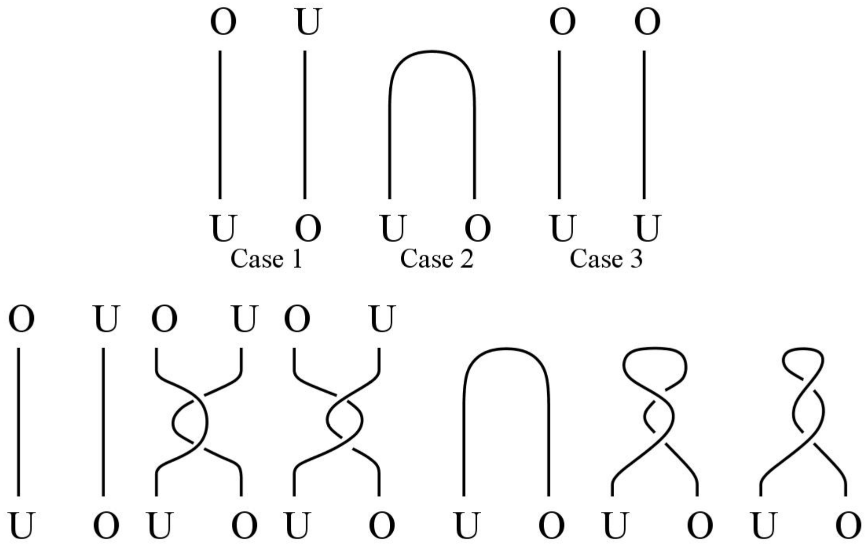


Figure 16: The three cases, with undercrossings marked U and overcrossings O, and demonstrations that after we perform a Type II Reidemeister move on either of the first two cases, one of the crossings may be switched to form an alternating projection.

and switching every crossing in the tangle, once we remove those that are non-alternating, unknotted, or with more than one component.

We must look more specifically at the third case, which cannot result in an alternating projection. To do this, we look at possible tangles formed by the two strands to be crossed and the previous crossings along each. There are three possible tangles, shown below in Figure 17, again up to rotation and switching every crossing in the tangle, and once we remove those that are non-alternating, unknotted, or with more than one component. The rest of the proof is to show that these cases are impossible in a single component alternating projection.

For the purposes of this discussion, we will refer to the undefined crossing exits as *open ends*. As we see in Figure 17, each crossing has either two or three open ends. It is important whether the strand the open end is associated with on the crossing is an overcrossing or an undercrossing.

Note that no strands may cross between the two strands we wish to push over each other with the Type II Reidemeister move. In such a configuration, the Type II move would not be valid. Also, no other strands may cross the strands of the tangle between the crossings shown — those are defined to be the nearest crossings to the Type II move.

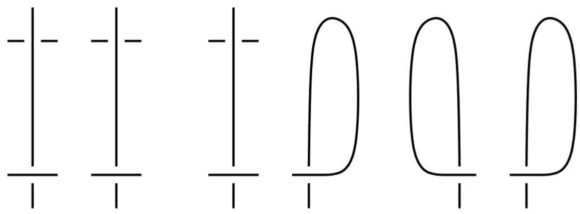


Figure 17: The impossible cases.

Consider the first case:

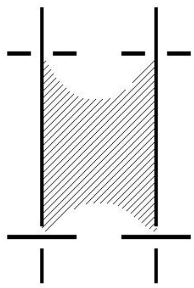


Figure 18: The first case to consider. Note that the crosshatched region cannot be crossed by strands.

Assume that there is an alternating knot projection where this case occurs. The knot projection would be the union of the two parts shown in Figure 18 and six connecting strands. As the projection is of a knot, it will have one component, so there must be some connecting strand between Part 1 and Part 2. Call this strand S.

We now consider the open ends on our tangle are connected together by S:

($*$) If similar open ends of this diagram are connected, i.e. undercrossing to undercrossing or overcrossing to overcrossing, there must be an odd number of crossings over the connecting strand in order for the knot projection to be alternating. If dissimilar ends are connected, then the connecting strand must have an even number of crossings along it.

Note that we may ignore the strand crossing over itself: while this can change the number of crossings along the strand, following the strand there will be an overcrossing and an undercrossing each time the strand crosses itself, therefore this will have no effect on the number of other crossings needed.

If we connect two open ends on different parts of the tangle with a strand, we can use it to define two sections of the diagram. We say that the region of $\mathbb{R}^2$ between the strand we have drawn and the inviolate region between the original strands, together with the inviolate region, is the *interior* of our new strand. Everything else is the *exterior* of the strand. Note that the interior can be covered by a compact set, while the exterior cannot be.

If S connects similar open ends on Part 1 and Part 2, there are an even

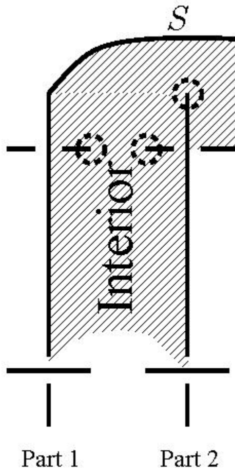


Figure 19: The interior of a connecting strand, with the interior open ends marked.

number of open ends in the interior. If S connects dissimilar open ends on Part 1 and Part 2, there are an odd number of open ends in the interior.

If two other open ends are to be connected, the connecting strand may cross S. As S is the only part of the boundary between the interior and exterior of the diagram that may be crossed, it is obvious that if an open end in the interior, and an open end in the exterior are to be joined, their connecting strand must cross S an odd number of times. If two open ends both in the interior, or both in the exterior are to be connected, the connection will cross S an even number of times.

So, if similar ends are connected by S there are an even number of open ends in the interior, and there will be an even number of crossings over S. If dissimilar ends are connected by S then there are an odd number of open ends in the interior, and there will be an odd number of crossings over S. But then our knot projection would not be alternating, by $(*)$, so S cannot exist.

Therefore no end from Part 1 connects next to Part 2. So the projection must have two components. Contradiction. Therefore this case cannot arise.

The other three cases of Figure 17 can be shown to be impossible by the same argument. All that is needed is to redefine the inviolate region, and so the interior of any connecting strands, for each case. We do so as in Figure 20.

None of these three cases can arise in a single component, knotted, alternating projection, and we have dealt with all other cases.

□

Corollary 3.7. *If we have a knotted alternating projection of a knot, and we perform a Type II Reidemeister move, then we will have increased the length of the projection's maximum knotted switching sequence by at least two.*

Sketch of Proof: This holds because the intermediate projection (after one of the two new crossings introduced by the Type II move has been switched to make an alternating projection) will always be knotted, as will be shown by the following. If we divide our our original alternating projection into nugatory

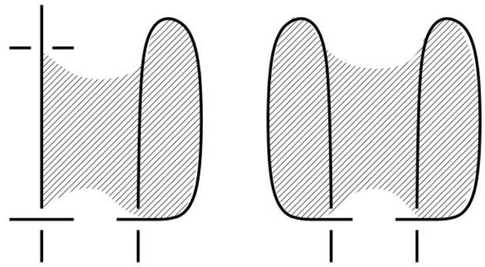


Figure 20: The other two impossible cases, with the inviolable regions shaded.

crossings and the disjoint reduced components, we will either (1) have added two more nugatory crossings, so that our reduced form of our new projection is the same of that of the original projection, or (2) added two more crossings within one of the reduced components, so that the new crossings cannot be nugatory as they will occur within one of the components that were reduced, or (3) connected two pieces of the original projection together in which case there are strands above and below our new crossings which would have to be crossed by any Jordan closed curve we attempted to draw through our new crossings, so that they cannot be nugatory. In any case, there are non-nugatory crossings in the new projection, and so it will not reduce to the unknot.

3.4 The Conway polynomial and Type II Reidemeister moves

The Conway polynomial is an invariant link polynomial of one variable of the form $\nabla = \sum_{i=0}^{d} \nabla_i z^i$ with integer coefficients and indeterminate z satisfying the following two rules:

Definitions 3.8.
1. The Conway polynomial of any projection of the unknot is 1.
2. The Conway polynomial satisfies the following skein relation:

$$\nabla(\times) = \nabla(\times) + z\nabla(\,)(\,)$$

That is, the Conway polynomial of a link projection is associated with the Conway polynomials of similar projections with changes to one of the crossings.

Remark 3.9. While calculation of the Conway polynomial is defined for a link projection, it is invariant under Reidemeister moves, and so all projections of a link have the same Conway polynomial. It is therefore an invariant of the links themselves [2].

As it is an invariant polynomial, each coefficient of the polynomial is also a link invariant, and the first two coefficients can be understood in the following manner [2]:

$$\nabla_0(L) = \begin{cases} 1 & \text{if } L \text{ has exactly one component} \\ 0 & \text{otherwise} \end{cases} \tag{10}$$

$$\nabla_1(L) = \begin{cases} lk(L) & \text{if } L \text{ has exactly two components} \\ 0 & \text{otherwise} \end{cases} \tag{11}$$

Definition 3.12. The linking number *lk* of two components of a link is the sum over all crossings between the two components of +1 if the crossing is right handed, and −1 if the crossing is left handed.

A result reached by an easy calculation is that the Conway polynomial of any split link (see Section 1.1) is 0.

With these tools, we can approach the same problem of the effects of Type II Reidemeister moves on the length of maximum knotted switching sequences, for all knots.

Theorem 3.13. *Given any projection of a knotted knot, and a possible Type II Reidemeister move in which both strands have the same orientation, at least one crossing of the resulting projection may be switched to form a knotted knot.*

Proof. Assume that this is not true. So for some knotted knot projection, after such a Type II move, switching either crossing results in the unknot.

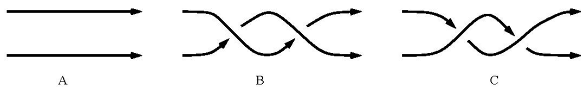
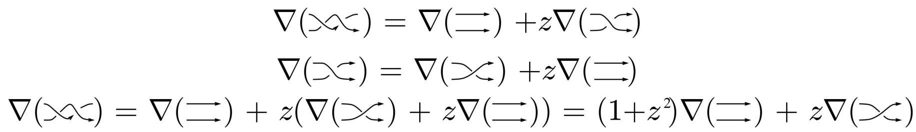
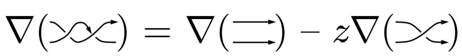


Figure 21: A before a Type II move, knotted; B right hand double twist, unknotted; C left hand double twist, unknotted.

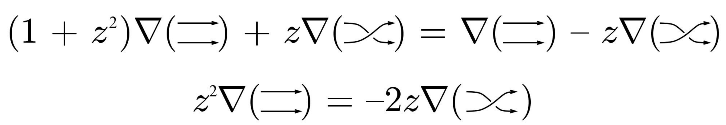
The Conway polynomial of the unknot is 1, so the Conway polynomials of the two projections resulting from the crossing switches must also be 1, and importantly they must be equal. First, let us examine the Conway polynomial of figure B:



Now figure C:



Setting the two polynomials to be equal, we have:

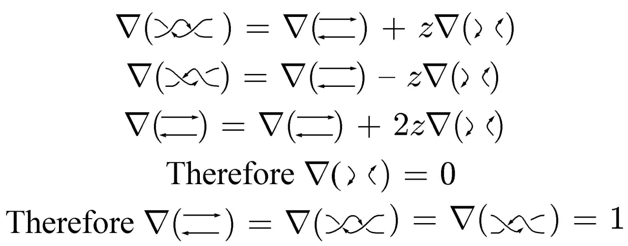


As the original projection is a knot, and thus has one component, $\nabla(\;)$ has constant term 1. Therefore $z^2\nabla(\;)$ has degree two term z^2. But as the coefficients of $\nabla(\;)$ must be integers, the coefficients of $z^2\nabla(\;)$ must be divisible by 2. Contradiction.

□

So now we have that any Type II Reidemeister move on strands of the same orientation will increase or decrease the maximum knotted switching sequence length of a knot projection by at least two. Because the Conway polynomial is not a complete invariant, we cannot use it to prove this for the other case, when the strands to be pulled across each other have opposite orientation. However, we can severely restrict possible counterexamples.

Following the method above, we get:



There are in fact knotted knots with Conway polynomial 1, but while ∇(⇄) = 1 is not a contradiction, it is a significant restriction.

Now, let us examine the consequences of ∇(› ‹) = 0. As the constant term is 0, we know that › ‹ must have more than one component, and therefore must have two components, as the original projection had only one. Then, as its first degree coefficient is also 0, it must have linking number 0.

However, it cannot be a split link. If it is, then it can be separated into two components, as seen in Figure 22. Then the original knot is a composite of those two components, so the three projections corresponding to ⇄ ⤫ and ⤫ are in fact of the same knot. And, as we require the original projection to be knotted, the other two are as well.

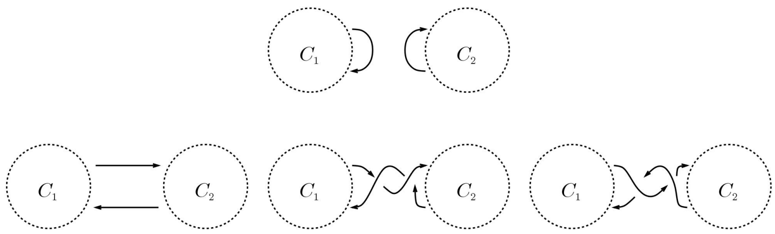


Figure 22: A split link of two components; Three projections of said split link despliced.

So, taken together, we have restricted the possible set of counterexamples to Conjecture 3.1 to the following: a Type II move on a projection of a non-alternating knotted knot with Conway polynomial 1, where the two strands to be crossed have opposite orientation, and where splicing the strands results in a linked link with linking number and Conway polynomial 0.

3.5 Type III Reidemeister moves

Type III Reidemeister moves also do not act predictably. In Figure 23, there are three examples of Type III moves. The knot projections are labeled with a

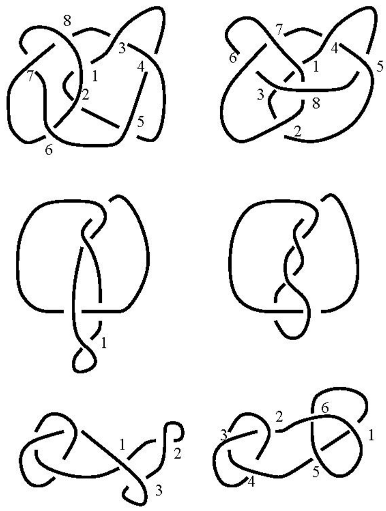


Figure 23: Examples of the effects of Type III Reidemeister moves on maximum switching sequences.

maximum knotted switching sequence. The first example is a composite knot made of two trefoils after a Type II move. In both projections, there is a full sequence, so the Type III move has not altered the length of the maximum knotted switching sequences.

The second example is a figure eight knot after a Type I move. The first has a one move sequence, the crossing of the Type I move, but the second has no switchable crossings, which is easy to check. So here the Type III move adds or subtracts one from the length of the maximum knotted switching sequences of the knot.

The third example is a trefoil after three Type I moves. The first projection therefore has a three length sequence — switching any of the other crossings will always unknot the projection. The second projection however has a full sequence, so the Type III move can also create or remove a full sequence.

4 The maximum sequence length of minimal alternating knot projections

4.1 Flype moves

Maximum knotted switching sequences are defined with respect to projections of a knot, and, as we have seen, allowing all projections of a knot permits a variety of maximum sequence lengths. It would be nice to be able to restrict the allowed projections in some way so that we could apply this property of a knot's projections to the knot itself. With alternating knots, we can.

In this section, we will prove that the maximum knotted switching sequences of any alternating minimal projection of an alternating knot have the same length. It would be very difficult to discuss this using Reidemeister moves. The only one which does not change the number of crossings is the third, and one can only use it in non-alternating diagrams. So with Reidemeister moves, to get between these minimal alternating projections, one has to increase the number of crossings and then decrease them again.

However, there is another type of projection alteration that does not alter the knot being projected, called a flype move. In a flype move a 2–tangle of the knot is flipped over a crossing, as shown below. Tait's flyping conjecture, which has been proven [5], is that in S^3 all minimal alternating projections of a knot can be reached in a finite number of flypes from any one alternating minimal projection of the knot.

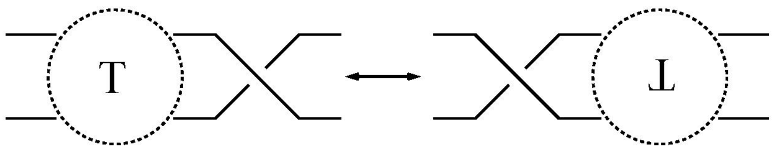


Figure 24: A flype move of tangle T.

Using this conjecture, we can prove the following result:

Theorem 4.1. *The length of any maximum knotted switching sequence is invariant for all minimal alternating projections of a knot.*

Proof. It is enough to show that the length of a projection's maximum knotted switching sequence is not altered by a flype move. Given a maximum knotted switching sequence in the original projection, the same sequence will work after a flype move (where for each crossing that would have been switched in the original projection, the crossing of the new projection corresponding to in under the flype move is switched) as at each stage, the two projections will still be equivalent by a flype move. □

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