STOKES' THEOREM ON MANIFOLDS

GIDEON DRESDNER

Abstract. The generalization of the Fundamental Theorem of Calculus to higher dimensions requires fairly sophisticated geometric and algebraic machinery. In this paper I sought to understand this important theorem without getting to sidetracked. I assume the reader has seen basic multivariable calculus.

Contents

1. The Fundamental Theorem of Calculus 1
2. Manifolds and Diffeomorphisms 2
3. Boundaries 5
4. Orientation 6
5. Forms 6
6. Integration and Stokes' Theorem 8
Acknowledgments 9
References 9

1. The Fundamental Theorem of Calculus

We begin by giving a quick statement and proof of the Fundamental Theorem of Calculus to demonstrate how different the flavor is from the things that follow.

Lemma 1.0.1. Given a Riemann integrable $f : [a, b] \to \mathbb{R}$,

$$(b-a)\inf f \leq \int_a^b f(x)dx \leq (b-a)\sup f$$

Proof. Just consider the partition $P = \{a, b\}$. Then $L(P, f)$ is precisely the left hand side of the inequality and $U(P, f)$ is the right hand side. $\square$

Theorem 1.0.2. ***(Fundamental Theorem of Calculus)***
If $f \in C[a, b]$ then $F(x) = \int_a^x f(t)dt$ is continuous and differentiable with the derivative $F'(x) = f(x)$

Proof. By definition, $F'(x_0)$ is the unique linear map such that,

$$F(x_0 + h) = F(x_0) + F'(x_0)(h) + R(h)$$

where $\lim_{h\to 0} R(h)/h = 0$. By the definition of F this is equivalent to,

$$\int_a^{x_0+h} f(t)dt = \int_a^{x_0} f(t)dt + F'(x_0)(h) + R(h)$$

Date: DEADLINE AUGUST 22, 2008.

Note that $\int_a^{x_0+h} f(t)dt - \int_a^{x_0} f(t)dt = \int_{x_0}^{x_0+h} f(t)dt$ (This should be verified). This gives,

$$\frac{\int_{x_0}^{x_0+h} f(t)dt}{h} = F'(x_0) + \frac{R(h)}{h}$$

By the lemma we know that

$$h \cdot \inf_{[x_0, x_0+h]} f \leq \int_{x_0}^{x_0+h} f(t)dt \leq h \cdot \sup_{[x_0, x_0+h]} f$$

This is equivalent to

$$\inf_{[x_0, x_0+h]} f \leq \frac{\int_{x_0}^{x_0+h} f(t)dt}{h} \leq \sup_{[x_0, x_0+h]} f$$

But the middle term is $F'(x_0) + R(h)/h$ so we have

$$\inf_{[x_0, x_0+h]} f \leq F'(x_0) + \frac{R(h)}{h} \leq \sup_{[x_0, x_0+h]} f$$

By continuity, letting $h \to 0$ gives

$$f(x_0) \leq F'(x_0) + 0 \leq f(x_0)$$

QED. $\square$

We will see that the correct understanding of the FTC considers the interval $[a, b]$ as a 1-dimensional manifold with boundary $\{a\}^- \cup \{b\}^+$ and that the object which is being integrated is a differential 1-form, the dual of a vector field.

2. Manifolds and Diffeomorphisms

Definition 2.0.1. A function f between open subsets $U \subset \mathbb{R}^k$ and $V \subset \mathbb{R}^l$ is smooth if all of its partial derivatives exist and are continuous.

In general given two arbitrary subsets $X \subset \mathbb{R}^k$ and $Y \subset \mathbb{R}^l$ we can say that $f : X \to Y$ is smooth if for every $x \in X$ there exists an open set $U \ni x$ and a smooth mapping $F : U \to Y$ such that F coincides with f on $U \cap X$.

Definition 2.0.2. A diffeomorphism is a smooth invertible function whose inverse is also smooth.

Definition 2.0.3. $M \subset \mathbb{R}^k$ is an m-dimensional manifold if every $x \in M$ has a neighborhood $W \ni x$ such that $W \cap M$ is diffeomorphic to an open subset of $\mathbb{R}^m$.

The diffeomorphism $f : \mathbb{R}^m \to W$ is call a coordinate system on M around the point x. A diffeomorphism going the other direction $g : W \to \mathbb{R}^m$ (you may as well choose f^{-1}) is call a parametricization of M around x.

Example 2.0.4. Unit sphere,

$$S^n = \{(x_1, \ldots, x_n) \mid \sum x_i^2 = 1\}$$

Given a point whose last coordinate is positive the diffeomorphism is

$$(x_1, \ldots, x_{n-1}) \mapsto (x_1, \ldots, x_{n-1}, \sqrt{1 - x_1^2 - \ldots - x_{n-1}^2})$$

Unsurprisingly if the last coordinate is negative than the the last coordinate becomes $-\sqrt{1 - x_1^2 - \ldots - x_{n-1}^2}$.

Example 2.0.5. $\mathbb{R}^n$. The diffeomorphism is the inclusion map.

Example 2.0.6. The Cartesian graph of any $f : [0,1]^k \to \mathbb{R}^k$ where f is a diffeomorphism. Since f is a diffeomorphism it serves as a universal parametricization for every point in the graph.

Given a smooth map $f : M \to N$ between manifolds we want to define the derivative $df_x : TM_x \to TM_{f(x)}$. To do this we need to the notion of tangent space. We can think of the tangent space to a manifold M at x, denoted TM_x as the (unique) m-dimensional plane in $\mathbb{R}^k$ which best approximates M near x (but translated to the origin). For an open set $U \subset R^k$ we define the tangent space $TU_x = \mathbb{R}^k$. Now we can define df_x for functions between open subset of $\mathbb{R}^k$.

Definition 2.0.7. Given $f : U \subset \mathbb{R}^k \to V \subset \mathbb{R}^l$ we can define df_x as the unique linear map T such that

$$f(x+h) = f(x) + Th + R(h) \quad \text{and} \quad \lim_{h\to 0} \frac{R(h)}{h} = 0$$

We call $R(h)$ the residue of the linear map L. This is simply formalizing what we mean by a linear approximation.

Remark 2.0.8. This is equivalent to the more conventional definition:

$$df_x(h) = \lim_{t\to 0} \frac{f(x+th) - f(x)}{t}$$

Theorem 2.0.9 (Basic Properties of the Derivative)**.**

*(1) **Chain Rule** Diffeomorphisms $f : V \to U$ and $g : U \to W$, $d(g \circ f)_x = dg_{f(x)} \circ df_x$. In other words, a commutative diagram of diffeomorphisms*

$$\begin{tikzcd} V \arrow[d, "f"'] \arrow[dr, "g\circ f"] & \\ U \arrow[r, "g"'] & W \end{tikzcd}$$

induces a commutative diagram of linear maps

$$\begin{tikzcd} \mathbb{R}^l \arrow[d, "df_x"'] \arrow[dr, "d(g\circ f)_v"] & \\ \mathbb{R}^k \arrow[r, "dg_{f(v)}"'] & \mathbb{R}^m \end{tikzcd}$$

(2) If $i : U \to U'$ is the inclusion map then $di_x = id$.
(3) If $L : \mathbb{R}^k \to \mathbb{R}^l$ is linear then $dL_x = L$

Proof. (Chain Rule) Since f and g are differentiable we can write

$$f(a+h) = f(a) + L_f h + R_f(h)$$

and

$$g(a+h) = g(a) + L_g h + R_g(h)$$

where $R_f(h)$ and $R_g(h)$ satisfy $\lim_{n\to\infty} R_f(h)/h = \lim_{n\to\infty} R_g(h)/h = 0$. Consider the following

$$= (g\circ f)(a+h) = g(f(a+h)) = g(f(a) + L_f h + R_f(h))$$

continue to expand

$$\begin{aligned}
=&g(f(a)) + L_g(L_f h + R_f(h)) + R_g(L_f h + R_f(h)) \\
=&g(f(a)) + (L_g L_f)h + L_g R_f(h) + R_g(L_f h + R_f(h))
\end{aligned}$$

It is clear by definition that $L_g = Dg_{f(a)}$ and $L_f = Df_a$. So we have our derivative. Our residue is everything to the right:

$$L_g R_f(h) + R_g(L_f h + R_f(h))$$

Since

$$\lim_{n\to\infty} \frac{L_f h + R(h)}{h} = 0 \implies \lim_{n\to\infty} \frac{R_g(L_f h + R_f(h))}{h} = 0$$

we have a residue satisfying $\lim_{n\to\infty} R(h)/h = 0$. $\square$

Parts (2) and (3) follow immediately from the conventional definition of the derivative in the remark.

Remark 2.0.10 (Neat Quick Application)**.** Suppose we have a diffeomorphism $f : U \subset \mathbb{R}^k \to V \subset \mathbb{R}^l$ then $k = l$ and in particular $df_x : \mathbb{R}^k \to \mathbb{R}^l$ is nonsingular.

Proof. Consider $f^{-1} \circ f = id$. $id = d(id) = d(f^{-1} \circ f)_x = df^{-1}_{f(x)} \circ df_x$ on $\mathbb{R}^k$ and similarly $df_x \circ df^{-1}_{f(x)} = id$ on $\mathbb{R}^l$. Thus df_x has a two-sided inverse which implies that $k = l$. $\square$

Now we can define TM_x. Take a parametricization $g : U \to M \subset \mathbb{R}^k$ $(g(u) = x)$. g is a diffeomorphism and U is open so we have the linear map $dg_u : \mathbb{R}^m \to \mathbb{R}^k$. We define

$$TM_g(u) = dg_u(\mathbb{R}^m)$$

We need to show that this definition is independent of the choice of g. So suppose we have another parametrization $h : V \to M \subset \mathbb{R}^k$ $(h(v) = x)$. Without loss of generality we can choose U and V to be sufficiently small so that we can draw the following diagram:

$$\begin{tikzcd}
U \arrow[d, "h^{-1}\circ g"'] \arrow[dr, "g"] & \\
V \arrow[r, "h"'] & M
\end{tikzcd}$$

induces

$$\begin{tikzcd}
\mathbb{R}^m \arrow[d, "d(h^{-1}\circ g)_u"'] \arrow[dr, "dg_u"] & \\
\mathbb{R}^m \arrow[r, "dh_v"'] & \mathbb{R}^k
\end{tikzcd}$$

It follows that $d(h^{-1} \circ g)$ is an isomorphism of vector spaces, and from this it follows that dh_v and dg_u have the same range.

Now we have come to the point where we can define the derivative in general between any two smooth manifolds M and N. Consider a smooth map $f : M \subset \mathbb{R}^k \to N \subset \mathbb{R}^l$. Since f is smooth $(\forall x \in M)$ There exists an neighborhood $W \ni x$ and a smooth function $F : W \to \mathbb{R}^l$ such that F coincides with f on $W \cap M$ (this is the definition of smooth on arbitrary sets). We define

$$df_x = DF_x$$

Now we need to justify this definition by proving that $df_x(v) \in TN_{f(x)}$ and that df_x is independent of the choice of F. Choose two parametrizations $g : U \to M \subset \mathbb{R}^k$ and $h : V \to N \subset \mathbb{R}^l$. Without loss of generality assume that $f : g(U) \to g(V)$. Thus we have a map $h^{-1} \circ F \circ g : U \to V$. Let's draw this in terms of diagrams:

$$\begin{tikzcd} U \arrow[r, "h^{-1}\circ f\circ g"] \arrow[d, "g"'] & V \arrow[d, "h"] \\ W \arrow[r, "F"] & \mathbb{R}^l \end{tikzcd}$$

this gives (by the chain rule):

$$\begin{tikzcd} \mathbb{R}^m \arrow[r, "d(h^{-1}\circ f\circ g)_u"] \arrow[d, "dg_u"'] & \mathbb{R}^n \arrow[d, "dh_v"] \\ \mathbb{R}^k \arrow[r, "dF_x"] & \mathbb{R}^l \end{tikzcd}$$

This essentially completes the proof since $DF_x = dh_v \circ d(h^{-1} \circ f \circ g)_u \circ dg_u^{-1}$, and the same for going the other direction on the diagram.

Theorem 2.0.11. *The basic three properties of derivatives of functions between open subsets of $\mathbb{R}^k$ hold for those between manifolds. The proof consists in bringing the problem back to the $\mathbb{R}^k$.*

Remark 2.0.12. Given a smooth manifold M, we can define a strange manifolds in terms of its tangent spaces.

$$TM := \{(x, v) \mid v \in TM_x\}$$

To see that this is a manifold suppose we take a point in it $p = (x, v)$. M is a manifold, and the tangent space TM_x is a manifold also. To get a parameterizable neighborhood of p, simply take the cross product of the neighborhoods of x and v. Define a parameterization in the obvious way.

This will be important later in conceptualizing differential forms.

3. Boundaries

Definition 3.0.1. $H^m = \{(x_1, \dots, x_n) \in \mathbb{R}^m \mid x_m \geq 0\}$ this is called the closed half space in $\mathbb{R}^m$. We can also define $\partial H^m = \mathbb{R}^{m-1} \times \{0\} = \{x \in H^m \mid x_m = 0\} \subset \mathbb{R}^m$. (Note that dimension respects Cartesian products so that ∂H^m has dimension $m - 1$)

Definition 3.0.2. A set $M \subset \mathbb{R}^k$ is an m-dimensional smooth manifold with boundary if each $x \in M$ has a neighborhood $U \ni x$ such that $U \cap M$ is diffeomorphic to H^m. ∂M is defined to be the set of all points that map to ∂H^m under such a diffeomorphism.

As usual, we need to justify this definition, which means using arguments based on our knowledge of Euclidean space. First of all there is the question of whether it is well defined: ∂M is independent of the parametricization.

Given a manifold with boundary M, suppose it has a point x which is mapped to ∂H^m under one diffeomorphism and to $H^m \setminus \partial H^m$ under another. Using a simple dimension argument this would mean that a neighborhood of x is diffeomorphic to both H^m and ∂H^m. But they have different dimension - contradiction.

By the definitions it is clear that ∂M has dimension $m-1$ and that $M \setminus \partial M$ has dimension m.

4. Orientation

Definition 4.0.1. We define the following equivalence relation on basi B_1 and B_2 of vector spaces:

$$B_1 \sim B_2 \quad \text{if} \quad \exists A : B_1 \to B_2 \text{ such that } det(A) > 0$$

It is easy to check that this is an equivalence relation. The classes are called orientations. Thus on finite dimensional vector spaces there are two orientations. When we choose an orientation on a vector space we often say this by denoting one class as positive $(+1)$ and the other as negative (-1).

Remark 4.0.2. Given a basis $B = \{b_1, \ldots, b_n\}$ we can permute its elements with an odd permutation σ: $B_\sigma = \{b_{\sigma(1)}, \ldots, b_{\sigma(n)}\}$. If B is in one orientation then B_σ is in the other one, i.e. $B \not\sim B_\sigma$.

To see this note that $B_\sigma = P_{odd} \circ B$ where P_{odd} is an permutation matrix of odd degree. $det(P_{odd}) = -1$, and since we are dealing with equivalence relations (i.e. this mapping is unique) this implies that $B \not\sim B_\sigma$.

This satisfies our intuition for what an orientation should be: an arrangement of a basis.

As for 0-dimensional vector spaces, points, we can define the orientation as the symbol $+1$ or -1.

With this basic notion of orientation on vector spaces we can talk about orientations on manifolds.

Definition 4.0.3. An orientation on a manifold M is an orientation on each one of its tangent spaces TM_x that "fit together"in the appropriate way: for every $x \in M$ there exists a neighborhood $U \ni x$ and an orientation preserving diffeomorphism to $\mathbb{R}^m$ (or H^m). Orientation preserving means exactly what you would think. If μ_x is an orientation on TM_x then the isomorphism df_x maps μ_x into the standard orientation on $\mathbb{R}^m$. This is to say that

$$[dh_x(v_1, \ldots, v_m)] = [e_1, \ldots, e_m]$$

(where $[\cdot]$ denotes the equivalence class containing $\cdot$)

There are a couple things to be said here about orientations and boundaries. If $x \in \partial M$ then there are three types of vectors in TM_x:

(1) vectors in $T(\partial M)_x$. By definition these are the vectors that are tangent to the boundary.
(2) "outward vectors"which form an open half space bounded by $T(\partial M)_x$.
(3) "inward vectors"which form the complementary half space to the "outward vectors"

An orientation μ on M induces an orientation on ∂M in the following way: choose $(v_1, \ldots, v_m) \in \mu_x$ such that $v_2, \ldots, v_m \in T(\partial M)_x$ and v_1 is an outward vector. Then $[v_2, \ldots, v_m]$ is an orientation for $T(\partial M)_x$.

Now that we have all the preliminaries we need about manifolds we move on to talk about differential forms on manifolds.

5. Forms

Definition 5.0.1. (Tangent Bundle) We define the following notation:

$$T^k M = \{(x, v_1, \ldots, v_k) \mid v_i \in TM_x\}$$

and $T^0 M = M$

A differential k-form on M then is a mapping

$$\omega : T^k M \to \mathbb{R}$$

.

But there are many such mappings, specifically there are many identifiable types of such mappings. They could be anything: discontinuous, not linear, and so on. So we have to be more specific.

First of all it should certainly be smooth, because that is the type of objects we are currently dealing with.

A differential k-form $\omega : T^k M \to \mathbb{R}$ is in a sense acting on two completely different objects at the same time: the manifold M itself and the tangent space TM_x. To understand what ω actually looks like we should break it up into its constituent parts.

Definition 5.0.2. (More specific than before) A differential k-form is smooth on M and multilinear on $T_x M$.

Smooth is a concept we are familiar with, multilinear is one which we are not, and we are going to try and avoid. Instead we define a canonical 1-form

Definition 5.0.3. We define a bunch of differential 1-forms in a canonical way (with respect to the usual basis) and call them dx_i:

$$dx_i(p)(v_1 e_1 + \ldots + v_m e_m) = v_i$$

where $p \in M$ and $e_i \in TM_p$ for each i. So what this notation does is simply separate out the point in the manifold from the tangent vectors to that point, because the rather abstract definition we gave covers up the fact that these two things, points on the manifold and vectors in the tangent space are radically different objects.

Next we define an orientation preserving way of combining dx_i called the wedge product.

Definition 5.0.4. The wedge product of dx_i and dx_j denoted $dx_i \wedge dx_j$ is defined in the following way:

$$dx_i \wedge dx_j(p)(v_1, v_2) = dx_1(v_1) \cdot dx_2(v_2)$$

which satisfies the skew-symmetric property: $dx_i \wedge dx_j = -dx_j \wedge dx_i$.

Note that this is in some sense orientation preserving since if you decide to flip the positive orientation from say $[e_1, e_2]$ to $[e_2, e_1]$ then a wedge combination of dx_i will switch signs also.

Remark 5.0.5. Important There is a lot of material not being treated here which involves the algebra of differential forms and multilinear forms. This can be done on many levels and is not within the scope of this paper. Without going into the details we will state that, by choosing the usual basis we can write every differential k-form in the form:

$$f dx_1 \wedge \cdots \wedge dx_k$$

where $f : M \to \mathbb{R}$ so that $(fdx_1 \wedge \cdots \wedge dx_k)(p)(v_1, \ldots, v_k) = f(p) \cdot (dx_1 \wedge \cdots \wedge dx_k)(v_1, \ldots, v_k) = f(p)dx_1(v_1) \cdots dx_k(v_k)$

Just to reiterate: Functions $f : M \to \mathbb{R}$ are 0-forms.
Mappings of the form $\sum f dx_i$ are 1-forms where f is are the 0-forms.
Mappings of the form $\sum f dx_i \wedge dx_j + \sum g dx_i$ are the 2-forms.
And so on, we can define m-forms on a manifold of dimension m. Note that all $m+1$-forms on an m-dimensional manifold are 0.

Forms are going to be the objects that we want to integrate over manifolds. The reason for this is because they behave well with respect to pullbacks, a convenient way of encoding the composition of a parametricization and a form, which will give a beautiful and coordinate-free way of expressing the change of variables formula for integrals.

Definition 5.0.6. A linear map $f : M \to N$ induces a pullback map $f^* : (k\text{-forms on } N) \to (k\text{-forms on } M)$ in the following way:

$$f^*\omega(x)(v_1, \ldots, v_m) = \omega(f(x))(df_x(v_1), \ldots, df_x(v_m))$$

While differential forms and pullbacks behave well with respect to the differential operator d, their most important purpose (in this paper) is to provide the machinery for an invariant notion of integration on manifolds.

6. Integration and Stokes' Theorem

Definition 6.0.1. The support of a differential form ω is the closure of the set of all points where ω is nonzero:

$$\text{supp } \omega = \overline{\{x \mid f(x) \neq 0\}}$$

Suppose that a differential form $\omega = f dx_1 \wedge \ldots dx_k$ in $\mathbb{R}^n$ is bounded. Without loss of generality we can assume that the support of ω is exactly the n-cube I^n. Then we define the integral of f as simply the standard multi-variable integral over I^k:

$$\int_{R^n} \omega = \int_{I^k} f dx_1 \wedge \ldots \wedge dx_k := \int_0^1 \cdots \int_0^1 f dx_1 \ldots dx_k$$

Say we want to integrate a differential k-form over something parametricizable to the k-cube I^k. We write the parametricization as $F : S \subset \mathbb{R}^n \to I^k$ (S for shape) and denote the integral as $\int_F \omega$. F can be thought of as a change of variables from multivariable calculus. Let's write ω in terms of the usual basis and some smooth function f. Then,

$$\int_F \omega = \int_F f dx_1 \wedge \cdots dx_k = \int_{I^k} f \circ F |det F'| dx_1 \wedge \cdots \wedge dx_k = \int_{I^k} F^*\omega$$

This we can use as a definition: $\int_F \omega = \int_{I^k} F^*\omega$. This is nothing more than a coordinate free way of writing the change of variables formula. Note that the determinant will keep the orientations and the $+1/-1$s correct.

With the above definition we know how to integrate differential forms over these little neighborhoods on the manifold. Now our task is to extend this to cover a compact manifold.

Remark 6.0.2. For every $x \in M$ we have a coordinate chart $f : V \subset \mathbb{R}^m \to U \ni x$. With a differential form ω on M we can define a new form ω_x by simply making the support of ω equal U. If M is compact then it is covered by a finite number of charts and every differential form ω can be written as the sum of ω_i where each ω_i is defined as we just did for a finite number of $i \in M$. Naturally, we want integration to respect finite sums so we may define:

$$\int_M \omega = \int_M \sum \omega_i := \sum \int_M \omega_i$$

(Note that there must be some normalization here to make this a true partition of unity, namely instead of ω_i what we really mean is $\dfrac{\omega_i}{\sum \omega_i}$)

Theorem 6.0.3. *Stokes' Theorem on Manifolds. Given a differential m-form ω whose support is the m-dimensional manifold M then*

$$\int_M d\omega = \int_{\partial M} \omega$$

Proof. First we can verify for the m-cube, $I^m := [0,1]^m$. Without loss of generality write the differential $(k-1)$-form ω as $f(x)dx_2 \wedge \cdots \wedge dx_m$. Then,

$$d\omega = \frac{\partial f}{\partial x_1} dx_1 \wedge \cdots \wedge dx_m$$

But we know how to integrate $\frac{\partial f}{\partial x_1} dx_1$ by the Fundamental Theorem of Calculus,

$$\int_{I^m} d\omega = \int_{I^m} \frac{\partial f}{\partial x_1} dx_1 \wedge dx_2 \wedge \cdots \wedge dx_m = \int_{I^m} \frac{\partial f}{\partial x_1} dx_1 dx_2 \cdots dx_m$$

$$= \int_{I^{m-1}} \left(f\Big|_{x_1=1} \right) dx_2 \cdots dx_m - \int_{I^{m-1}} \left(f\Big|_{x_1=0} \right) dx_2 \cdots dx_m$$

So what we have is f evaluated on the "left face"and the "right face"of the m-cube, with every other term getting canceled, which completes this portion of the proof.

The general case is proved by reducing it to the m-cube case by using a partition of unity. Given a differential form ω on a compact differentiable manifold M we can write it as a finite sum little differential forms ω_i where each ω_i is supported on some coordinate system. Since ω_i is supported by a coordinate system, and since integration is independent of such coordinate systems we can write things just in terms of cubes.

$$\int_M d\omega_i = \int_{I^n} d\omega_i$$

by Stokes' Theorem on cubes as we just have done,

$$\int_{I^n} d\omega_i = \int_{\partial I^n} \omega_i = \int_M \omega_i$$

the last equality is because it is coordinate independent (you use a parametricization to get back to M). QED □

Acknowledgments. I am indebted to both of my mentors Ian Biringer and Hyomin Choi for their help, but especially to Hyomin Choi who helped me a lot.

References

[1] John M. Lee. Introduction to Smooth Manifolds. Springer 2000.
[2] John W. Milnor. Topology from the Differentiable Viewpoint. The University Press of Virginia 1965.
[3] Michael Spivak. Calculus on Manifolds. Perseus Books Publishing 1965.
[4] Ferit Öztürk. Notes from Calculus on Manifolds Class Fall 2003. math.boun.edu.tr/instructors/ozturk/eskiders/fall04math488/moscova.pdf