ELEMENTARY APPLICATIONS OF FOURIER ANALYSIS

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ABSTRACT. This paper is intended as a brief introduction to one of the very first applications of Fourier analysis: the study of heat conduction. We derive the steady state equation, which serves as our motivating PDE, and then identify solutions to two fundamental scenarios: the first, a heat distribution on a circle; the second, one on a rod. In doing all of this there is much more room for depth that simply isn't explored for brevity's sake. A more comprehensive paper could explicitly define what constitutes a good kernel, the limitations of convergence of the Fourier transform, and different ways to circumvent these limitations. Theoretically, there's room to explore harmonic functions explicitly, as well as Hilbert spaces and the L^2 space. That would be the more analytic route to take; on the other hand there are many fascinating applications that aren't broached either. In math, application is nearly synonymous with PDEs, and as far as partial differential equations are concerned, the most important feature that the Fourier transform has is the property of exchanging differentiation for multiplication. This property is really what makes bothering with the Fourier transforms of most functions worth it at all. Another equally important property of Fourier transforms is called scaling, which, in physics, directly leads to a concept called the Heisenberg Uncertainty Principle, which, like the whole of quantum physics, is very good at bothering people.

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1. INTRODUCTION

The mathematical field of Fourier Analysis was born out of the search for a general solution to the heat equation near the turn of the 19th century. Because we seek to explore several applications of Fourier Analysis, it will be necessary to redevelop the techniques and properties that combine to make this subject area illustrative and useful. As far as a starting point: there's no better place to start than the very problem that motivated this powerful technique - the heat equation.

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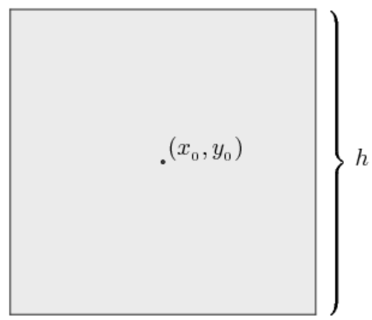
2. The heat equation

Consider an infinite metal plate which we model as the plane $\mathbb{R}^2$, and suppose we start with an initial heat distribution at time $t = 0$. Let the temperature at point (x, y) at time $t = 0$ be denoted by $u(x, y, t)$.

Consider a small square centered at (x_0, y_0) with sides parallel to the axis and of side length h, as shown in the picture below.



The amount of heat energy in S at time t is given by

$$H(t) = \sigma \iint_S u(x, y, t)\, dx\, dy,$$

where σ is a constant called the specific heat of the material. Therefore, the heat flow into S is

$$\begin{aligned}\frac{\partial H}{\partial t} &= \sigma \iint_S \frac{\partial u}{\partial t}\, dx\, dy, \\ &\approx \sigma\, h^2\, \frac{\partial u}{\partial t}(x_o, y_o, t)\end{aligned}$$

(... as S has an area of h^2.) Given two bodies with a common surface K, Newton's law of cooling states that heat flows at a rate proportional to the difference in temperature between the bodies integrated over K. Note that κ is a proportionality constant named the conductivity.

$$\frac{dH}{dt} = \kappa \iint_K \nabla u \cdot \hat{\mathbf{n}}\, dK,$$

For the vertical side on the right of our square S, this becomes

$$-\kappa h \frac{\partial u}{\partial x}(x_o - h/2, y_o, t),$$

We can find the net rate of heat change on the square by adding up the other sides as well.

$$\frac{dH}{dt} = \kappa h\left[\frac{\partial u}{\partial x}(x_o + h/2, y_o, t) - \frac{\partial u}{\partial x}(x_o - h/2, y_o, t)\right.$$
$$\left. + \frac{\partial u}{\partial x}(x_o, y_o + h/2, t) - \frac{\partial u}{\partial x}(x_o, y_o - h/2, t)\right]$$

Then, by the mean value theorem and letting h tend to 0, we derive

$$\frac{\sigma}{\kappa}\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \tag{2.1}$$

a partial differential equation affectionately called the heat equation.

3. The Steady-state equation and the Laplacian

If we wait long enough, the system will reach thermal equilibrium and the time derivative will reach zero. Our heat equation becomes

$$0 = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \tag{3.1}$$

This is called *Laplace's equation.* Laplace's equation is the specific case of the Laplacian of a function being equal to zero. The Laplacian is a differential operator, denoted by either ∇^2 or Δ. For a function f,

$$\Delta f = \frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} + \frac{\partial^2 f}{\partial x_3^2} + \cdots + \frac{\partial^2 f}{\partial x_n^2} \tag{3.2}$$

Laplace's equation, $\Delta f = 0$, is a partial equation with solutions that are important in many fields of science, such as electromagnetism, astronomy, and fluid dynamics. These solutions are important because they describe the behavior of electromagnetic, gravitational, and fluid potentials. In heat conduction specifically, Laplace's equation is called the steady state equation, representing a final heat distribution that is no longer changing, equation 3.1 above.

Solutions of Laplace's equation are also of interest in pure mathematics. For L, an open subset of $\mathbb{R}^n$, a *harmonic function* is a twice continuously differentiable function $f : L \to \mathbb{R}$, that satisfies Laplace's equation, that is $\Delta f = 0$. *Harmonic analysis* is a field of mathematics that studies various properties of these harmonic functions, as well as those of the Fourier series and the Fourier transform.

4. The Dirichlet Problem on the disc

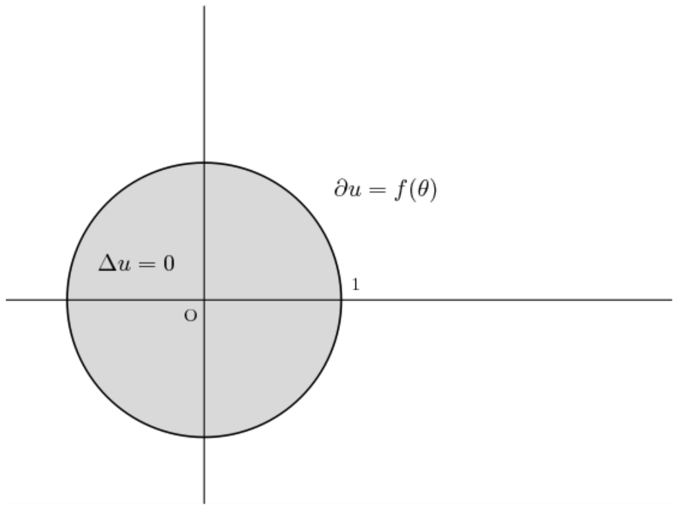
Consider the unit disc and its boundary, the unit circle:

$$D = \{(r, \theta) \in \mathbb{R}^2 : r < 1\}, \tag{4.1}$$
$$C = \{(r, \theta) \in \mathbb{R}^2 : r = 1\} \tag{4.2}$$

The Dirichlet problem (for the Laplacian on the unit disc) concerns the search for solutions that satisfy the following conditions:

i) u is a harmonic function on D,
ii) u is equal to a scalar function $f(\theta)$ on C.

In the context of heat conduction, this involves fixing a predetermined temperature distribution on the unit circle, waiting a long time, and then examining the final distribution of temperature.



The boundary, $f(\theta)$, is not easily expressible in Cartesian coordinates and so it's helpful to express the Laplacian in terms of polar coordinates. This is done by an application of the chain rule for partial derivatives. The end result:

$$\Delta u = 0 = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

Rearranging the equation and multiplying by r^2,

$$-\frac{\partial^2 u}{\partial \theta^2} = r^2\frac{\partial^2 u}{\partial r^2} + r\frac{\partial u}{\partial r}$$

We now separate variables and look for a solution of the form $u(r,\theta) = F(r)G(\theta)$. We find:

$$\frac{r^2F''(r) + rF'(r)}{F(r)} = -\frac{G''(\theta)}{G(\theta)}$$

Because each side depends on a different variable, they must each be equal to a constant, call this λ.

$$\begin{cases} G''(\theta) + \lambda G(\theta) = 0 \\ r^2F''(r) + rF'(r) - \lambda F(r) = 0 \end{cases}$$

Functions $\sin(\theta)$, $\cos(\theta)$, and $e^{i\theta}$ are all valid solutions for G. Because Laplace's equation is *linear*, a full solution can be expressed as a superimposition of all possible solutions and, indeed, this sum for $G(\theta)$ is

$$
\begin{aligned}
G(\theta) &= A_n \cos(n\theta) + B_n \sin(n\theta), \;\; n \in \mathbb{Z} \\
&\rightsquigarrow \sum_{n=-\infty}^{\infty} a_n e^{in\theta} \qquad (4.3)
\end{aligned}
$$

The only solution for $F(r)$ defined at the origin turns out to be $F(r) = r^{|n|}$. Therefore, we are left with the following solution:

$$
u(r, \theta) = \sum_{n=-\infty}^{\infty} a_n r^{|n|} e^{in\theta} \qquad (4.4)
$$

This is not just a Fourier series, but in fact the *Abel means* of our function u.

5. Abel means and the Poisson kernel

Fourier series are very delicate creatures. Given functions that are too unruly, the Fourier sum may not converge in places.

$$
S_n(f)(\theta) \neq f(\theta)
$$

Figure 1. castastrophe!

For people like you and I who are interested in expressing functions in terms of their Fourier series, this can be a very bad thing. If we add the constraint that our functions be of the class C^2, our Fourier series will naturally uniformly converge everywhere.

That said, the more accommodating the domain of an operation, the stronger it's generally considered. The simple Fourier sum of a function f is of the form

$$
S_n(f)(\theta) = \sum_{n=-\infty}^{\infty} a_n e^{in\theta},
$$

We can extend the class of functions whose Fourier series converge by looking at our sums in terms of Abel summation. A sum is *Abel summable* to s if for $0 \leq r < 1$,

$$
A_n(r, \theta) = \sum_{n=0}^{\infty} a_n r^n \text{ converges and } \lim_{r \to 1} A(r) = s
$$

By looking at our Fourier series in terms of Abel summability, we can guarantee pointwise convergence of Fourier series anywhere a function is continuous. Let's play around with our Abel sum!

$$
\begin{aligned}
A_r(f)(\theta) &= \sum_{n=-\infty}^{\infty} r^{|n|} a_n e^{in\theta} \\
&= \sum_{n=-\infty}^{\infty} r^{|n|} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} f(\varphi) e^{-in\varphi} \, d\varphi \right) e^{in\theta} \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\varphi) \left(\sum_{n=-\infty}^{\infty} r^{|n|} e^{-in(\varphi - \theta)} \right) d\varphi \qquad (5.1)
\end{aligned}
$$

This form of Abel summation becomes particularly illustrative if we introduce a new operation. For two functions $r(t)$ and $s(t)$, their convolution, denoted $(r*s)(t)$ is defined:

$$(r*s)(t)=\int_{\infty}^{\infty} r(\tau)\cdot s(t-\tau)\,d\tau$$

Convolution is a binary operation that performs a sort of "sliding integral of two functions." The feature of convolutions that currently interests us is called *approximation to the identity*. Given a function f and a *good kernel* K_n,

$$\lim_{n\to\infty}(f*K_n)(x)=f(x)$$

wherever f is continuous. Functions must have a certain set of properties to qualify as *good kernels* but that exact definition is tangential to this paper. Instead, if we examine (5.1), it is evident that our Abel means is actually the convolution of two functions. In fact,

$$A_r(f)(\theta)=(f*P_r)(\theta)=\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\varphi)P_r(\theta-\varphi)\,d\varphi \tag{5.2}$$

where

$$P_r(\theta)=\sum_{n=-\infty}^{\infty} r^{|n|}e^{in\theta}$$

is the *Poisson kernel*, which is not only a good kernel, but special enough to have merit its own name. We can obtain a prettier version of the Poisson kernel by parsing our infinite sums into a positive sum and a negative sum like so:

$$P_r(\theta)=\sum_{n=0}^{\infty} re^{in\theta}+\sum_{n=1}^{\infty} re^{-in\theta}$$

These are infinite geometric sequences and thus their limit has an exact algebraic representation. This limit is:

$$P_r(\theta)=\frac{1-r^2}{1-2r\cos\theta+r^2} \tag{5.3}$$

Theorem 5.4. *Let f be an integrable function defined on the unit circle. Then the function u defined in the unit disc by the Poisson integral*

$$u(r,\theta) = (f * P_r)(\theta)$$

has the following properties:

i) u has two continuous derivatives in the unit disc and satisfies $\Delta u = 0$.

ii) If θ is any point of continuity of f, then

$$\lim_{r\to 1} u(r,\theta) = f(\theta)$$

If f is continuous everywhere, this limit is uniform.

iii) If f is continuous, then $u(r,\theta)$ is the unique solution to the steady state heat equation in the disc which satisfies i) *and* ii)*.*

To prove *i)*, recall that u is given by the series (4.3). Fix $\rho < 1$; inside each disc of radius $r < \rho < 1$ centered at the origin, the series for u can be differentiated term by term, and the differential series is uniformly and absolutely convergent. Thus u can be differentiated twice and since this holds for all $\rho < 1$, we conclude that u is twice differentiable in the unit disc. Moreover, by using term by term differentiation and Laplace's equation in polar coordinates, we find that $\Delta u = 0$.

Proving *ii)* is an easy affair. Express $u(r,\theta)$ as in (5.3). Taking $\lim_{r\to 1}$,

$$u(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\varphi)\underbrace{\left(\sum_{n=-\infty}^{\infty} \cdot 1 \cdot e^{-in(\varphi-\theta)}\right)}_{\text{equals } 2\pi \text{ if } \varphi=\theta, \text{ else } 0} d\varphi$$

$$\rightsquigarrow \frac{1}{2\pi}\cdot f(\theta)\cdot 2\pi = f(\theta)$$

The proof for *iii)* is a uniqueness proof: it consists in showing that given a solution to this problem v, it will be equal to the solution u that we've actually derived. Unfortunately, such proofs take up a good amount of space and aren't particularly illustrative or groundbreaking given all of the work we've already done finding the solution, so this proof will be omitted. Otherwise, we've found the exact solution to Dirichlet's problem on the disc:

$$u(r,\theta) = \begin{cases} \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} f(\varphi)\frac{(1-r^2)}{1-2r\cos(\varphi-\theta)+r^2}\,d\varphi & \text{if } 0 \le r < 1, \\ f(\theta) & \text{if } r = 1. \end{cases}$$

6. The time-dependant heat equation on the real line

Now consider an infinite rod which we model by the real line and suppose we are given a temperature distribution $f(x)$ on the rod at time $t = 0$. The same considerations given in section 2 that generate the heat equation for $\mathbb{R}^2$, give a one dimensional equivalent for $\mathbb{R}$

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \tag{6.1}$$

The initial condition that we impose is $u(x, 0) = f(x)$. Let's solve this directly using a Fourier transform. The transform rule that we'll need is:

$$f'(x) \longrightarrow 2\pi i \xi \hat{f}(\xi).$$

Taking the Fourier transform of (6.1) in the x variable, we get

$$\frac{\partial \hat{u}}{\partial t}(\xi, t) = -4\pi^2\xi^2\hat{u}(\xi, t).$$

Fixing ξ, this is an ordinary differential equation in the variable t with unknown $\hat{u}(\xi, \cdot)$, so there exists a constant $A(\xi)$ so that

$$\hat{u}(\xi, t) = A(\xi)e^{-4\pi^2\xi^2 t}$$

By transforming the initial conditions as well, we remark that $\hat{u}(\xi, 0) = \hat{f}(\xi)$. This leads us to conclude that $A(\xi) = \hat{f}(\xi)$. Thus,

$$\hat{u}(\xi, t) = \hat{f}(\xi)e^{-4\pi^2\xi^2 t} \tag{6.2}$$

The final step is to find a way to inverse-transform our solution back to the time-domain. This can be done by taking advantage of the behavior of Fourier transformed convolutions. One property of convolutions is that given functions f and g elements of the Schwartz space,

$$\widehat{(f * g)}(\xi) = \hat{f}(\xi)\hat{g}(\xi).$$

Looking at (6.2), we can see that $\hat{u} = \hat{f} \cdot e^{-4\pi^2\xi^2 t}$. This implies that there exists a g such that

$$g(x) \longrightarrow \hat{g}(\xi) = e^{-4\pi^2\xi^2 t} \tag{6.3}$$

$$(f * g)(x) \longrightarrow \hat{u}(\xi, t) = \hat{f}(\xi) \cdot e^{-4\pi^2\xi^2 t}$$

Take (6.3) and perform a Fourier inversion on it, that is

$$g(x) = \int_{\infty}^{\infty} e^{-4\pi^2 t\xi^2} \cdot e^{2\pi i\xi x}\, d\xi \tag{6.4}$$

$$= \frac{1}{(4\pi t)^{1/2}} e^{-x^2/4t} \tag{6.5}$$

In fact, this function that our solution dictates we convolve f with is a special kernel called *the heat kernel on the real line.* The heat kernel on the real line and its transform are formally denoted

$$\mathcal{H}_t(x) = \frac{1}{(4\pi t)^{1/2}} e^{-x^2/4t} \quad \longrightarrow \quad \hat{\mathcal{H}}_t(\xi) = e^{-4\pi^2 t \xi^2}$$

As it turns out, the heat kernel is another good kernel, so its discovery signifies the end of our search for a solution.

The final solution for our heat problem on the real line is therefore

$$u(x,t) = (f * \mathcal{H}_t)(x) = \frac{1}{(4\pi t)^{1/2}} \int_{-\infty}^{\infty} f(x) \cdot e^{-x^2/4t}\, dx \tag{6.6}$$

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