ON GRAPH CONNECTIVITY AFTER PATH REMOVAL

GRAHAM CASEY GIBSON

1. Abstract

In this paper we show that for any two vertices x, y of a 6-connected graph G, there exists a path between them whose removal leaves G 3-connected. This proves the case of Lovasz's path removal conjecture (1975), where $f(3) = 6$.

This article examines the relationship between connectivity and path removal. The following definition of connectivity is used frequently

Definition 1. *A graph G is $k-$connected if a set S of k vertices disconnects the graph into at least two components C_1 and C_2 and S and no set S' of $k-1$ vertices disconnects the graph.*

Definition 2. *A k vertex cut in a graph G is a set of k vertices which disconnects the graph into at least two components C_1 and C_2.*

Thus any k-connected graph must contain a k-vertex cut which disconnects the graph into at least two components.

By definition, the complete graph on n vertices, K_n, is $n-1$-connected.

In order to describe a graph in terms of sets we need the following definitions.

Definition 3. *The set $E(G)$ is the set consisting of edges of G.*

Definition 4. *The set $V(G)$ is the set consisting of vertices of G.*

The central question arising from connectivity is if a graph G is k-connected and a path P connecting two vertices is removed how well connected is the resulting graph $G - V(P)$. This is a crucial question in building fail-safe networks because one may increase the connectivity accordingly such that any path removed will yield a connected graph with the desired connectivity. The given conjecture was made by Lovász (1975)

Conjecture 1 (Lovász)**.** *For each natural number k, there exists a least natural number $f(k)$ such that, for any two vertices u, v in any f(k)-connected graph G, there exists a path P with end points u and v such that $G - V(P)$ is k-connected.*

Two cases of this conjecture have thus far been proven. It is well known that as a consequence of a theorem of Tutte that $f(1) = 3$, because all 3-connected graphs contain a $non - separating\ path$, a path whose removal does not disconnect the graph, between any two vertices. The case $f(2) = 5$ was proven by Chen, Gould, and Yu (1998).

2. The Case of f(3) = 6

A quick counter example shows that $f(3) \neq 5$ and makes use of the Turán graph.

Definition 5. *A Turán graph $T(n, r)$ is graph with n vertices that may be partitioned into r subsets where no two vertices in the same partition are adjacent. In addition, each partition will have size $\lfloor n/r \rfloor$ or $\lceil n/r \rceil$ implying the Turń graph is a complete r-partite graph.*

First, we show that indeed $f(3) \neq 5$ due to the turan graph on 8 vertices that does not contain K_4. Let $T(8, 3)$ be the turan graph that does not contain K_4. Because $|V(T)| = 8$ and there are 3 partitions. $\lfloor 8/3 \rfloor = 2$ and $\lceil 8/3 \rceil = 3$, so the size of the partitions R_1, R_2, R_3 are $2, 3, 3$ respectively.

Theorem 1. *Let $x, y \in R_1$ of $T(8, 3)$ then there does not exist a path P with endpoints x, y such that $T(8, 3) - V(P)$ is 3-connected.*

Proof. By $x, y \in R_1$ there does not exist an edge joining them. Thus, a path P joining them would have to include at least one vertex of a second partition R_2. Thus in $G - V(P)$ $|R_2| \leq 2$. Let $v_1, v_2\ inR_2$ in $G - V(P)$. Then v_1, v_2 clearly induce a 2 vertex cut in the graph because $G - (V(P) \cup (v_1, v_2)) = R_3$ and by definition no vertices of R_3 are adjacent so the graph is disconnected. □

Thus, $f(3) \neq 5$.

Theorem 2. *Any $6-connected$ graph contains a path P between any vertices $x, y \in V(G)$ such that $G - V(P)$ is 3-connected.*

The following definition leads to a useful theorem about 6-connected graphs.

Definition 6. *Let $x \in G$ then $d(x)$ is the number of edges incident with the vertex x.*

In addition, there is a fundamental operation on graphs, the edge contraction.

Definition 7. *Let G be a graph. Let $x, y \in V(G)$ and let e be an edge incident with x and y. Let G' be the graph obtained by contracting the edge e. Then G' may be formed by identifying all edges incident with x and incident with y and directing these to a new vertex v while also deleting the vertices x, y and the edge e.*

If G is a graph, the graph G' obtained by an edge contraction of an edge xy is denoted G/xy. The proof relies on the following characterization of 3-connected offered by Tutte (See [2].

Theorem (Tutte 1961)**.** *A graph G is 3-connected if and only if there exists a sequence $G_0, ..., G_n$ of graphs that have the following two properties*
1) $G_0 = K_4$ and $G_n = G$
2) G_{i+1} has an edge xy with $d(x)$, $d(y) \geq 3$ and $G_i = G_{i+1}/xy$.

The concept of containing K_4 may be captured by the following definition.

Definition 8. *A graph $H \subset G$ is an induced subgraph if $V(H) \subset V(G)$ and $E(H) \subset E(G)$.*

This characterization shows that if G is a 6-connected graph and there exists a path P between any two vertices such that $G - V(P)$ contains K_4 as an induced subgraph then the graph is 3-connected by $G_0 \in G - V(P)$. In light of this, we split the proof into two cases. First G is a 6-connected graph the does not contain K_4, and second G is a 6-connected graph the does contain K_4.

Consider two paths P_1 and P_2 both with endpoints x, y.

Definition 9. *The paths P_1 and P_2 are internally disjoint if $V(P_1) \cap V(P_2) = (x, y)$.*

A tool that is used frequently to examine connectivity is Menger's theorem (See [2]).

Theorem (Menger's Theorem)**.** *In a k-connected graph there exist at least k-internally disjoint paths between any two vertices.*

We may establish a useful counting principle to employ Menger's thoerem.

Definition 10. *Given a graph G and two vertices x, y in G the path count is the number of paths $P_1, P_2, ...P_n$ between x, y where $V(P_m) \in G$ for $m = 1, 2, ...n$.*

The following definition is quite useful in gaining information about a vertex.

Definition 11. *Let G be a graph. Let $x \in G$. Then $N(x)$ is defined to be the set of vertices in G who are adjacent to x.*

Claim 1. *If a graph G is k-connected and there exists a path P between any two vertices x, y and $(G - V(P)) \cup (x, y)$ is (k-1)-connected then P is one of the k paths identified by Menger's theorem.*

Proof. Let G be a k-connected graph and let $x, y \in V(G)$. Let S be the set of the paths between x, y. By Menger's theorem $|S| = k$ in G. If P is a path with endpoints x, y then in $G - V(P)$ let $S' = S \setminus P$. We know $|S'| = k - 1$ so the graph $G - V(P)$ is (k-1)-connected and thus P is a path in G identified by Menger's theorem. $\square$

Let T be a tree with a root r and let $N(r) \neq \emptyset$. Consider a vertex $v \in N(r)$. The set of all children of v is called a branch of T. This is true $\forall v \in N(r)$, therefore the number of branches of T is simply the size of $N(r)$.

In addition we define a tree that hits all vertices as the following.

Definition 12. *A spanning tree of a graph G is a tree T such that $V(T) = V(G)$.*

Definition 13. *The level of v, denoted $l(v)$ is the length of the shortest path P with end points r and v such that $E(P) \subset E(T)$.*

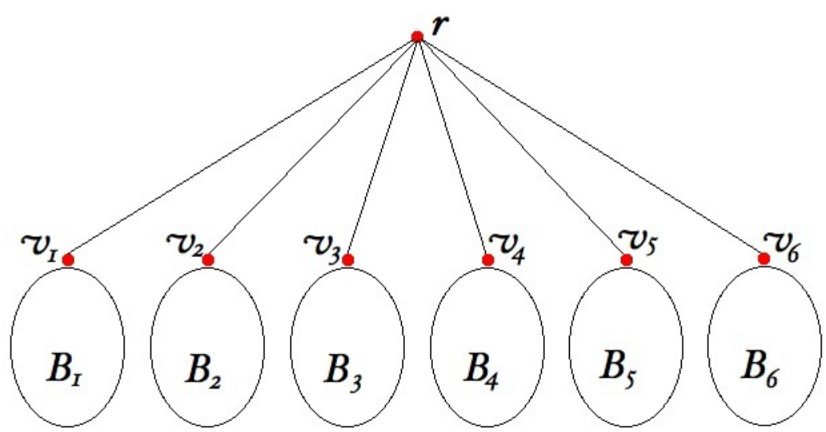
Definition 14. *Let T be a tree with $(B_1, B_2, ..., B_n)$ branches from a root r. A pseudo-tree on T is the graph obtained from T by adding a collection of edges, called* cross edges*, that are disjoint from the edges connecting $(B_1, B_2, ..., B_n)$ to the root r of T. Let $d(B_n)$ is the number of cross edges of the branch B_n. Let $D(B_n)$ denote the set of cross edges which intersect nontrivially with B_n. Finally, for notational convenience let $d(B_n) = |D(B_n)|$.*

Definition 15. *Two branches B_n and B_m of a pseudo-tree are linked if there exists a path P with endpoints x, y with $x \in B_n$ and $y \in B_m$ such that x is the only vertex of $B_n \in P$ and y is the only vertex of $B_m \in P$ where the root r of T is not in P.*

Case 1. *G is a 6-connected graph that does not contain K_4 as an induced subgraph.*

Theorem 3. *Any 6-connected bipartite graph G contains a path P between any two vertices such that $G - V(P)$ is 3-connected.*

Proof. Consider the graph G. Because G is bipartite we may consider the tree spanning tree T grown from a root r we pick such that $d(r) = 6$. Thus this tree T has precisely 6 branches from the root r. Let these branches be labelled $(B_1, B_2, ..., B_6)$. Let G be the pseudo-tree obtained from T by adding cross edges to T such that for $x \in T$ $d(x) \geq 6$ and $d(B_n) \geq 6$. Because G is 6-connected we know by Menger's Theorem that there exist 6 internally disjoint paths between any two vertices x, y of G. The tree T clearly has only one path between any of the branches, the path through the root r. Thus in order for G to be 6-connected each branch $(B_1, B_2, ..., B_6)$ in G must have at least 5 edges leaving it, in other words $\forall B \in (B_1, B_2, ..., B_6)$ we have $d(B) \geq 5$.



Claim 2. *There exists a path P between any two branches B_1, B_2 of the pseudo-tree G such that B_1, B_2 are linked by P.*

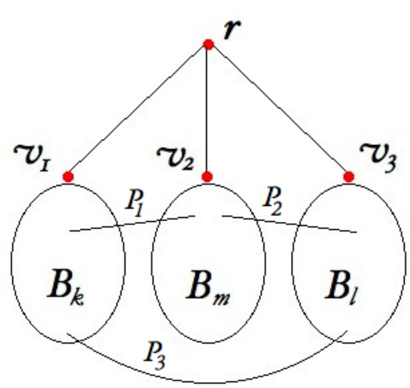
Proof. Let G be a 6-connected graph with 6-branches $(B_1, B_2, ..., B_6)$. Let B_k and B_n be two branches. By Menger's theorem there must exist 6-internally disjoint paths between the branch B_k and the branch B_n. Let $b_k \in B_k$ be a vertex incident with a cross edge of B_k. Let $b_n \in B_n$. In order to show B_k and B_n are linked it suffices to show that there exists a path P with end points b_n and b_k such that $V(P) \cap B_n = b_n$ and $V(P) \cap B_k = b_k$.

Consider a path P from B_n to B_k. It is clear that one can truncate P to get a path P' so that $V(P') \cap B_n = b_n$ and $V(P') \cap B_k = b_k$. □

Because Claim 1 was shown with two arbitrary branches we know the same is also true for a third branch $B_m \in G$. Then for the three branches B_n, B_m, B_k we know any two are linked. In addition we may show this useful property of the construction.

Claim 3. *If G is a 6-connected pseudotree obtained from a tree T, then a path P defined by $E(P) \subset E(T)$ is one of the paths identified by Menger's theorem.*

Proof. By Claim 1 it suffices to show that there exists a path $P \in G$ between any two vertices x, y such that $(G - V(P)) \cup (x, y)$ is 5-connected. Consider the path P in G such that $E(P) \subset V(P)$. Then $V(P) \subset V(B_n)$ for some n and where the root r might also be in P. However, consider the graph $G - V(P)$. Because $V(P) \subset V(B_n)$ for some n we know $d(B_k) \geq 5$ for all $k \neq n$. Thus it suffices to show that B_n itself is 5-connected. Without loss of generality assume $l(x) < l(y)$. Consider a vertex $v \in B_n$ such that $l(y) = l(v) + 1$. □



As can be seen there is a cycle induced by the paths (P_1, P_2, P_3) within the branches B_k, B_m, B_L in the graph G.

Consider a vertex v in the pseudo-tree G with an underlying tree T containing a root r. In addition, for every B_n there exists a vertex v_n such that $v_n \in N(r)$. Because G is 6-connected $d(v_n) \geq 6$. Clearly, one edge of v_n is incident with the root. Suppose $v_n \in B_n$ is incident with a cross edge e. Then clearly e is directed to some branch B_k. By above, we know there exists a branch B_m such that B_k and B_n are linked by a path P_2 and B_m and B_n are linked by a path P_3. Then we may choose $P_1 = e$ and the three paths identified by Claim 1 become e, P_2, P_3. However, suppose $\forall v_n \in G$ all edges of v_n are incident with vertices in B_n. In this case the following claim holds.

Claim 4. *At least 5 edges of v_n must connect into the same branch, B_n that contains v_n.*

Proof. There exists an ordering of vertices that are contained in the same branch. Consider two vertices of B_n, x, y. Either $l(x) < l(y)$ or $l(x) > l(y)$ or $l(x) = l(y)$ We know B_n must have 5 distinct cross edges so let $(w_1, w_2, ..., w_5)$ be the vertices incident with the cross edges the branch B_n. There must exist five disjoint paths $(P_1, P_2, ..., P_5)$ connecting v_n to $(w_1, w_2, ..., w_n)$ for the graph to be 6-connected. Because for all w_n, w_k we know $l(w_n) \neq l(w_k)$ choose $(w_1, w_2, ..., w_5)$ such that $(l(w_1) < l(w_2) < l(w_3) < l(w_4) < l(w_5))$. Let P_1 be one of the five paths $(P_1, P_2, ..., P_5)$ identified by Menger's theorem with endpoints v_1 and w_1. Because there must exist 6-internally disjoint paths from v_1 to all vertices $v \in B_2, B_3, B_4, B_5, B_6$ we can choose P_1 such that it does not contain (w_2, w_3, w_4, w_5). Therefore, there must be an edge e' connecting v_1 to a vertex c where $l(v_1) < l(c) < l(w_1)$. In addition, v_1 is connected to a vertex x where $l(x) = l(v_1) + 1$ by an edge e. Thus the set of edges, e, e', e'', e''', e'''' identify the 5 edges of v_n that connect to vertices in B_n. □

In fact, taking any vertex $v \in B_n$ as v_n demonstrates the inequality $l(v_n) < l(w_n) < l(c)$ must hold for all vertices in B_n, otherwise there would not exist 6-internally disjoint paths between $v \in B_n$ and $v \in B_m$ for $n \neq m$. In addition, we see that the case where v_n does contain a cross edge e is actually just the case where $v_n = w_1$ in the proof of Claim 2 and thus $E(P_1) = e$. Consider, again, the construction derived above of the branches containing the cycle defined by (P_1, P_2, P_3). By the Claim 2, in B_k there exists an edge g_1 leaving v_k and connecting below the vertex in B_k which is an endpoint of P_1. Similarly, there exists an edge g_1' leaving v_k that connects below the vertex in B_k which is an endpoint of P_3. A similar argument implies the existence of edges $g_2, g_2' \in B_m \, and \, g_3, g_3' \in B_l$. Thus given any two vertices $(x, y) \in G$ a path P connecting them such that $(v_k, v_m, v_l) \in P$ can be found by the following algorithm.

Step 1

Identify which branch x and y belong to. By Claim 1, there must exist a third branch such that both branches any two of the three branches are adjacent.

Step 2

Identify the paths P_1, P_2, P_3 by Claim 1 (where, as described above, it might be that all of the paths are cross edges incident with v_n). For each B_k, B_m, B_l identify each vertex in B_k, B_m, B_l which is an endpoint of P_1, P_2, P_3 and label them $w_1, w_1', w_2, w_2', w_3, w_3'$ since there will be at most 2 for each branch.

Step 4

Compute $l(w_n)$ and $l(w_n')$ for $n = 1, 2, 3$ and label w_n such that $l(w_m) < l(w_m')$

Step 5

Without loss of generality assume $x \in B_k$ and $y \in B_l$ Compute $l(x)$.

(Case1) If $l(x) > l(w_1')$ let $P = (x, v_1, w_1, w_2, v_2, w_2', w_3', v_3, y)$ plus any vertices$\in (B_1, B_2, ..., B_6)$ needed to reach any vertices of P

(Case 2) If $l(x) < l(w_1)$ let $P = (x, v_1, w_1', w_2', v_2, w_2, w_3, v_3, y)$ plus any vertices $\in (B_1, B_2, ..., B_6)$ needed to reach any vertices of P

(Case3) If $l(w_1) < l(x)$ the same of case 2. □

Without loss of generality assume $x \in B_1$ such that $x \neq w_1, w_1' or v_1$. Let $(y_1, y_2, ..., y_5)$ be the vertices incident with the cross edges leaving the branch B_1 such that $l(y_1) < l(y_2) <$

$l(y_3) < l(y_4) < l(y_5)$. By G is 6-connected $d(x) \geq 6$ and there must be 5 edges leaving x that connect to 5 distinct vertices $(q_1, q_2, q_3, q_4, q_5$ where $l(q_1) < l(q_2) < l(q_3) < l(q_4) < l(q_5)$. By similar argument we know $l(q_1) < l(y_1) < l(q_2) < l(y_2) < l(q_3) < l(y_3) < l(q_4) < l(y_4) < l(q_5) < l(y_5)$.

Claim 5. *Given the path P returned by Algorithm 1, at most 2 of $(y_1, y_2, ..., y_5)$ are in P*

Using the above inequality we may prove the claim.

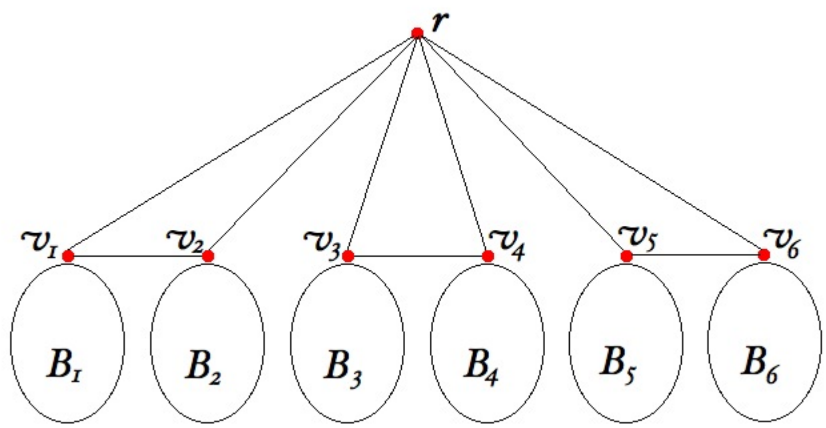
Proof. As above, let $x \in B_1$.
If P is the path returned by Case 1 we know $l(x) > l(y_n)$ for $n = 2, 3, 4, 5$. However, $n \neq 1$ because w_1 must also exist and $l(w_1) < l(w_1')$ and $l(y_1) < l(y_n)$ for $n = 2, 3, 4, 5$. By Algorithm 1 $x, v_1, w_1 \in P$. Consider the maximum number of $y_n \in P$. First, by Claim 3 $l(x) > l(y_n)$ for $n = 2, 3, 4, 5$ there must exist an edge e incident with v_1 and a vertex c where $l(c) > l(y_n)$ for $n = 2, 3, 4, 5$. Thus the path Q_1 from x to v_1 may be defined by $Q_1 = x, c, v_1$ and may reach v_1 without the removal of y_n. By $l(v_1) < l(y_1)$ and $l(y_1) < l(y_n)$ for $n = 2, 3, 4, 5$ there exists a path $Q_2 = v_1, y_1$ that contains only y_1. Thus the path $P_1 = Q_1 \cup Q_2$ defines a path leaving B_1 that contains only y_1. Suppose, the target vertex x' was also in B_1. Let P_2 be the path into B_1. Clearly, y_n for $n = 2, 3, 4, 5 \in P_2$. Let $l(y_n) < l(x') < l(y_m)$ where only one side of the inequality necessarily holds. Let c' be a vertex $\in N(y_n)$. Because Claim 3 also applies to $c' \in B_1$ there exists an edge e' from c' to a vertex c'' such that $l(y_n) < l(c'') < l(y_m)$. Therefore, the path $P_2 = y_n, c', c'', x'$ contains only y_n. The path $P = P_1 \cup P_2$ contains only y_1, y_n.

Case 2 may be shown in the exact same manner except with the inequality reversed.

Thus, given a path P returned by Algorithm 1 in the graph G, the graph $G' = G - V(P)$ is 3-connected. □

Claim 6. *Algorithm 1 also returns a 3-connected graph if G is a 6-connected 3-partite graph.*

Proof. Consider the graph G generated by first selecting a vertex r such that $d(r) = 6$ and subsequently identifying branches $B_1, B_2, B_3, B_4, B_5, B_6$. However, the difference between the bipartite case and the tripartite case is that in a tripartite graph G is $3 - colorable$ so there may now exist edges connecting $(v_1, v_2, v_3, v_4, v_5, v_6)$. However, because r clearly has a color c, $(v_1, v_2, v_3, v_4, v_5, v_6)$ can have at most 2 colors which means there can exist at most 3 edges between $(v_1, v_2, v_3, v_4, v_5, v_6)$. Thus, at most 3 of $(v_1, v_2, v_3, v_4, v_5, v_6)$ are in P returned by Algorithm 1. Thus $G - V(P)$ will again yield a 3-connected graph. □



Therefore, if G is a 6-connected *bipartite* or *tripartite* graph there exists a path P between any two vertices such that $G - V(P)$ is 3-connected. The next case, where G does contain K_4, follows.

Case 2. *The graph G is a 6-connected graph that contains K_4.*

Let G be a 6-connected graph that contains K_4. We know that if G is 6-connected the minimum degree is 6. Therefore, each vertex of K_4 must be incident with at least three other distinct vertices outside of K_4.

Definition 16. *Let $x \in K_4$ where $K_4 \in G$. Then let $N'(x) = N(x) \setminus V(K_4)$.*

Thus, for G to be 6-connected $d(v_n) \geq 6$ where $v_n \in K_4$. Let $B = \cup N'(v_n) \ \forall \ v_n \in K_4$. Also, let a path P be a *neighbor path* if P has endpoints x, y such that $x, y \in B$ and $K_4 \not\subset V(P)$. Thus, a neighbor path is simply a path incident with the neighbors of $K_4 \in G$.

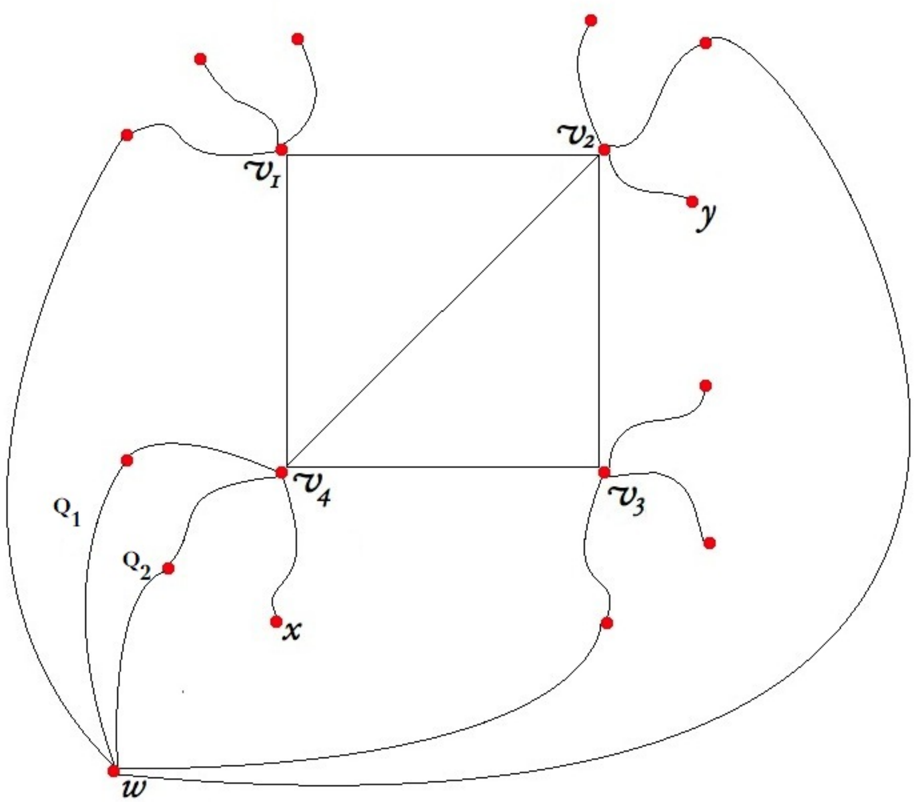
Claim 7. *Let G be a 6-connected graph containing K_4. Let $v_n \in K_4$. For any vertex $w \in G$ such that $w \notin (B \cup K_4)$ there must exist 2 paths, P and P', such that both P and P' have endpoints w and $x, x' \in N'(v_n)$ respectively.*

Proof. Let $x, y \in B$. Consider a vertex $w \in G$ incident with $K_4 \cup B$. By Menger's theorem there must exist 6-internally disjoint paths between w and $V(K_4)$. Let (v_1, v_2, v_3, v_4) denote $V(K_4)$. Let P_1 be the path identified by Menger's theorem with endpoints w and v_1 and P_2 the path identified by Menger's theorem with endpoints w and v_2. Clearly there exists an edge e with endpoints v_1 and v_2. Thus, we may identify a path $P_1' = P_1 \cup e$ with endpoints v_2 such that $v_1 \in V(P_1')$. In addition, we may identify a path $P_2' = P_2 \cup e$ with endpoints v_1 such that $v_2 \in V(P_2')$. Similarly, we may identify a set of three paths, denoted S for each $v_n \in K_4$ such that $V(K_4) \setminus v_n \in S$. Consider a path Q from w to v_n. We know Q must include some $x \in B$ because Q reached $V(K_4)$ via $N'(v_n)$ for each $v_n \in K_4$. In fact, since Menger's theorem identifies at least 6 paths from w to each $v_n \in K_4$. Because there exist at most 4 disjoint sets of $N'(v_n)$ We know by the Pigeon Hole principle that there must exist some v_n such that two paths identified by Menger's theorem P and P' have end points w and $x, x' \in N'(v_n)$. Thus the claim is proven. □

Thus by Claim 2 we may identify the vertices x and y such that $x, y \in N'(v_n)$. Let Q_1 be the path defined by $Q_1 = w, x, v_1$ and let Q_2 be the path defined by $Q_2 = w, y, v_1$

Claim 8. *The path Q_2 may only increase the path count of v_1 by one and does not change the path count of v_2, v_3, v_4.*

Proof. We know Q_1 and Q_2 may contribute 2-internally disjoint paths between w and v_1. However, consider the path Q_3 defined by $Q_3 = w, x, v_1, v_2$ between w and v_2. Thus $V(Q_1) \subset V(Q_3)$. However, by $v_1 \in Q_3$, $V(Q_2) \subset V(Q_3)$ and Q_1 and Q_2 do not contribute 2-internally disjoint paths between w and v_2, but only contribute one, namely the path Q_3. Thus, in order for Q_1 and Q_2 to contribute 2-internally disjoint paths between w and v_2 there must exist a path P from $y \in N'(v_1)$ to v_2. Thus Q_2 may be replaced by $(Q_2 \setminus v_1) \cup P$. □



Consider the path $(Q_2 \setminus v_1) \cup P$ described in Claim 4. There must exist at least three such paths with endpoints $x \in N'(v_1)$ to $v_n \in K_4$ where $n \neq 1$ in order to account for at least 6-internally disjoint paths given by Menger's theorem. In addition, three such paths must exist $\forall v \in K_4$. Thus let $x \in N'(v_n)$ and let $y \in N'(v_k)$. Then we know there exists a path P with endpoints x, y such that $V(P) \cap V(K_4) = \emptyset$.

Claim 9. *Given a 6-connected graph G for any two vertices $x, y \in G$ there exists a path P with endpoints x and y such that for some $v_n \in K_4$ we know $N'(v_n) \subset P$.*

Proof. Given $(x, y)\, inG$. Let $q \in N'(v_n)$. By Claim 5 there exists a path Q_1 with endpoints x and q. By above there must exist a path from $N'(v_n)$ to $N'(v_k)$. Let this path be called Q_2. Again, there exists a path from $N'(v_k)$ back to $N'(v_n)$. Let this path be called Q_3. There again exists an path from $N'(v_n)$ to $N'(v_m)$ (where $N'(v_m)$ and $N'(v_k)$ are not necessarily distinct). Let this path be called Q_4. Also, there must exist a path from $N'(v_m)$ to $N'(v_n)$. Let this path be called Q_5. In addition, y must have a path to $N'(v_n)$. Let this path be called Q_6. We know $(Q_1, Q_2, Q_3, Q_4, Q_5, Q_6)$ are internally disjoint. Thus the path $P = (Q_1, Q_2, Q_3, Q_4, Q_5, Q_6)$ defines a path such that $N'(v_n) \subset P$. $\square$

Claim 10. *Given P above, $G - V(P)$ is 3-connected.*

Because $N'(v_n) \subset P$, $N'(v_n)$ in $G - V(P)$ is the $\emptyset$. Thus $N(v_n) = V(K_4) \setminus v_n$. We know $V(K_4) \setminus v_n$ induces a 3-vertex cut in the graph $G - V(P)$. Thus, $G - V(P)$ is 3-connected.

Acknowledgments. It is a pleasure to thank my mentor, Khalid, for his extremely helpful advice and guidance throughout the paper process. In addition, I would like to thank Erica Fagin for illustrating the diagrams throughout the paper.

References

[1] G. Chen, R. Gould, X. Yu, Graph connectivity after path removal, Combinatorica 23 (2003) 185203.
[2] R. Diestel, Graph Theory, third ed., Springer-Verlag, 2005
[3] K. Kawarabayashi et al, A weaker version of Lovszs path removal conjecture, Journal of Combinatorial Theory, Series B 98 (2008) 972979
[4] L. Lovsz, Problems in graph theory, in: M. Fielder (Ed.), Recent Advances in Graph Theory, Acadamia Prague, 1975