UNDERSTANDING IRREDUCIBLE REPRESENTATIONS

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Abstract. This paper acts as an introduction to representation theory. In addition to defining representations of Lie algebras and of finite groups, this paper will go through two standard examples, the Lie algebra $\mathfrak{sl}_2\mathbb{C}$ and the finite group S_n, to illustrate techniques for their construction and use. In addition, those examples will develop some important theory behind representations of Lie algebras and finite groups.

Contents

1. Introduction

Groups are structures that are defined by very simple rules, but are varied in description and activity. On the other hand, vector spaces are very easily described in very plain terms. Thus, to get more information about groups, or other mysterious structures like Lie algebras, we consider them as vector spaces. We do this with something called a "representation". Representations take the operation of a group or Lie algebra or other complicated structure into an action on a vector space.

To get a good grasp of how these representations work, this paper will examine the Lie algebra $\mathfrak{sl}_2\mathbb{C}$ and the finite group S_n. $\mathfrak{sl}_2\mathbb{C}$ is a Lie algebra whose representations are very easy to describe. Even more than this, the development of the representations of $\mathfrak{sl}_2\mathbb{C}$ will act as an entryway into understanding the representations of many other Lie algebras, such as $\mathfrak{sl}_n\mathbb{C}$ and $\mathfrak{gl}_n\mathbb{C}$.

S_n does not generalize as well as $\mathfrak{sl}_2\mathbb{C}$. The techniques used to discover the irreducible representations of S_n are very specific to S_n. However, what is lost in generalizability is gained in understandability as many questions about S_n can be answered relatively simply. For example, each irreducible representation corresponds with a conjugacy class of S_n. This correspondence allows a very easy construction of the irreducible representations from the well known conjugacy classes.

Even though it is hard to describe the irreducible representations of a finite group in general, breaking down a representation into those representations is a breeze with characters. Characters are very simple, very powerful tools that in some sense

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"encode" a representation as something similar to a characteristic function. This code uniquely identifies a representation and can be used to decompose a representation as a direct sum of irreducible representations. Note that these techniques work for every finite group and so will be used to decompose S_n.

Although representation theory is ubiquitous in mathematics, we are especially interested in the field because it is being used in complexity theory for an attempt to prove $P \neq NP$. However, that topic is outside the scope of this paper and so will not be discussed.

2. The Irreducible Representations of $\mathfrak{sl}_2\mathbb{C}$

The following are some basic definitions needed for describing the representations of Lie algebras.

Definition 2.1. A *Lie algebra* is a vector space $\mathfrak{g}$ equipped with a bilinear, skew-symmetric map $[,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, called the Lie bracket, satisfying the Jacobi identity: $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ for all x, y, and z in $\mathfrak{g}$.

Example 2.2. If V is a vector space, then the endomorphisms of V, labeled $\mathfrak{gl}(V)$, forms a Lie algebra with bracket $[x, y] := x \circ y - y \circ x$ for all x and y in $\mathfrak{g}$.

Definition 2.3. For Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ a *homomorphism* from $\mathfrak{g}$ to $\mathfrak{h}$ is a linear map $\rho : \mathfrak{g} \to \mathfrak{h}$ where $\rho([x, y]) = [\rho(x), \rho(y)]$.

Definition 2.4. A *representation* of a Lie algebra $\mathfrak{g}$ on a finite-dimensional complex vector space V is a homomorphism $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$.

Given a representation ρ, we can define an action of $\mathfrak{g}$ on V by $X(v) = \rho(X)(v)$

Definition 2.5. A representation is *irreducible* if there is no proper, nontrivial subspace of V that is invariant under $\mathfrak{g}$.

A very important starting point in the description of representations of Lie algebras is $\mathfrak{sl}_2\mathbb{C}$. The description of the representations on many different Lie algebras is analogous to the process used below, especially $\mathfrak{sl}_n(\mathbb{C})$ and $\mathfrak{gl}_n(\mathbb{C})$.

Definition 2.6. $\mathfrak{sl}_2\mathbb{C}$ is the Lie algebra $\{A \in M_2(\mathbb{C}) | \mathrm{tr}(A) = 0\}$ with bracket $[X, Y] = XY - YX$.

A useful basis of $\mathfrak{sl}_2\mathbb{C}$ as a vector space is given by

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, F = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \text{ and } B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

It is clear that these form a basis of $\mathfrak{sl}_2\mathbb{C}$. By simple computation, we have $[H, F] = 2F$, $[H, B] = -2B$, and $[F, B] = H$.

So far we have defined representations and $\mathfrak{sl}_2\mathbb{C}$. Now, we need to show such representations exist for $\mathfrak{sl}_2\mathbb{C}$.

Example 2.7. Let $n \in \mathbb{N}$. Let W_n be an n-dimensional complex vector space. We define a representation $\phi : \mathfrak{sl}_2\mathbb{C} \to \mathfrak{gl}(W_n)$ by

$$\phi(H) = \begin{pmatrix} n-1 & & & & & & \\ & n-3 & & & & & \\ & & n-5 & & & & \\ & & & \ddots & & & \\ & & & & -n+5 & & \\ & & & & & -n+3 & \\ & & & & & & -n+1 \end{pmatrix},$$

$$\phi(F) = \begin{pmatrix} 0 & 1 & & & & & \\ & 0 & 1 & & & & \\ & & 0 & 1 & & & \\ & & & \ddots & \ddots & & \\ & & & & 0 & 1 & \\ & & & & & 0 & 1 \\ & & & & & & 0 \end{pmatrix}, \text{ and } \phi(B) = \begin{pmatrix} 0 & & & & & & \\ 1 & 0 & & & & & \\ & 1 & 0 & & & & \\ & & \ddots & \ddots & & & \\ & & & 1 & 0 & & \\ & & & & 1 & 0 & \\ & & & & & 1 & 0 \end{pmatrix}.$$

It is easily checked that $\phi(H)\phi(F) - \phi(F)\phi(H) = 2\phi(F)$, $\phi(H)\phi(B) - \phi(B)\phi(H) = -2\phi(B)$, and $\phi(F)\phi(B) - \phi(B)\phi(F) = \phi(H)$ showing that ϕ is a representation. Also, note that, by our construction, there exists a representation for every dimension.

And it turns out that W_n is the "canonical" representation of $\mathfrak{sl}_2\mathbb{C}$ as follows from the following theorem.

Theorem 2.8. *Every n-dimensional irreducible representation of $\mathfrak{sl}_2\mathbb{C}$ is isomorphic to W_n.*

Proof. Let ρ be an irreducible representation of $\mathfrak{sl}_2\mathbb{C}$ on a finite-dimensional complex vector space V. Since $\mathbb{C}$ is an algebraically closed field, $\rho(H) \in \mathfrak{gl}(V)$ has n eigenvalues (counting multiplicities). Let $\alpha \in \mathbb{C}$ be such an eigenvalue with eigenvector v, i.e., $H(v) = \alpha \cdot v$.

The bracket relations between H, B, and F determine how B and F act on v. For example, $H(F(v)) = [H, F](v) + F(H(v)) = 2F(v) + F(\alpha v) = (\alpha + 2)F(v)$. Similarly, $H(B(v)) = (\alpha - 2)B(v)$.

Let $V_\alpha = \{w \in \mathfrak{gl}(V) : H(w) = \alpha \cdot w\}$ be the eigenspace of α. The above facts can be restated as: $\rho(F)$ maps V_α into $V_{\alpha+2}$, $\rho(B)$ maps V_α into $V_{\alpha-2}$, and, clearly, $\rho(H)$ maps V_α into V_α

Let z be an eigenvalue of $\rho(H)$ such that $z + 2$ is not an eigenvalue of $\rho(H)$ (such a z must exist since V is finite dimensional) and let v be a corresponding eigenvector. Thus, $F(v) \in V_{z+2} = (0)$ and so $F(v) = 0$.

Let $W = \mathrm{Span}\{v, B(v), B^2(v), \dots\}$.

Lemma 2.9. *$W = V$.*

Proof of lemma. Since ρ is irreducible by assumption, it suffices to show that W is invariant under the action of $\mathfrak{sl}_2\mathbb{C}$. Since $\mathfrak{sl}_2\mathbb{C}$ is generated by H, F, and B, it suffces to show that W is invariant under the action of these three elements.

To show that W is invariant under H, consider $B^m(v)$. By induction, $H(B^m(v)) = (z - 2m)B^m(v)$ which is clearly in W.

To show that W is invariant under F, recall that $F(v) = 0$. Thus, $F(v)$ is contained in W. For $m > 0$, $F(B^m(v)) = [F, B](B^{m-1}(v) + B(F(B^{m-1}(v)))$. Thus, by induction, $F(B^m(v)) = m(z - m + 1)B^{m-1}(v)$, which is in W.

Finally, it is clear from the definition of W that it is invariant under B.

Thus, we have proved that $W = V$, which concludes the proof of the lemma.

□

Let $n = \min\{m : B^m(v) = 0\}$. We know $F(B^n(v)) = F(0) = 0$ and, by a simple induction argument, $F(B^n(v)) = n(z-n+1)B^{n-1}(v)$. Thus, $n(z-n+1)B^{n-1}(v) = 0$. Since $n > 0$ and $B^{n-1}(v) \neq 0$, $z = n - 1$. Thus, z, our chosen eigenvalue, is an integer.

Since $B^m(v) = 0$ for all $m \geq n$, $W = \text{Span}(\{v, B(v), ..., B^{n-1}(v)\})$. Since $H(B^m(v)) = (z - 2m)B^m(v)$, each of the vectors $B^m(v)$ is in a different eigenspace of $\rho(H)$ and thus they are linearly independent. Therefore, $\{v, B(v), ..., B^{n-1}(v)\}$ is a basis for V. In particular, n is the dimension of V. Thus, $V = V_{-z} \oplus V_{-z+2} \oplus ... \oplus V_{z-2} \oplus V_z$. Since $z = n - 1$, we have $V = V_{-n+1} \oplus V_{-n+3} \oplus ... \oplus V_{n-3} \oplus V_{n-1}$, which is clearly isomorphic to the representation W_n. □

Note that $W_n = V_{-n+1} \oplus V_{-n+3} \oplus ... \oplus V_{n-3} \oplus V_{n-1}$. Sometimes it is more convenient to refer to an irreducible representation by its largest eigenvalue rather than its dimension, so we define $V^{(\alpha)}$ by $V^{(\alpha)} := W_{\alpha+1} = V_{-\alpha} \oplus V_{-\alpha+2} \oplus ... \oplus V_{\alpha-2} \oplus V_\alpha$.

As mentioned earlier, irreducible representations are the building blocks of representations. The nice thing about $\mathfrak{sl}_2\mathbb{C}$ is that it is easy to determine the irreducible subrepresentations of any representation.

Let ρ be any representation. Find the eigenvalues of $\rho(H)$ and their multiplicities. We take the largest eigenvalue, say n, and note that $V^{(n)}$ is a part of our vector space. We then remove 1 from the multiplicities of each eigenvalue of $V^{(n)}$ and repeat the process until all the multiplicities are 0. Those $V^{(n)}$ are the irreducible representations of ρ.

For example, let the following diagram be the eigenvalues and multiplicities of $\rho(H)$.

multiplicities	1	0	3	1	3	0	1
eigenvalues	-3	-2	-1	0	1	2	3

The largest eigenvalue is 3, so we know that $V^{(3)}$ is part of our representation, so remove the eigenvalues of $V^{(3)}$, -3, -1, 1, and 3, and get:

multiplicities	0	0	2	1	2	0	0
eigenvalues	-3	-2	-1	0	1	2	3

The next highest is 1, so we know $V^{(1)}$ is a part of our representation. We remove the eigenvalues of $V^{(1)}$ to get:

multiplicities	0	0	1	1	1	0	0
eigenvalues	-3	-2	-1	0	1	2	3

We repeat the process to find that again $V^{(1)}$ is a part of our representation and are left with:

multiplicities	0	0	0	1	0	0	0
eigenvalues	-3	-2	-1	0	1	2	3

All that's left to remove is the one where the eigenvalue is 0, so $V^{(0)}$ is also part of our representation.

Thus, $V^{(3)}$, $V^{(1)}$, $V^{(1)}$, and $V^{(0)}$ are the irreducible representations in ρ.

It won't be proven in this paper, but, in general, a representation is equal to the direct sum of its irreducible representations. Thus, the space associated to ρ is $V^{(3)} \oplus V^{(1)} \oplus V^{(1)} \oplus V^{(0)}$.

3. The Irreducible Representations of S_n

In addition to Lie algebras, representations are often used to characterize finite groups. Like Lie algebras, finite groups also have irreducible representations and, as in the case of $\mathfrak{sl}_2\mathbb{C}$, each finite-dimensional representation can be split as a direct sum of a finite number of irreducible representations.

Definition 3.1. A *representation* of a finite group G on a finite-dimensional vector space V is a homomorphism from G to $GL(V) = \{A \in M_n(\mathbb{C}) : \det(A) \neq 0\}$.

Definition 3.2. A representation is *irreducible* if there is no proper, nontrivial subspace of V that is invariant under the action of G.

Both definitions are very similar to those used for Lie algebras. The idea is the same: a representation is still an action on a vector space and the irreducible representations form the building blocks of arbitrary representations.

Proposition 3.3. *The number of irreducible representations for a finite group is equal to the number of conjugacy classes.*

Example 3.4. Since S_n has at least three conjugacy classes, it has at least three irreducible representations:

One is called the *trivial representation* which is on $\mathbb{C}$ and acts by $\sigma(v) = v$ for $\sigma \in S_n$ and $v \in \mathbb{C}$.

Another one is called the *alternating representation* which is also on $\mathbb{C}$, but acts by $\sigma(v) = \text{sign}(\sigma)v$ for $\sigma \in S_n$ and $v \in \mathbb{C}$.

A third irreducible representation of S_n is called the *standard representation* and it is on $V = \{(z_1, z_2, z_3) \in \mathbb{C}^3 : z_1 + z_2 + z_3 = 0\}$ acting by $\sigma((z_1, z_2, z_3)) = (z_{\sigma(1)}, z_{\sigma(2)}, z_{\sigma(3)})$ for $\sigma \in S_n$ and $(z_1, z_2, z_3) \in \mathbb{C}^3$.

Note that for $n > 3$ there are more irreducible representations than just these three.

We will focus on S_n, the symmetric group on n elements, and its irreducible representations.

So, the question becomes: how many conjugacy classes of S_n are there? It turns out that the number of conjugacy classes of S_n is the number of ways of writing n as a sum of a sequence of n descending, nonnegative numbers, which is elaborated in the following proposition.

Definition 3.5. A *partition* of $n \in \mathbb{N}$ is an n-tuple $(\lambda_1, \lambda_2, ..., \lambda_{n-1}, \lambda_n) \in \mathbb{Z}$ such that $n = \lambda_1 + \lambda_2 + ... + \lambda_{n-1} + \lambda_n$ and $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_{n-1} \geq \lambda_n \geq 0$.

Proposition 3.6. *The conjugacy classes of S_n correspond to partitions of n.*

Proof. The conjugacy classes of S_n are uniquely determined by the cycle type of their elements, and each class contains only one cycle type. Thus, the conjugacy classes correspond to a partition of n into the cycle type of the class. $\square$

In fact, for S_n the irreducible representations are naturally in bijective correspondence with the conjugacy classes; we will demonstrate this correspondence in the remainder of this section. Young diagrams are a fundamental tool for keeping track of combinatorial data related to partitions, and are especially relevant to the representation theory of the symmetric groups. The Young diagram corresponding to the partition $(\lambda_1, \ldots, \lambda_n)$ of n is an arrangement of boxes—typically aligned to the top and to the left—with the λ_1 boxes in the first row, λ_2 boxes in the second row, and so on. For example, the Young diagram for a partition $\mu = (3,3,2,1,0,0,0,0,0)$ is the following:

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 & 6 \\ \hline 7 & 8 & \multicolumn{1}{c}{} \\ \cline{1-2} 9 & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array}$$

Note: Parts of size 0 are not displayed in a Young diagram. In order to reference certain boxes, the Young diagram has its boxes labeled with numbers 1 through n in reading order.

Example 3.7. For S_3, $\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline \end{array}$ is the Young diagram of the partition $(3,0,0)$, $\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array}$ is the Young diagram of the partition $(1,1,1)$, and $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$ is the Young diagram of the partition $(2,1,0)$.

Numbering the boxes allows us to define an action of S_n. Let $R_\lambda = \{g \in S_n : g$ preserves each row$\}$ and let $C_\lambda = \{g \in S_n : g$ preserves each column$\}$. For example, if $\mu = (3,3,2,1,0,0,0,0,0)$, $(13) \in R_\mu$ because 1 and 3 are both in the first row but $(13) \notin C_\mu$ because 1 is in the first column and 3 is in the third column, thereby switching the columns of those two numbers.

Example 3.8. For the Young diagram $\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline \end{array}$, every permutation preserves the rows. Thus, $R_{(3,0,0)} = S_n = \{1, (12), (13), (23), (123), (321)\}$. However, only the identity preserves the columns, so $C_{(3,0,0)} = \{1\}$.

For the Young diagram $\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array}$, only the identity preserves the rows, so $R_{(1,1,1)} = \{1\}$. On the other hand, every permutation preserves the column, so $C_{(1,1,1)} = S_n = \{1, (12), (13), (23), (123), (321)\}$.

For the Young diagram $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$, only the identity and (12) preserve the rows, so $R_{(2,1,0)} = \{1, (12)\}$. The identity and (13) preserve the column, so $C_{(2,1,0)} = \{1, (13)\}$.

In order to use R_λ and C_λ to determine the irreducible representations of S_n, it's important to know the space of those representations. To that end, let $\mathbb{C}S_n$ be the group algebra of S_n, i.e., the vector space with basis $\{e_g : g \in S_n\}$ with multiplication given by $e_g \cdot e_h = e_{gh}$. $\mathbb{C}S_n$ is part of the vector space used in the construction of the irreducible representations from R_λ and C_λ.

Given a partition λ, we define $a_\lambda, b_\lambda \in \mathbb{C}S_n$ by $a_\lambda = \sum_{g \in R_\lambda} e_g$ and $b_\lambda = \sum_{g \in C_\lambda} \operatorname{sign}(g) e_g$.

Example 3.9. The R_λ and C_λ for the Young diagrams of S_3 are:

$R_{(3,0,0)} = \{1, (12), (13), (23), (123), (321)\}$, so $a_{(3,0,0)} = e_1 + e_{(12)} + e_{(13)} + e_{(32)} + e_{(321)} + e_{(123)}$. $C_{(3,0,0)} = \{1\}$, so $b_{(3,0,0)} = e_1$.

$R_{(1,1,1)} = \{1\}$, so $a_{(1,1,1)} = e_1$. $C_{(1,1,1)} = \{1, (12), (13), (23), (123), (321)\}$, so $b_{(1,1,1)} = e_1 - e_{(12)} - e_{(13)} - e_{(23)} + e_{(321)} + e_{(123)}$.

$R_{(2,1,0)} = \{1, (12)\}$, so $a_{(2,1,0)} = e_1 + e_{(12)}$. $C_{(2,1,0)} = \{1, (13)\}$, so $b_{(2,1,0)} = e_1 - e_{(13)}$.

Definition 3.10. The *"Young symmetrizer"* c_λ for a partition λ is defined to be $c_\lambda = a_\lambda \cdot b_\lambda$.

Example 3.11. The Young symmetrizers of the Young diagrams of S_3 are:

$c_{(3,0,0)} = a_{(3,0,0)} b_{(3,0,0)} = (e_1 + e_{(12)} + e_{(13)} + e_{(32)} + e_{(321)} + e_{(123)})e_1 = e_1 + e_{(12)} + e_{(13)} + e_{(32)} + e_{(321)} + e_{(123)}$.

Similarly, $c_{(1,1,1)} = a_{(1,1,1)} b_{(1,1,1)} = e_1(e_1 - e_{(12)} - e_{(13)} - e_{(23)} + e_{(321)} + e_{(123)}) = e_1 - e_{(12)} - e_{(13)} - e_{(23)} + e_{(321)} + e_{(123)}$.

For $(2,1,0)$, $c_{(2,1,0)} = a_{(2,1,0)} b_{(2,1,0)} = (e_1 + e_{(12)})(e_1 - e_{(13)}) = e_1 + e_{(12)} - e_{(13)} - e_{(123)}$.

As the following theorem states (proof omitted), these Young symmetrizers determine the irreducible representations of S_n.

Theorem 3.12. *Let $\mathbb{C}S_n \cdot c_\lambda = \{z \cdot c_\lambda : z \in \mathbb{C}S_n\}$. The representation ρ of S_n on $\mathbb{C}S_n \cdot c_\lambda$ defined by $\rho(\sigma)(z) := e_\sigma \cdot z$ for $\sigma \in S_n$ and $z \in \mathbb{C}S_n \cdot c_\lambda$ is irreducible.*

In addition, every irreducible representation of S_n is of this form for some partition λ.

Example 3.13. Let σ be in S_3.

For the partition $(3,0,0)$, $c_{(3,0,0)} = e_1 + e_{(12)} + e_{(13)} + e_{(32)} + e_{(321)} + e_{(123)} = a_{(3,0,0)}$. Since $a_{(3,0,0)}$ is the same if you rearrange the terms, $\mathbb{C}S_n \cdot c_{(3,0,0)} = \mathbb{C}c_{(3,0,0)}$. Thus, for $\alpha \in \mathbb{C}$, $\sigma(\alpha c_{(3,0,0)}) = \alpha(e_{\sigma\cdot 1} + e_{\sigma\cdot(12)} + e_{\sigma\cdot(13)} + e_{\sigma\cdot(32)} + e_{\sigma\cdot(321)} + e_{\sigma\cdot(123)}) = \alpha(e_1 + e_{(12)} + e_{(13)} + e_{(32)} + e_{(321)} + e_{(123)}) = \alpha c_{(3,0,0)}$. In other words, for $v \in \mathbb{C}S_n \cdot c_\lambda$, $\sigma(v) = v$. Thus, this is the trivial representation.

By similar computation, $\mathbb{C}S_n \cdot c_{(1,1,1)} = \mathbb{C}c_{(1,1,1)}$. Thus, for $v \in \mathbb{C}S_n \cdot c_\lambda$, $\sigma(v) = \mathrm{sign}(\sigma)v$ making this the alternating representation.

The partition (2,1,0) is not as simple to analyze as the other representations, so a different approach is useful. Consider the vectors $v_1 = e_1 + e_{(12)} - e_{(13)} - e_{(132)}$, $v_2 = e_{23} + e_{(132)} - e_{(123)} - e_{(12)}$, and $v_3 = e_{13} + e_{(123)} - e_{(1)} - e_{(23)}$. It is easily verified that $v_1 = (13)c_{(2,1,0)}, v_2 = (23)c_{(2,1,0)}, v_3 = c_{(2,1,0)}$, and that these three vectors span $\mathbb{C}S_n \cdot c_{(2,1,0)}$. Through simple computation, it is easy to see that $\sigma(v_1) = v_{\sigma(1)}$, $\sigma(v_2) = v_{\sigma(2)}$, and $\sigma(v_3) = v_{\sigma(3)}$. Since $v_1 + v_2 + v_3 = 0$, it follows that this representation is isomorphic to the standard representation.

More than the above, we also can find a nice formula for the dimension of an irreducible representation. To each box in the Young diagram of λ, we can associate a number called the "hook length". The hook length for a box is calculated by taking the number of boxes below that box plus the number of boxes right of the box plus 1. The stars in the following figure give an example of a hook:

$$\begin{array}{|c|c|c|}\hline o & * & * \\ \hline o & * & o \\ \hline o & * \\ \cline{1-2} o \\ \cline{1-1}\end{array}$$

For example, if $\mu = (3, 3, 2, 1, 0, 0, 0, 0, 0)$ and if we label each box with the hook length, we have the following figure:

6	4	2
5	3	1
3	1	
1		

Proposition 3.14. *Let h be the product of all hook lengths in the Young diagram. The dimension of V_λ is $\frac{d!}{h}$.*

Thus, the dimension of an irreducible representation of S_n can be easily determined from the corresponding Young diagram.

4. Characters of Representations of Finite Groups

It was noted in the previous section that there are the same number of irreducible representations as conjugacy classes. Characters are functions, which are constant on conjugacy classes, that correspond to a representation. These functions are analogous to the process used for Lie algebras since representations of finite groups are decomposed by characters as representations of Lie algebras are decomposed by eigenvalue.

Definition 4.1. Let ρ be a representation of a finite group G on a finite-dimensional complex vector space V. The character of V, χ_V, is defined to be $\chi_V(g) = \mathrm{tr}(\rho(g))$ for $g \in G$.

Remark 4.2. Since trace is constant under conjugation, $\chi_V(hgh^{-1}) = \chi_V(g)$. That is, characters are class functions.

The interesting thing about characters is that they allow an easy formula to describe how many of an irreducible representation are in an arbitrary representation. The following statements will establish a formula.

Definition 4.3. Let (α, β) be the Hermitian inner product on the vector space of class functions from G to $\mathbb{C}$ defined by $(\alpha, \beta) = \frac{1}{|G|} \sum_{g \in G} \overline{\alpha(g)} \beta(g)$.

Theorem 4.4. *The characters of irreducible representations of a finite group G are orthonormal under this inner product.*

Although it will not be proven here, any representation of a finite group G can be written as a direct sum of irreducible representations. Therefore, the following proposition is a formula for the character of a direct sum in terms of the characters of its summands.

Proposition 4.5. $\chi_{V \oplus W} = \chi_V + \chi_W$.

Corollary 4.6. *For any representation V and irreducible representation W of a finite group G, (χ_V, χ_W) is equal to the multiplicity of W in V.*

Proof. Proof of Corollary 4.6. Let $V_1, \ldots, V_n$ be the irreducible representations of G. Since a representation can be written as a sum of irreducible representations, $V = V_1^{\oplus a_1} \oplus \cdots \oplus V_n^{\oplus a_n}$ where a_i is the multiplicity of V_i in V. Thus, $\chi_V = a_1 \chi_{V_1} + \cdots + a_n \chi_{V_n}$. Therefore, $(\chi_V, \chi_W) = a_1(\chi_{V_1}, \chi_W) + \cdots + a_n(\chi_{V_n}, \chi_W)$. Since characters of irreducible representations are orthonormal under this product, $(\chi_{V_i}, \chi_W) = 1$ if $V_i = W$ and $(\chi_{V_i}, \chi_W) = 0$ otherwise. Thus, $(\chi_V, \chi_W) = a_i$, which is the multiplicity of $V_i = W$ in V. $\square$

Therefore, any representation can be decomposed uniquely, up to isomorphism, into irreducible representations by studying the characters of the representation.

However, knowing what these specific characters look like still needs to be addressed. In general, characters for representations of groups are hard to describe. However, one of the special things about S_n is that the characters follow a specific, though not necessarily nice formula as shown by the following theorem.

Theorem 4.7. *If σ is an element of S_n and $\lambda = (\lambda_1, \ldots, \lambda_n)$ is a partition for the irreducible representation V, then $\chi_V(\sigma)$ is the coefficient of $x_1^{\lambda_1} x_2^{\lambda_2} \ldots x_n^{\lambda_n}$ in $\prod_{i<j}(x_i - x_j) \prod_{j=1}^{n}(x_1^j + \ldots x_n^j)^{i_j}$ where i_j is the number of j-cycles in σ.*

Example 4.8. Using the above formula, it is easy to describe the characters of the irreducible representations.

Let σ_1 be the identity of S_3, let σ_2 be a representative of the conjugacy class of elements of S_3 that have a 2-cycle, and let σ_3 be a representative of the conjugacy class of elements of S_3 that have a 3-cycle.

The characters of the irreducible representations are as follows:

Let T be the trivial representation. From the formula, $\chi_T(\sigma_1) = 1$, $\chi_T(\sigma_2) = 1$, and $\chi_T(\sigma_3) = 1$. Let A be the alternating representation. From the formula, $\chi_A(\sigma_1) = 1$, $\chi_A(\sigma_2) = -1$, and $\chi_A(\sigma_3) = 1$. Let S be the standard representation. From the formula, $\chi_S(\sigma_1) = 2$, $\chi_S(\sigma_2) = 0$, and $\chi_S(\sigma_3) = -1$.

As an example of how to decompose a given representation into irreducible representations, let V be a representation with characters $\chi_V(\sigma_1) = 5$, $\chi_V(\sigma_2) = 1$, and $\chi_V(\sigma_3) = -1$.

The following comes from the use of the corollary above.

$$(\chi_V, \chi_T) = \frac{1}{3!}(\overline{\chi_V(\sigma_1)}\chi_T(\sigma_1) + 3(\overline{\chi_V(\sigma_2)}\chi_T(\sigma_2)) + 2(\overline{\chi_V(\sigma_3)}\chi_T(\sigma_3))) =$$
$$\frac{1}{6}(5 \cdot 1 + 3 \cdot 1 \cdot 1 + 2 \cdot (-1) \cdot 1) = 1$$

Therefore, V contains one copy of the trivial representation.

$$(\chi_V, \chi_A) = \frac{1}{3!}(\overline{\chi_V(\sigma_1)}\chi_A(\sigma_1) + 3(\overline{\chi_V(\sigma_2)}\chi_A(\sigma_2)) + 2(\overline{\chi_V(\sigma_3)}\chi_A(\sigma_3))) =$$
$$\frac{1}{6}(5 \cdot 1 + 3 \cdot 1 \cdot (-1) + 2 \cdot (-1) \cdot 1) = 0$$

Therefore, V does not contain any copies of the alternating representation.

$$(\chi_V, \chi_S) = \frac{1}{3!}(\overline{\chi_V(\sigma_1)}\chi_S(\sigma_1) + 3(\overline{\chi_V(\sigma_2)}\chi_S(\sigma_2)) + 2(\overline{\chi_V(\sigma_3)}\chi_S(\sigma_3))) =$$
$$\frac{1}{6}(5 \cdot 2 + 3 \cdot 1 \cdot 0 + 2 \cdot (-1) \cdot (-1)) = 2$$

Therefore, V contains two copies of the standard representation.

Therefore, $V = T \oplus S \oplus S$.

This shows how through characters, it is easy to decompose representations into irreducible representations.

References

[1] W. Fulton and J. Harris. Representation Theory: A First Course. Springer-Verlag. 1991.