REPRESENTATION THEORY OF SL_2 OVER A P-ADIC FIELD: THE PRINCIPAL SERIES

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Abstract. A result concerning the irreducibility (and reducibility) of the principal series of representations of SL_2 over a p-adic field is presented (see Theorem 4.16). An overview of the structure of p-adic fields precedes the demonstration, as does the introduction of certain special functions on these fields.

Contents

1. Introduction

Let K be a locally compact Hausdorff nondiscrete topological field. The principal series of representations of $SL_2(K)$ is a collection of continuous unitary representations of $SL_2(K)$ on $L^2(K)$. Its properties are well understood. We give here a brief analysis of the irreducibility (and reducibility) of this series in the special case where K is a p-adic field and p is odd. The computations and techniques employed follow [1], [2], and [3].

Section 2 contains necessary results concerning p-adic fields. The account is terse and proofs are not provided. Readers desirous of further information should consult [3].

Section 3 treats certain special functions on p-adic fields. Proofs can be found in [3].

Section 4 introduces the principal series of representations of SL_2 over a p-adic field F. A convenient unitarily equivalent representation is thoroughly studied. The paper concludes with the main result: Theorem 4.16.

2. p-adic Fields

Let p be an odd prime and F a finite algebraic extension of $\mathbb{Q}_p$. Denote the additive and multiplicative groups of F by F^+ and $F^\times$, respectively. Let dx be a fixed Haar measure on F^+.

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Definition 2.1. Define $|\cdot| : F \longrightarrow \mathbb{R}_{\geq 0}$ such that

(i) $|\cdot|$ is a non-Archimedean norm on F;
(ii) For any $a \in F^{\times}$, $d(ax) = |a|\, dx$.

Note 2.2. There exists exactly one such function.

Observe that $dx/|x|$ is a Haar measure on $F^{\times}$. Give F the topology induced by this norm. Define subsets $\mathcal{O}$ and $\mathfrak{p}$ of F by

$$\mathcal{O} = \{x : |x| \leq 1\}$$
$$\mathfrak{p} = \{x : |x| < 1\}.$$

$\mathcal{O}$ is the maximal compact subring of F; $\mathfrak{p}$ is the unique maximal ideal of $\mathcal{O}$. Moreover, $\mathfrak{p}$ is principal. Let τ be a generator for $\mathfrak{p}$. $\mathcal{O}/\mathfrak{p}$ is a field with q elements, where q is some power of p. It can be shown that $|\tau| = q^{-1}$ and for all $a \in F^{\times}$, $|a| = q^n$, for some $n \in \mathbb{Z}$.

Let

$$U = \{x : |x| = 1\}$$

be the group of units in $F^{\times}$. U contains an element ϵ such that

(i) ϵ has order $q-1$
(ii)
$$F^{\times} = \left(F^{\times}\right)^2 \bigcup (-\tau)\left(F^{\times}\right)^2 \bigcup (-\epsilon\tau)\left(F^{\times}\right)^2 \bigcup (\epsilon)\left(F^{\times}\right)^2.$$

Definition 2.3. Let $F_1 = F^{+}$, $F^{\times}$, or U. A *character* of F_1 is a continuous homorphism

$$\psi : F_1 \longrightarrow \mathbb{T},$$

where $\mathbb{T}$ is the group of complex numbers with norm one. Denote the set characters of F_1 by $\hat{F}_1$.

Define the following sets:

$$\mathfrak{p}^n = \left\{x \in F : |x| \leq q^{-n}\right\}, \quad n \in \mathbb{Z}$$
$$U_n = \left\{x \in U : |1-x| \leq q^{-n}\right\} = 1 + \mathfrak{p}^n, \quad n \geq 1.$$

If $\chi \in \hat{F}^{\times}$, there exists $s \in \mathbb{R}$ with

$$\frac{-\pi}{\ln q} < s \leq \frac{\pi}{\ln q}$$

and $\chi^* \in \hat{U}$ such that

$$\chi(x) = |x|^{is}\, \chi^*(u)$$

for all $x \in F^{\times}$, where $|x| = q^{-n}$ and $xq^{-n} = u$. For any nontrivial $\chi^* \in \hat{U}$, there exists $l \geq 1$ such that χ^* is trivial on U_l and nontrivial on U_{l-1}. For any nontrivial $\psi \in \hat{F}^{+}$, there is an $m \in \mathbb{Z}$ such that ψ is trivial on $\mathfrak{p}^m$ and nontrivial on $\mathfrak{p}^{m-1}$. The following definitions are thus sensible.

Definition 2.4. Let $\chi \in \hat{F}^{\times}$. χ is said to be *unramified* if χ^* is the trivial character, and *ramified of degree* l otherwise, where χ^* and l are as above.

Definition 2.5. Let $\psi \in \hat{F}^{+}$ be nontrivial. Then $\mathfrak{p}^m$ is said to be the *conductor* of ψ, where m is as above.

The three characters of degree two in $\hat{F}^{\times}$ figure prominently below. Denote them by sgn_{ϵ}, sgn_{τ}, and $\mathrm{sgn}_{\epsilon\tau}$, where

$$\mathrm{sgn}_{\epsilon}(x) = \begin{cases} 1 & \text{if } x \in (F^{\times})^2 \bigcup (\epsilon)(F^{\times})^2 \\ -1 & \text{otherwise} \end{cases}$$

and

$$\mathrm{sgn}_{\theta}(x) = \begin{cases} 1 & \text{if } x \in (F^{\times})^2 \bigcup (-\theta)(F^{\times})^2 \\ -1 & \text{otherwise} \end{cases}$$

for $\theta = \tau$ or $\epsilon\tau$.

Fix $\psi \in \hat{F}^{+}$ with conductor $\mathcal{O}$. For all $u \in F$, define $\psi_u \in \hat{F}^{+}$ by

$$\psi_u(x) = \psi(ux)$$

for all $x \in F$.

Definition 2.6. Let $f \in L^1(F)$. The *Fourier transform* of f, $\mathcal{F}f = \hat{f}$, is defined by

$$\hat{f}(u) = \int_F f(x)\,\psi_u(x)\,dx$$

for all $u \in F$.

$\mathcal{F}$ restricted to $L^1(F) \bigcap L^2(F)$ extends to an isometry of $L^2(F)$. Denote this extension by $\mathcal{F}$ as well. Without loss of generality, assume that dx is normalized so that $\hat{\hat{f}}(x) = f(-x)$ for all $f \in L^2(F)$ and $x \in F$.

Definition 2.7. The *Schwarz-Bruhat space* of F, $\mathcal{S}$, is the set of all complex-valued, compactly supported, locally constant functions on F.

Theorem 2.8. *$\mathcal{S}$ is dense in $L^p(F)$, for $1 \leq p < \infty$.*

Theorem 2.9. *The map defined by $\varphi \mapsto \hat{\varphi}$ for all $\varphi \in \mathcal{S}$ is a bijection of $\mathcal{S}$ onto itself.*

3. Special Functions on p-adic Fields

Note 3.1. The special functions below are vital to section 4. This section characterizes them more fully than required there, as they are also of independent interest.

Definition 3.2. Let $f : F \longrightarrow \mathbb{C}$ be locally integrable, except (possibly) at 0. For all $n \geq 0$, $[f]_n : F \longrightarrow \mathbb{C}$ is defined by

$$[f]_n(x) = \begin{cases} f(x) & \text{if } q^{-n} \leq x \leq q^n \\ 0 & \text{otherwise.} \end{cases}$$

If the limit in $(*)$ exists, define the *principal value integral* of f by

$$\text{P.V.} \int_F f(x)\,dx = \lim_{n \to \infty} \int_F [f]_n(x)\,dx. \qquad (*)$$

Theorem 3.3. *Let $f \in L^2(F)$. Suppose*

$$P.V. \int_F f(x)\,\psi_u(x)\,dx$$

exists for almost all $u \in F$. Then

$$\hat{f}(u) = P.V. \int_F f(x)\,\psi_u(x)\,dx$$

for almost all $u \in F$.

Note 3.4. This is part of a result known as *Plancherel's theorem.*

Definition 3.5. Let χ be a nontrivial character of $F^{\times}$. Then

$$\Gamma(\chi) = \Gamma(\chi^* \, |\cdot|^s) = \Gamma_{\chi^*}(s)$$

is defined as follows:

(i) if χ is ramified,

$$\Gamma(\chi) = \Gamma_{\chi^*}(s) = \text{P.V.} \int_F \psi(x)\, \chi(x)\, \frac{dx}{|x|}.$$

(ii) if χ is unramified and $\Re(s) > 0$,

$$\Gamma(\chi) = \Gamma_1(s) = \text{P.V.} \int_F \psi(x)\, \chi(x)\, \frac{dx}{|x|}. \qquad (**)$$

(iii) if χ is unramified and $\Re(s) \leq 0$,

$$\Gamma(\chi) = \Gamma_1(s)$$

is given by the analytic continuation of $(**)$.

Note 3.6. The above is well-defined. It is known as the *gamma function.* See [3].

Definition 3.7. Define $q' \in \mathbb{R}$ by

$$\frac{1}{q} + \frac{1}{q'} = 1.$$

Theorem 3.8. *Let $\chi = \chi^* \, |\cdot|^s$ be a nontrivial multiplicative character on $F^{\times}$.*

(i) If χ is ramified of degree $h \geq 1$,

$$\Gamma(\chi) = \Gamma_{\chi^*}(s) = C_{\chi^*}\, q^{h\left(s - \frac{1}{2}\right)},$$

where

$$C_{\chi^*} = \Gamma_{\chi^*}(1/2).$$

Note that

$$|C_{\chi^*}| = 1$$

and

$$C_{(\chi^*)^{-1}} C_{\chi^*} = \chi^*(-1).$$

(ii) If χ is unramified,

$$\Gamma(\chi) = \Gamma_1(s) = \frac{1 - q^{s-1}}{1 - q^{-s}}.$$

$\Gamma_1(s)$ has a simple pole at $s = 0$ with residue

$$\frac{1}{q' \ln q}.$$

$1/(\Gamma_1(s))$ has a simple pole at $s = 1$ with residue

$$\frac{-1}{q' \ln q}.$$

The only singularity of $\Gamma_1(s)$ occurs at $s = 0$; the only zero occurs at $s = 1$.

(iii) If $\chi(x) \neq |x|$, then

$$\Gamma_{\chi^*}(s) = \chi^*(-1)\,\overline{\Gamma_{(\chi^*)^{-1}}(\overline{s})}$$
$$\Gamma_{\chi^*}(s)\,\Gamma_{(\chi^*)^{-1}}(1-s) = \chi^*(-1).$$

Hence,

$$\Gamma_{\chi^*}(s)\,\overline{\Gamma_{\chi^*}(1-\overline{s})} = 1.$$

Definition 3.9. For $\chi \in \hat{F}^\times$ and $u, v \in F^\times$, define the *Bessel function* $J_\chi(u,v)$ as follows:

$$J_\chi(u,v) = \text{P.V.} \int_F \psi\left(ux + \frac{v}{x}\right)\chi(x)\,|x|^{-1}\,dx.$$

Note 3.10. The Bessel function is well-defined. See [3].

Lemma 3.11. *Let $u, v \in F^\times$. Then*
(i) $J_\chi(u,v) = J_{\chi^{-1}}(v,u)$.
(ii) $\chi(u)\,J_\chi(u,v) = \chi(v)\,J_\chi(v,u)$.
(iii) $J_\chi(u,v) = \overline{J_{\chi^{-1}}(-u,-v)} = \chi(-1)\,\overline{J_{\chi^{-1}}(u,v)}$.
(iv) If $\chi(-1) = 1$ (resp. -1), then $J_\chi(u,u)$ is real-valued (resp. pure imaginary-valued).

Definition 3.12. Let $k \in \mathbb{Z}_{>0}$, $\chi \in \hat{F}^\times$, and $v \in F^\times$. Then

$$F_\chi(k,v) = \int_{|x|=q^k} \psi(x)\,\psi\left(\frac{v}{x}\right)\chi(x)\,|x|^{-1}\,dx.$$

Lemma 3.13. *Suppose that $|v| = q^m$ and $1 \leq k < m$.*
(i) If χ is unramified, then $F_\chi(k,v) \neq 0$ if and only if m is even and $k = m/2$.
(ii) If χ is ramified of degree $h \geq 1$, then $F_\chi(k,v) \neq 0$ if and only if one of the following holds:
(a) m is even, $m \geq h$, and $k = m/2$
(b) $m < 2h < 2m$ and $k = h$ or $k = m - h$.

Theorem 3.14. *If $\chi \in \hat{F}^\times$ is unramified, $\chi \not\equiv 1$, and $u, v \in F^\times$, then*

$$J_\chi(u,v) = \begin{cases} \chi(v)\,\Gamma\left(\chi^{-1}\right) + \chi^{-1}(u)\,\Gamma(\chi) & |uv| \leq q \\ \chi^{-1}(u)\,F_\chi\left(\frac{m}{2}, uv\right) & |uv| = q^m,\ m > 1,\ m \text{ even} \\ 0 & |uv| = q^m,\ m > 1,\ m \text{ odd.} \end{cases}$$

If $\chi \equiv 1$, then the first case becomes

$$J_1(u,v) = \frac{m+1}{q'} - \frac{2}{q} = \frac{1}{q'}\left[-\frac{\ln|uv|}{\ln q} + 1\right] - \frac{2}{q}$$

for $|uv| = q^{-m} \leq q$. The other cases remain valid as stated.

Theorem 3.15. *If $\chi \in \hat{F}^\times$ is ramified of degree $h \geq 1$, and $u, v \in F^\times$, then*

$$J_\chi(u,v) = \begin{cases} \chi(v)\,\Gamma\left(\chi^{-1}\right) + \chi^{-1}(u)\,\Gamma(\chi) & |uv| \leq q^h \\ \chi^{-1}(u)\left[F_\chi(h,uv) + F_\chi(m-h,uv)\right] & |uv| = q^m,\ h < m < 2h \\ \chi^{-1}(u)\,F_\chi\left(\frac{m}{2}, uv\right) & |uv| = q^m,\ m \geq 2h,\ m \text{ even} \\ 0 & |uv| = q^m,\ m > 2h, m \text{ odd.} \end{cases}$$

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4. The Principal Series of Representations of $SL_2(F)$

Theorem 4.1. *Let $\chi \in \hat{F}^\times$, $f \in L^2(F)$, $x \in F$, and*

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL_2(F).$$

If

$$\left[\pi_\chi \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} f\right](x) = \chi(\beta x + \delta)\,|\beta x + \delta|^{-1} f\left(\frac{\alpha x + \gamma}{\beta x + \delta}\right)$$

for all $f \in L^2(F)$ and (almost all) $x \in F$, then π_χ is a continuous unitary representation of $SL_2(F)$ on $L^2(F)$.

Proof. See [1] or [3]. □

Definition 4.2. The collection of representations

$$\left\{\pi_\chi : \chi \in \hat{F}^\times\right\}$$

is called the *principal series* of representations of $SL_2(F)$.

Definition 4.3. For all $\chi \in \hat{F}^\times$ and $g \in SL_2(F)$, set

$$\hat{\pi}_\chi(g) = \mathcal{F}\pi_\chi(g)\,\mathcal{F}^{-1}.$$

Note 4.4. Let $\chi \in \hat{F}^\times$. $\hat{\pi}_\chi$ is a representation of $SL_2(F)$ on $L^2(F)$. It is unitarily equivalent to π_χ and more tractable, computationally.

Lemma 4.5. *Let $\gamma \in F$, $\varphi \in \mathcal{S}$, and $\chi \in \hat{F}^\times$. Then for all $x \in F$,*

$$\hat{\pi}_\chi \begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix} \varphi(x) = \psi(-\gamma x)\,\varphi(x).$$

Proof. Suppose $\varphi = \hat{f}$. For all $x \in F$,

$$\begin{aligned} \pi_\chi \begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix} \mathcal{F}^{-1}\varphi(x) =& \mathcal{F}^{-1}\varphi(x+\gamma). \\ =& f(x+\gamma). \end{aligned}$$

Hence,

$$\begin{aligned} \hat{\pi}_\chi \begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix} \varphi(x) =& \mathcal{F}\pi_\chi \begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix} \mathcal{F}^{-1}\varphi(x) \\ =& \int_F \pi_\chi \begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix} \mathcal{F}^{-1}\varphi(y)\,\psi_x(y)\,dy \\ =& \int_F f(y+\gamma)\,\psi_x(y)\,dy \\ =& \psi_x(-\gamma)\,\hat{f}(x) \\ =& \psi(-\gamma x)\,\varphi(x) \end{aligned}$$

for all $x \in F$. □

Lemma 4.6. *Let $\alpha \in F^\times$, $\varphi \in \mathcal{S}$, and $\chi \in \hat{F}^\times$. Then for all $x \in F$,*

$$\hat{\pi}_\chi \begin{pmatrix} \alpha^{-1} & 0 \\ 0 & \alpha \end{pmatrix} \varphi(x) = \chi(\alpha)\,|\alpha|\,\varphi\left(\alpha^2 x\right).$$

Proof. Suppose $\varphi = \hat{f}$. For all $x \in F$,

$$\begin{aligned}\pi_\chi \begin{pmatrix} \alpha^{-1} & 0 \\ 0 & \alpha \end{pmatrix} \mathcal{F}^{-1}\varphi(x) =& \chi(\alpha)|\alpha|^{-1}\mathcal{F}^{-1}\varphi\left(\alpha^{-2}x\right) \\ =& \chi(\alpha)|\alpha|^{-1} f\left(\alpha^{-2}x\right).\end{aligned}$$

Hence,

$$\begin{aligned}\hat{\pi}_\chi \begin{pmatrix} \alpha^{-1} & 0 \\ 0 & \alpha \end{pmatrix} \varphi(x) =& \mathcal{F}\pi_\chi \begin{pmatrix} \alpha^{-1} & 0 \\ 0 & \alpha \end{pmatrix} \mathcal{F}^{-1}\varphi(x) \\ =& \int_F \pi_\chi \begin{pmatrix} \alpha^{-1} & 0 \\ 0 & \alpha \end{pmatrix} \mathcal{F}^{-1}\varphi(y)\,\psi_x(y)\ dy \\ =& \chi(\alpha)|\alpha|^{-1}\int_F f\left(\alpha^{-2}y\right)\psi_x(y)\ dy \\ =& \chi(\alpha)|\alpha|\int_F f(y)\,\psi_x\left(\alpha^2 y\right)\ dy \\ =& \chi(\alpha)|\alpha|\,\varphi\left(\alpha^2 x\right)\end{aligned}$$

for all $x \in F$. □

Lemma 4.7. *Suppose $\mathcal{L} : L^2(F) \longrightarrow L^2(F)$ is a bounded linear operator such that for all $\gamma \in F$, $\alpha \in F^\times$, and $\chi \in \hat{F}^\times$,*

$$\mathcal{L}\hat{\pi}_\chi \begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix} = \hat{\pi}_\chi \begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix} \mathcal{L}$$

and

$$\mathcal{L}\hat{\pi}_\chi \begin{pmatrix} \alpha^{-1} & 0 \\ 0 & \alpha \end{pmatrix} = \hat{\pi}_\chi \begin{pmatrix} \alpha^{-1} & 0 \\ 0 & \alpha \end{pmatrix} \mathcal{L}.$$

Then there exists $m \in L^\infty(F)$ such that for all $f \in L^2(F)$,

$$\mathcal{L}f(x) = m(x)\,f(x)$$

for almost all $x \in F$ and m is almost everywhere constant on the cosets of $\left(F^\times\right)^2$ in $F^\times$.

Proof. Lemma 4.5 implies the first half. The required details are intricate. See [1] or [3]. To obtain the second half, take the first as given and note that for all $\alpha \in F^\times$, $\varphi \in \mathcal{S}$ and almost all $x \in F$,

$$\left[\mathcal{L}\hat{\pi}_\chi \begin{pmatrix} \alpha^{-1} & 0 \\ 0 & \alpha \end{pmatrix} \varphi\right](x) = m(x)\,\chi(\alpha)|\alpha|\,\varphi\left(\alpha^2 x\right),$$

while

$$\left[\hat{\pi}_\chi \begin{pmatrix} \alpha^{-1} & 0 \\ 0 & \alpha \end{pmatrix} \mathcal{L}\varphi\right](x) = \chi(\alpha)|\alpha|\,m\left(\alpha^2 x\right)\varphi\left(\alpha^2 x\right)$$

by Lemma 4.6. □

Note 4.8. Examination of the algebra of bounded linear operators on $L^2(F)$ commuting with $\hat{\pi}_\chi$, for $\chi \in \hat{F}^\times$ not of order 2, ultimately yields the proof of the irreducibility of $\hat{\pi}_\chi$.

Recall the following from [4]:

Theorem 4.9. *Let $\chi \in \hat{F}^{\times}$, $x \in F^{\times}$, and $\varphi \in \mathcal{S}$. The principal value integral*

$$P.V. \int_F \psi\left(\frac{x}{y}\right) \chi(y) |y|^{-1} \varphi(y)\, dy$$

exists. Moreover,

$$P.V. \int_F \psi\left(\frac{x}{y}\right) \chi(y) |y|^{-1} \hat{\varphi}(y)\, dy = \int_F \varphi(u) J_\chi(u,x)\, du.$$

Note 4.10. $f \in \mathcal{S}$ implies that

$$\int_F f(u) J_\chi(u,x)\, du$$

is absolutely convergent for all $\chi \in \hat{F}^{\times}$ and $x \in F^{\times}$. See [4].

Lemma 4.11. *Let $\varphi \in \mathcal{S}$ and $\chi \in \hat{F}^{\times}$. For almost all $x \in F$,*

$$\hat{\pi}_\chi \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \varphi(x) = \int_F \varphi(u) J_\chi(x,u)\, du$$

Proof. Suppose $\varphi = \hat{f}$. By Theorems 2.9 and 4.9,

$$\text{P.V.} \int_F \chi\left(\frac{1}{y}\right) f(-y) \psi\left(\frac{x}{y}\right) |y|^{-1}\, dy$$

exists for all $x \in F^{\times}$. Clearly,

$$\text{P.V.} \int_F \chi\left(\frac{1}{y}\right) f(-y) \psi\left(\frac{x}{y}\right) |y|^{-1}\, dy = \text{P.V.} \int_F \chi(y) f\left(-y^{-1}\right) \psi(xy) |y|^{-1}\, dy.$$

Theorem 3.3 implies

$$\mathcal{F}\left(\pi_\chi \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} f\right)(x) = \text{P.V.} \int_F \chi(y) f\left(-y^{-1}\right) \psi(xy) |y|^{-1}\, dy$$

for almost all $x \in F$. Theorem 4.9 and Lemma 3.11 give

$$\begin{aligned} &\text{P.V.} \int_F \chi\left(\frac{1}{y}\right) f(-y) \psi\left(\frac{x}{y}\right) |y|^{-1}\, dy \\ &= \text{P.V.} \int_F \chi^{-1}(y) \hat{\hat{f}}(y) \psi\left(\frac{x}{y}\right) |y|^{-1}\, dy \\ &= \int_F \hat{f}(u) J_{\chi^{-1}}(u,x)\, du \\ &= \int_F \hat{f}(u) J_\chi(x,u)\, du \end{aligned}$$

for all $x \in F^{\times}$. Hence,

$$\hat{\pi}_\chi \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \varphi(x) = \int_F \varphi(u) J_\chi(x,u)\, du$$

for almost all $x \in F$. □

Lemma 4.12. *Let $\chi \in \hat{F}^{\times}$ have order different from 2. If C_1 and C_2 are distinct cosets of $\left(F^{\times}\right)^2$ in $F^{\times}$, then there exist sets $A_1 \subset C_1$ and $A_2 \subset C_2$ of positive measure such that $x \in A_1$ and $u \in A_2$ implies $J_\chi(x,u) \neq 0$.*

Proof. Take $C_1 = (F^\times)^2$ and $C_2 = (\epsilon)(F^\times)^2$. The other cases are similar.

Suppose $\chi \equiv 1$. Let $N = \max\left(0, \frac{2q'}{q} - 1\right)$. Set

$$A_1 = \left\{y \in F : |y| \leq q^{\frac{-N}{2}}\right\} \bigcap \left(F^\times\right)^2$$

and

$$A_2 = \left\{y \in F : |y| \leq q^{\frac{-N}{2}}\right\} \bigcap (\epsilon)\left(F^\times\right)^2.$$

Theorem 3.14 gives the result.

Suppose $\chi \not\equiv 1$ is unramified. Set

$$A_1 = \{y \in F : |y| \geq q\} \bigcap \left(F^\times\right)^2$$

and

$$A_2 = \{y \in F : |y| \geq q\} \bigcap (\epsilon)\left(F^\times\right)^2.$$

Lemma 3.13 and Theorem 3.14 finish this case.

Suppose χ is ramified of degree $h \geq 1$. Set

$$A_1 = \left\{y \in F : |y| \geq q^h\right\} \bigcap \left(F^\times\right)^2$$

and

$$A_2 = \left\{y \in F : |y| \geq q^h\right\} \bigcap (\epsilon)\left(F^\times\right)^2.$$

Lemma 3.13 and Theorem 3.15 complete the proof. □

Note 4.13. The result above requires the hypothesis concerning the order of χ. Take $C_1 = (F^\times)^2$, $C_2 = (-\tau)(F^\times)^2$, and $\chi = \mathrm{sgn}_\epsilon$ to observe this. The computation is easy.

Theorem 4.14. *Let $\chi \in \hat{F}^\times$. Suppose that χ is not of order 2. Then $\hat{\pi}_\chi$ is irreducible.*

Proof. Suppose $\mathcal{L} : L^2(F) \longrightarrow L^2(F)$ is a bounded linear operator such that $\mathcal{L}$ commutes with

$$\hat{\pi}_\chi \begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix}, \quad \hat{\pi}_\chi \begin{pmatrix} \alpha^{-1} & 0 \\ 0 & \alpha \end{pmatrix}, \quad \text{and} \quad \hat{\pi}_\chi \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

for all $\gamma \in F$ and $\alpha \in F^\times$. By Lemma 4.7, there exists $m \in L^\infty(F)$ such that for all $f \in L^2(F)$,

$$\mathcal{L}f(x) = m(x) f(x)$$

for almost all $x \in F$ and m is almost everywhere constant on the cosets of $(F^\times)^2$ in $F^\times$. It suffices to show that m is almost everywhere constant on F.

For $\varphi \in \mathcal{S}$ and almost all $x \in F$,

$$\int_F \varphi(u)\, m(u)\, J_\chi(x,u)\; du = \int_F \varphi(u)\, m(x)\, J_\chi(x,u)\; du$$

by Lemma 4.11. Hence,

$$m(u)\, J_\chi(x,u) = m(x)\, J_\chi(x,u)$$

for almost all $x, u \in F$. The result follows by Lemma 4.12. □

Theorem 4.15. *Let $\chi \in \hat{F}^\times$ have order 2. Then $\hat{\pi}_\chi$ is reducible.*

Proof. The characters of order 2 on F are sgn_ϵ, sgn_τ, and $\mathrm{sgn}_{\epsilon\tau}$. sgn_τ and $\mathrm{sgn}_{\epsilon\tau}$ are ramified of degree 1. sgn_ϵ is unramified. Let $\theta = \epsilon, \tau$, or $\epsilon\tau$. Let $\varphi \in \mathcal{S}$. Theorems 3.14 and 3.15 and Lemma 4.11 give

$$\begin{aligned}\hat{\pi}_\chi \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \varphi(x) &= \int_F \varphi(u)\, J_\chi(x,u)\, du \\ &= \int_{|xu|\le q} \varphi(u)\,\Gamma(\mathrm{sgn}_\theta)\left[\mathrm{sgn}_\theta(u) + \mathrm{sgn}_\theta(x)\right] du \\ &\quad + \sum_{m>0,\, m \text{ even}} \int_{|xu|=q^m} \varphi(u)\,\mathrm{sgn}_\theta(x)\, F_{\mathrm{sgn}_\theta}\left(\frac{m}{2}, xu\right) du\end{aligned}$$

for almost all $x \in F$.

Suppose that φ is supported on $\ker(\mathrm{sgn}_\theta)$. Then

$$\begin{aligned}&\int_{|xu|\le q} \varphi(u)\,\Gamma(\mathrm{sgn}_\theta)\left[\mathrm{sgn}_\theta(u) + \mathrm{sgn}_\theta(x)\right] du \\ &= \Gamma(\mathrm{sgn}_\theta)\left[1 + \mathrm{sgn}_\theta(x)\right] \int_A \varphi(u)\, du \qquad (*)\end{aligned}$$

where

$$A = \ker(\mathrm{sgn}_\theta) \bigcap \{u \in F : |xu| \le q\}.$$

If $x \notin \ker(\mathrm{sgn}_\theta)$, $(*)$ vanishes.

Let $n > 0$, n even. Fix $u \in \ker(\mathrm{sgn}_\theta)$ and suppose $|xu| = q^n$, where $x \in F^\times$. Then

$$\begin{aligned}&\int_{|y|=q^{n/2}} \psi(y)\,\psi\left(\frac{xu}{y}\right) \mathrm{sgn}_\theta(y)\, \frac{dy}{|y|} \\ &= \int_{|y|=q^{n/2}} \psi\left(\frac{xu}{y}\right) \psi(y)\, \mathrm{sgn}_\theta\left(\frac{xu}{y}\right) \frac{dy}{|y|} \\ &= \mathrm{sgn}_\theta(x) \int_{|y|=q^{n/2}} \psi(y)\,\psi\left(\frac{xu}{y}\right) \mathrm{sgn}_\theta(y)\, \frac{dy}{|y|}.\end{aligned}$$

Hence, if $x \notin \ker(\mathrm{sgn}_\theta)$,

$$\int_{|y|=q^{n/2}} \psi(y)\,\psi\left(\frac{xu}{y}\right) \mathrm{sgn}_\theta(y)\, \frac{dy}{|y|} = 0.$$

But

$$\begin{aligned}&\int_{|xu|=q^n} \varphi(u)\,\mathrm{sgn}_\theta(x)\, F_{\mathrm{sgn}_\theta}\left(\frac{n}{2}, xu\right) du \\ &= \int_{|xu|=q^n} \varphi(u)\,\mathrm{sgn}_\theta(x) \left(\int_{|y|=q^{n/2}} \psi(y)\,\psi\left(\frac{xu}{y}\right) \mathrm{sgn}_\theta(y)\, \frac{dy}{|y|}\right) du.\end{aligned}$$

Hence, $x \notin \ker(\mathrm{sgn}_\theta)$, implies that

$$\sum_{m>0,\, m \text{ even}} \int_{|xu|=q^m} \varphi(u)\,\mathrm{sgn}_\theta(x)\, F_{\mathrm{sgn}_\theta}\left(\frac{m}{2}, xu\right) du = 0,$$

so

$$L^2(\ker(\mathrm{sgn}_\theta))$$

is invariant under the action of

$$\hat{\pi}_\chi \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

by Theorem 2.8.

Theorem 2.8 and Lemmas 4.5 and 4.6 indicate that

$$\hat{\pi}_\chi \begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix} \quad \text{and} \quad \hat{\pi}_\chi \begin{pmatrix} \alpha^{-1} & 0 \\ 0 & \alpha \end{pmatrix}$$

also fix this space. Since matrices of the form

$$\begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix}, \quad \begin{pmatrix} \alpha^{-1} & 0 \\ 0 & \alpha \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

where $\gamma \in F$ and $\alpha \in F^\times$, generate $SL_2(F)$, the result follows. □

Theorem 4.16. *Let $\chi \in \hat{F}^\times$. If χ has order 2, then π_χ is reducible. Otherwise, π_χ is irreducible.*

Proof. $\hat{\pi}_\chi$ is unitarily equivalent to π_χ. Cite Theorems 4.14 and 4.15. □

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6. References

[1] Boyarchenko, Dmitriy. "Irreducibility of the Principal Series Representations of SL_2 over a Local Field." *Unpublished Notes.* (2008), 1-19.

[2] Gel'fand, I.M. and Graev, M.I. "Representations of a Group of Matrices of the Second Order with Elements from a Locally Compact Field, and Special Functions on Locally Compact Fields." *Uspehi Matematichesm Nauk.* **18** (1963), 29-100.

[3] Sally, Jr. P.J. "An Introduction to p-adic Fields, Harmonic Analysis and the Representation Theory of SL_2." *Letters in Mathematical Physics.* **46** (1998), 1-47.

[4] Sally, Jr. P.J. and Taibleson, M.H. "Special Functions on Locally Compact Fields." *Acta. Math.* **116** (1966), 279-309.