BEHIND THE INTUITION OF TILINGS

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Abstract. It may seem visually intuitive that certain sets of tiles can be used to cover the entire plane without gaps or overlaps. However, it is often much more challenging to prove such statements rigorously. The Extension Theorem justifies the visual intuition. It allows us to prove the existence of a tiling by covering a circle of arbitrarily large finite radius. We clarify the proof of the Extension Theorem, consider its necessary assumptions, and present some interesting generalizations.

Contents

1. Introduction

Consider a jigsaw puzzle assembled on an infinite tabletop. We shall look at a finite set of potential puzzle pieces and see if the tabletop can be covered, without gaps or overlaps, using only pieces identical to those in the set. In many cases, it may be easy to believe that such a covering by puzzle pieces is possible. We will discuss the machinery that enables us to prove just by looking at a finitely covered surface that a tiling exists.

But first, let us drop the metaphors in favor of several definitions.

Definition 1.1. Let $T_1, T_2, T_3...$ be closed subsets of the Euclidean plane topologically equivalent - that is, homeomorphic - to a disc. These subsets form a *tiling* if their union is the Euclidean plane, and their interiors are pairwise disjoint. Every such T_i is called a *tile*. In other words, the tiles cover the plane without gaps or overlaps.

Remark 1.2. The standard definition of a tiling does not require each tile to be homeomorphic to a to a closed disk [1, p.16]. We include this condition for simplicity. Moreover, we will be dealing predominantly with straight-edged polygons whose edges touch each other completely. In other words, if the two edges of polygons touch, they are in fact the same line segment. Such tilings are called *edge-to-edge* [1, p.58].

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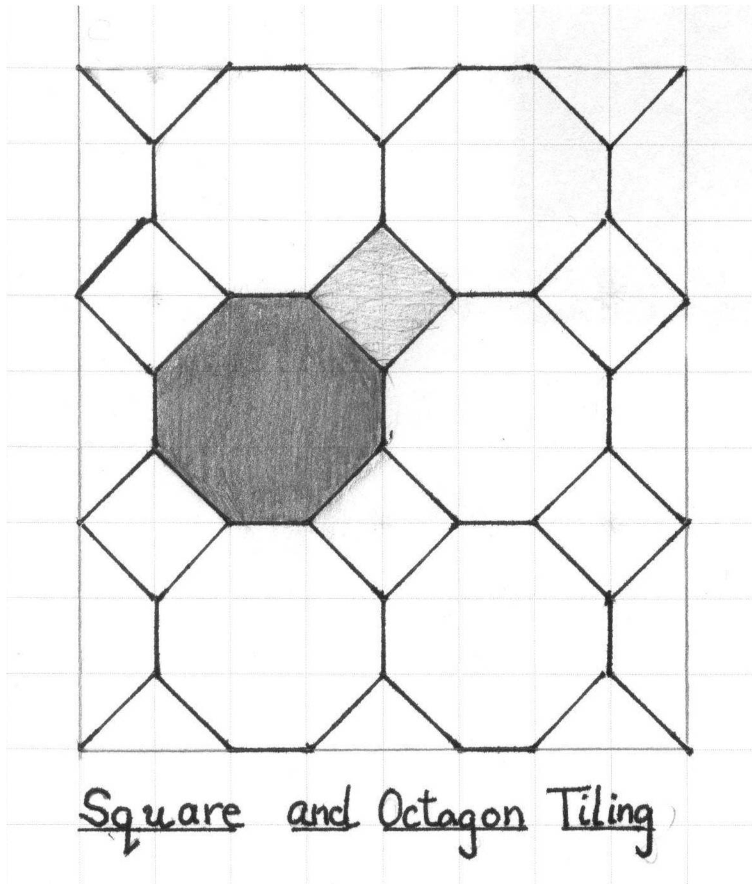

FIGURE 1. The shaded portion represents the "gluing" discussed below. Note that the octagons are not regular.

Definition 1.3. Take a tiling $\mathscr{T}$. The set $\mathscr{S} \subset \mathscr{T}$ is a set of *prototiles* if, for every tile T_i, T_i is congruent to some S_i in $\mathscr{S}$. Unless otherwise indicated, we will refer to finite prototile sets. The set $\mathscr{S}$ is called *minimal* if for all S_i, $S_j \in \mathscr{S}$, the tiles S_i and S_j are congruent if and only if $S_i = S_j$ [1, p.20]. (We will let the symbol $\cong$ denote congruence.)

Some examples of tilings include the square grid and the honeycomb, where the prototiles are the regular quadrilateral and regular hexagon, respectively. The only other regular polygon that can tile the plane by itself is the regular triangle. No other regular polygons have the necesary interior angle measure. For an example of a tiling with two prototiles, consider the tiling of an octagon and a square, as seen in Figure 1. This picture requires little explanation to show that it will tile the entire plane. To see this another way, we can "glue" an octagon and a square together to obtain a new tile that easily translates horizontally and vertically. However, not all tilings of the plane have such apparent symmetries. For such cases we have the Extension Theorem.

2. The Extension Theorem

Before stating the theorem and giving its proof, we present some additional definitions.

Definition 2.1. A *patch* is the union of a finite number of tiles that is topoloically equivalent to a disk. The interiors of the individual tiles making up the patch are pairwise disjoint [1, p.19].

Definition 2.2. Let $\mathscr{S}$ be a set of tiles. $\mathscr{S}$ *tiles over* a subset X of the plane if the there exists a patch $P_{\mathscr{S}}(X)$ such that $X \subset P_{\mathscr{S}}$ and every component tile is congruent to an element of $\mathscr{S}$ [1, p.151].

Definition 2.3. The *Hausdorff distance* between two tiles X and Y, denoted $h(X, Y)$, is defined as:

$$\max\{\sup_{y\in Y}\inf_{x\in X}\|x-y\|, \sup_{x\in X}\inf_{y\in Y}\|x-y\|\}$$

When $h(X, Y) = 0$ for X, Y closed, X and Y are the same set [1, p.153].

Definition 2.4. A sequence of tiles $T_1, T_2, T_3...$ *converges* to a limit tile T if $\lim_{i\to\infty} h(T_i, T) = 0$ [1, p.154].

Definition 2.5. U is a *circumparameter* of a prototile set $\mathscr{S}$ if for every $T \in \mathscr{S}$, T is contained in some circle of radius U. Analogously, u is an *inparameter* of $\mathscr{S}$ if for each $T \in \mathscr{S}$, there exists a circle of raidus u contained in T [1, p.122].

We require an additional fact:

Fact 2.6. *Take an infinite sequence of tiles $T_1, T_2...$ such that all T_i are congruent to some fixed tile T_0, where T_0 is bounded. Suppose that every T_i contains the point P_0. Then the sequence contains a convergent subsequence whose limit tile T is congruent to T_0 and contains P_0* [1, p.154].

Remark 2.7. For an outline of the proof, see [1, p.156].

Now, the statement of the Extension Theorem:

Theorem 2.8. *Let $\mathscr{S}$ be a finite set of tiles. If $\mathscr{S}$ tiles over arbitrarily large disks D, $\mathscr{S}$ admits a tiling of the plane* [1, p.151].

Proof. Let $\mathscr{S}$ be a finite and minimal set of prototiles. Let u be an inparapeter and U a circumparameter of the set. Because $\mathscr{S}$ is finite and the tiles are closed topological disks, U exists and u is greater than 0. Now, we construct a lattice Λ out of points with the cartesian coordinates $(nu, mu), n, m \in \mathbb{Z}$. Since there are countably many points, we can assign an order to them. Let L_0 be the origin, and then continue the labeling in a spiral as shown in Figure 2. Note that any disk of radius u will contain at least one point of Λ. Since any tile in $\mathscr{S}$ contains such a disk, any tile congruent to an element of $\mathscr{S}$ will also contain at least one point of the lattice.

With that in mind, let $D(x, r)$ denote the disk of radius r around a point x. Consider $D(L_0, r)$ for every positive integer r. By the conditions of the theorem, for every $r \in \mathbb{N}$, there exists a patch $P(r)$ that covers $D(L_0, r)$. If r is large enough for $D(L_0, r)$ to contain some L_s, then let the tile $T_{r,s}$ be that tile of $P(r)$ which contains L_s. If L_s is located on an edge or a vertex, there will be more than one possible choice of $T_{r,s}$. In that case, we select any one of them. The

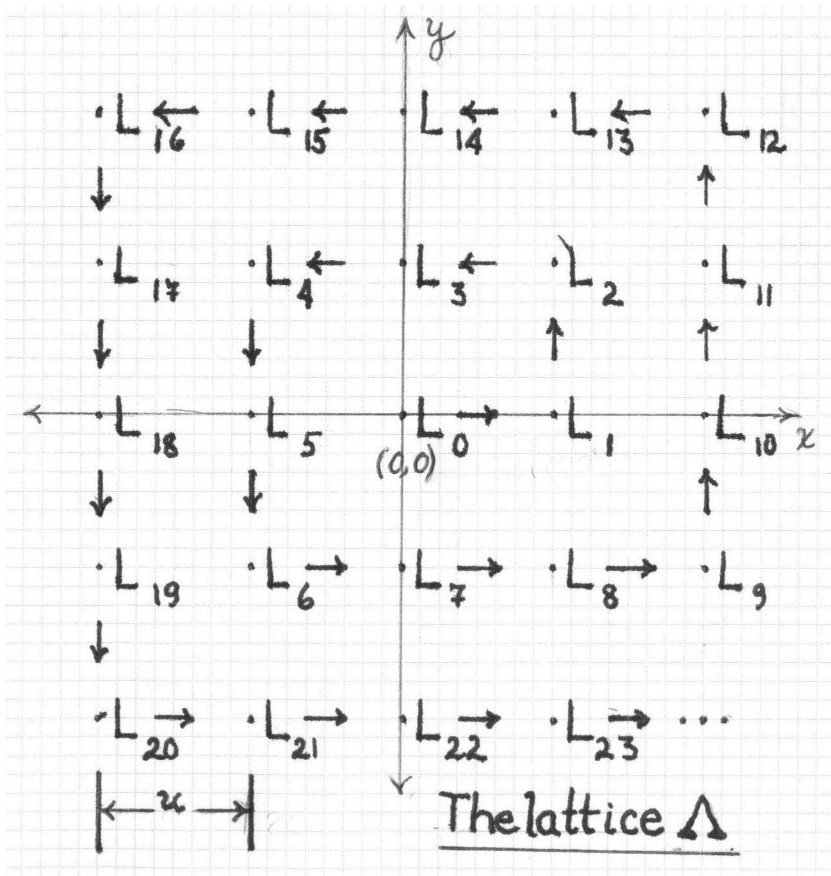

FIGURE 2. Every point on the lattice Λ has the coordinates (nu, mu) - integer multiples of the chosen inparameter of prototile set $\mathscr{S}$. As a result, any tile congruent to an element of $\mathscr{S}$ contains at least one point of Λ.

sequence $T_{1,0}, T_{2,0}, T_{3,0}$... contains a subsequence A_0 of tiles congruent to one tile in $\mathscr{S}$. Because all the terms of A_0 contain the common point L_0, the fact mentioned earlier states that a subsequence of A_0, call it B_0, converges to a limit tile T_0 containing L_0 and congruent to a prototile $S_0 \in \mathscr{S}$. For future reference, let I_{B_0} represent the set of r values such that for every $r \in I_{B_0}, T_{r,0}$ is a term in the sequence B_0.

We can follow a similar procedure with the point L_1. Consider the sequence of tiles $T_{r,1}$, where $r \in I_{B_0}$ and r large enough for L_1 to be in $D(L_0, r)$. This sequence too will have a subsequence A_1 of tiles that are all congruent to some single prototile in $\mathscr{S}$ which we will call S_1. Note that it is possible to have $S_1 = S_0$, but it is not important to know whether this is the case or not. In any event, by the aforementioned fact, A_1 has a subsequence B_1 that converges to a limit tile T_1 such that $T_1 \cong S_1$ and $L_1 \in T_1$. Let the definition of the set I_{B_1} be analogous to that of I_{B_0}. Using this technique, for the n^{th} sequence we consider all $r \in I_{B_{n-1}}$ and produce a subsequence B_n with a limit tile T_n containing L_n congruent to a prototile S_n. Now we will show that set $X = \{T_1, T_2, T_3...\}$ (with T_i the limit tile of the sequence B_i) is a tiling of the plane.

Because every T_i is congruent to a tile in $\mathscr{S}$, we know T_i s a closed topological disk. We need to show that $\bigcup_{i=0}^{\infty} T_i$ is the plane and that for all T_i, T_j the intersection of their interiors is empty, except when $T_i = T_j$. This follows if we take an arbitrary point y and show that it belongs to at least one T_i (more than one if y falls on an edge or vertex), and to the interior of at most one T_i (none if y falls on an edge or vertex). Taking such an arbitrary point y, consider the disk $D(y, U)$, where U is the circumparameter of the tiles in $\mathscr{S}$. From among the points L_i such that $L_i \in D(y, U)$ choose the one with the maximal index, and call that index k. Then consider the set I_{B_k}. Let $X_r = \{T_{r,0}, T_{r,1}...T_{r,k}\}$ for all $r \in I_{B_k}$. This means that for every r in I_{B_k} we consider the set of tiles containing the lattice point L_i up to L_k. (Some of these tiles may in fact be the same tile - a tile may contain more than one lattice point.)

As $r \to \infty$ every $T_{r,j} \to T_j \in X$. So we can say that when $r \to \infty, X_r \to X$. In every X_r, the $T_{r,j}$ make up the patch $P(r)$. Then every $T_{r,i}$ and $T_{r,j}$ have disjoint interiors unless they are simply the same tile. So if y is in the patch, then it belongs to the interior of no more than one tile. To show that y is indeed contained in the patch, we need to show that the center of $D(y, U) \subset P(r)$. The point y lies in the interior, on an edge, or on the a vertex of a square that connects four points of Λ in the plane. At any time a disk of radius u about y will enclose or touch at least two points of Λ, one of which has a bigger index than the other. For a disk of radius U, which is greater than u, that will certainly be the case as well. Consider the line segment between the two lattice points, say L_a and L_b. Without loss of generality let $a > b$. If the the patch $P(r)$ contains L_k, it also contains all L_i with $i < k$. Thus if it contains L_a, it will also contain L_b. Then the entire line segment is contained, including the point y. Since L_k has the maximal index, $a \leq k$, so $P(r)$ contains L_a, and therefore L_b, and therefore y. This holds for each of the r we are considering.

If, for every $r \in I_{B_k}$, at least one tile in X_r contains y and the interior of no more than one tile X_r contains it, the same can be said for the limit set. This means that the limit set X also forms a tiling. Thus we obtain a tiling of the entire plane - a circle with a radius of infinity, so to speak - while starting with a patch of tiles covering an arbitrarily large finite circle [1, pp.153-5]. □

3. Extensions of the Extension Theorem

In some cases, the Extension Theorem seems self-evident. Consider, for example, the square grid and tilings built up from the center. The theorem gains significance when the tiling *cannot* be achieved by an immediate and obvious construction from the center outwards. More complex sets of prototiles allow multiple combinations around each vertex, which means that there is no simple way of finding a radial expansion guaranteed to continue indefinitely. To see how tricky finding one may be, consider the tiles shown in Figure 3. This particular set of six prototiles, discovered by Roger Penrose, admits only *nonperiodic* tilings - that is, tilings that lack translational symmetry.

Definition 3.1. Let $\mathscr{S}$ be a prototile set. $\mathscr{S}$ is *aperiodic* if it admits *only* nonperiodic tilings.

The reader is invited to make physical copies of this set and try to devise a simple rule for tiling the plane with them [2, p.33].

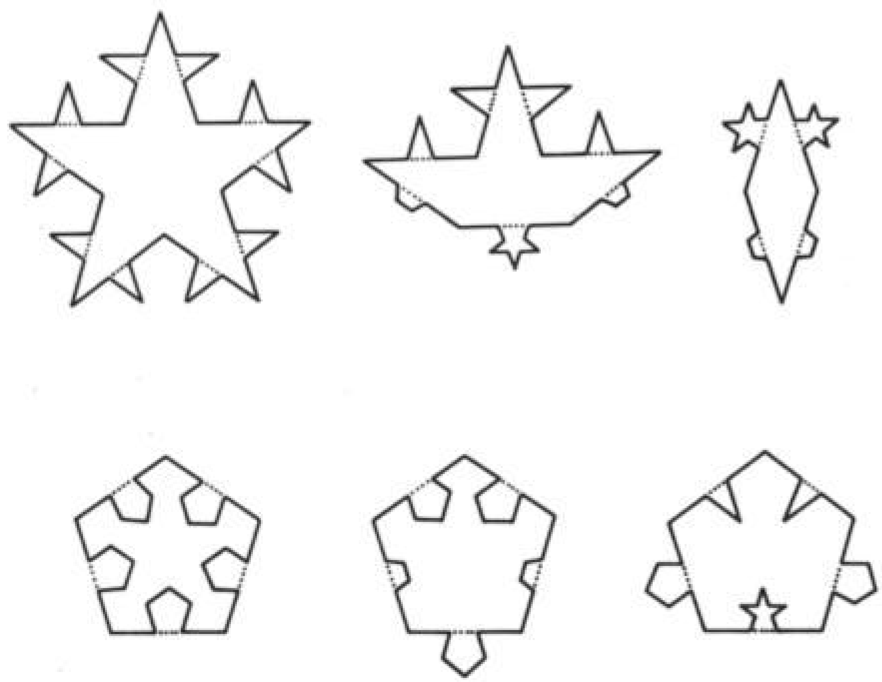

FIGURE 3. The above is the first set of Penrose tiles. These are often referred to as P1, the first of three surprisingly simple aperiodic tile sets discovered by physicist Roger Penrose [2, p.33]. All three sets, especially the other two, have numerous astonishing properties worth reading about. For a lively and engaging discussion, see [3, p.73-93]. For a more rigorous approach, see [1, pp.531-39].

Remark 3.2. A word of caution: some authors use the terms “nonperiodic” and “aperiodic” interchangeably, while others favor one exclusively.

Exercise 3.3. Consider the set of tiles in Figure 3, which Grünbaum and Shepard call P1, the first set of Penrose tiles [1, p.531]. By gluing together elements of P1, find a set of five prototiles that admits exactly the same set of possible tilings as P1 does. Is it possible reduce the number still futher to four prototiles [2, p.33]?

All this time we have dealt with a finite set of nice, bounded prototiles. Would the Extension Theorem still hold if we relaxed some of these conditions?

If we remove the restriction of a finite prototile set and make no other modification, we lose the basis of the entire proof. Without a finite set of prototiles, the existence of a positive inparameter is no longer certain. Also, the sequence A_i, as defined in the proof, is a sequence of tiles containing the lattice point L_i, all congruent to one prototile. The existence of this sequence is guarnateed only when the prototile set is finite. Grünbaum and Shepard give an example of the theorem’s failure under these conditions: prototiles whose sides consist of arcs from circles of varius radii. These tile a quadrant of the plane, but not the entire plane.

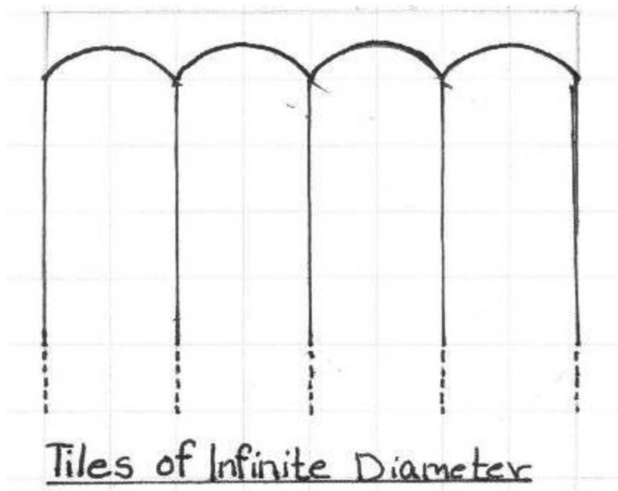

FIGURE 4. Tiles with infinite diameter violate the extension theorem. This example shows why the Extension Theorem requires tiles to be bounded.

Also, if we let the tiles have unbounded diameter, such as an infinite stripe with one end rounded off, as shown in Figure 4, then the conditions of the Extension Theorem are fulfilled but the conclusion is false. Such tiles cannot tile the plane because becuase nothing fits about the rounded edges. Given that they the tiles are infinite in one direction and can be placed next to each other indefinitely, copies of this figure can tile over a circle of arbitrarily large radius.

One generalization of the Extension Theorem, however, allows infinite prototile sets that are compact.

Definition 3.4. A set $\mathscr{S}$ of prototiles is *compact* if there exists a bounded set containing all elements of $\mathscr{S}$, and every infinite sequence of tiles in $\mathscr{S}$ converge to a limit tile that is also in $\mathscr{S}$.

This generalization is the following:

Theorem 3.5. *Let $\mathscr{S}$ be a set of prototiles in which each tile is a closed topological disk. If $\mathscr{S}$ is compact, and if for every disk of radius r there exists a patch $P(r)$ made of tiles of $\mathscr{S}$ that covers the disk, then $\mathscr{S}$ admits a tiling of the entire plane* [1, p.155].

Another interesting generalization regards the preservation of symmetry of an admitted tiling.

Theorem 3.6. *Let $\mathscr{S}$ be a finite set of prototiles, each of them a closed topological disk. If there exist patches covering arbitrarily large disks so that each patch has a particular symmetry σ,then $\mathscr{S}$ admits a tiling of the plane that also possesses the symmetry σ* [1, p.155].

There are other interesting generalizations and applications of this theorem. As we have seen, it plays a significant role in rigorously proving something that is often left to intuition.

4. Acknowledgments

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References

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