EXPLAINING BIFURCATIONS

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Abstract. This paper will introduce the topic of dynamical systems with both discrete and continuous time variables. Fixed points will be discussed, along with their properties such as stability or topological type. The paper will continue on to define the concept of hyperbolicity and its relevance in determining the structural stability of the system. It will conclude with a definition of a bifurcation as well as a brief description of bifurcation theory and its applications.

1. Introduction

A dynamical system is a system that changes with time. The time variable can either be discrete or continuous. With a discrete time variable, the system can be an iterated function based on time t in the integers ($\mathbb{Z}$) or the natural numbers ($\mathbb{N}$). With a continuous time variable, the system can be viewed as a differential equation with time t in the reals ($\mathbb{R}$). With such systems, variable x refers to the state of the system at a given time t.

Definition 1.1. A *homeomorphism* f is a bijection where both f and f^{-1} are continuous mappings.

Definition 1.2. A *diffeomorphism* f is a homeomorphism that is not only continuous, but differentiable. If f and f^{-1} are C^k, f is called a *C^k-diffeomorphism*.

These dynamical systems occur in differentiable manifolds, which resemble Euclidean spaces in small neighborhoods. This means that we can locally map an n-dimensional manifold onto $\mathbb{R}^n$ using a homeomorphism to construct a *chart*.

Definition 1.3. Given an n-dimensional manifold M, one can define a homeomorphism f from a neighborhood $U \subseteq M$ around a point $x \in M$ to a neighborhood $V \subseteq \mathbb{R}^n$ around $f(x) \in \mathbb{R}^n$. (V, f) is a *chart* that defines differentiability on U.

Diffeomorphisms can be used to represent systems with a discrete time variable by taking $f : M \to M$ with the initial value of $x_0 \in M$. When looking at a diffeomorphism $f : M \to M$, each point $x \in M$ is moved along an orbit under f. Since diffeomorphisms concern discrete units of time, for $s \in \mathbb{Z}$, f^s is:

- f composed with itself s times for $s \in \mathbb{Z}^+$
- $(f^{-1})^s$ for $s \in \mathbb{Z}^-$
- identity map on M (id_M) for $s = 0$.

The inverse of f, f^{-1} is known to exist by the definition of a diffeomorphism. This creates an action of $\mathbb{Z}$ on the differentiable manifold M generated by f. For a continuous time variable, one needs to create an action of $\mathbb{R}$ on M, or a *flow*.

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Definition 1.4. A *flow* on M is a continuously differentiable function $\phi : \mathbb{R} \times M \to M$ such that for every $t \in \mathbb{R}$, the restriction $\phi(t, \cdot) = \phi_t(\cdot)$ has the properties that $\phi_0 = id_M$ and $\phi_t * \phi_s = \phi_{t+s}$ for $t, s \in \mathbb{R}$.

The definition of flow implies that the inverse of ϕ_t exists, which is ϕ_{-t}. Given that ϕ is C^1, one can conclude that for each $t \in \mathbb{R}$, $\phi_t : M \to M$ is a diffeomorphism.

Definition 1.5. The *orbit* of a point $x \in M$ under f is the set of points that f sends x to, or $\{f^m(x) | m \in \mathbb{Z}\}$. For a flow, the *orbit* of ϕ through x is $\{\phi_t(x) | t \in \mathbb{R}\}$, oriented in the direction that t is increasing.

Definition 1.6. A *fixed point* of f is a point $x \in M$ such that $f^s(x) = x$ for all $s \in \mathbb{Z}$. For a flow, if $\phi_t(x) = x$ for all $t \in \mathbb{R}$, x is a *fixed point*.

The orbit of a fixed point is just the point itself.

Definition 1.7. A *periodic point* of f is a point $x \in M$ such that $f^r(x) = x$ for some integer $r \geq 1$.

The smallest r such that $f^r(x) = x$ is referred to as the *period* of x. The orbit of this x is then an r-cycle, with each point in it having period r. In a similar manner, closed orbits of flows give rise to periodic points.

Definition 1.8. A *closed orbit* of a flow is an orbit γ which is not a fixed point, but satisfies $\phi_t(x) = x$ for some $x \in \gamma$ and some $t \neq 0$.

If τ is the smallest time after which $\phi_t(x)$ returns to x, then x is periodic with period τ. In fact, the entire orbit γ is periodic of order τ. A concept similar to fixed points can be created for sets in M. Invariant sets are not moved by the diffeomorphism or flow in question.

Definition 1.9. A set $A \subseteq M$ is *invariant* under the diffeomorphism f if $F^s(x) \in A$ for each $x \in A$ and all $s \in \mathbb{Z}$. Similarly, A would be invariant under the flow ϕ if $\phi_t(x) \in A$ for all $x \in A$ and all $t \in \mathbb{R}$. This is denoted by:

$$f^s(A) \subseteq A, \quad \forall s \in \mathbb{Z} \tag{1.10}$$

or

$$\phi_t(A) \subseteq A, \quad \forall t \in \mathbb{R} \tag{1.11}$$

The orbit of any point is always an example of an invariant set. Closed orbits and fixed points are special examples of invariant sets that are periodic and do not have invariant subsets.

2. Stability and Topological Type

Since the object of this paper is to study bifurcations, or points with neighborhoods containing diffeomorphisms that are distinct topologically, the notions of fixed point stability and topological types must be explained. The stability of a point is established by looking at the orbit of points around it.

Definition 2.1. A fixed point x is said to be *stable* if for every neighborhood N of x, there is a neighborhood $N' \subseteq N$ of x such that if $x \in N'$ then $f^s(x) \in N$ for all $s \geq 1$.

If y is a stable fixed point and $\lim_{s\to\infty} f^s(x) = y$ for $x \in \mathbb{N}$, then y is referred to as being *asymptotically stable*. As s approaches infinity, the orbits of points in the neighborhood of y approach y. If a point is stable but not asymptotically so, it is referred to as being *neutrally stable*. Fixed points that do not satisfy the definition of stable are called *unstable*.

Flows become solutions to a differential equation by defining the vector field X of a flow ϕ for every $x \in M$ to be

$$X(x) = \frac{d\phi_t}{dt}(x)|_{t=0} = \lim_{\varepsilon\to 0} \frac{\phi(\varepsilon, x) - \phi(0, x)}{\varepsilon}. \tag{2.2}$$

A flow is just a curve through a point $x \in M$. The vector $X(x)$ is like a velocity curve in that it is tangent to the manifold M at the point x and has a magnitude proportional to speed under parametrization by t. Essentially, the differential equation, vector field, and flow represent the same system in different ways.

Definition 2.3. Two diffeomorphisms, $f, g : M \to M$ are *topologically conjugate* if there exists a homeomorphism $h : M \to M$ such that

$$h \circ f = g \circ h$$

Topological conjugacy is similarly defined for two flows, ϕ_t, ψ_t, but this definition is not as good for the continuous time variable. When looking at the orbits of a diffeomorphism, given the discrete time variable, it makes sense to require that two orbits be considered alike only if they match up completely in size. However, with a continuous time variable, intuitively, a stretched out orbit (i.e. differing by multiplication of a constant) should be considered similar to the original. So, another definition for comparison is made for flows.

Definition 2.4. Two flows, ϕ_t and ψ_t are *topologically equivalent* if there exists a homeomorphism h that takes orbits of ϕ_t to orbits of ψ_t while preserving their orientation.

Given two fixed points, y_1 and y_2 associated with diffeomorphisms f_1 and f_2 respectively, the fixed points are of the same *topological type* if the associated diffeomorphisms are topologically conjugate. Similarly, two fixed points of flows are of the same topological type if the flows are topologically equivalent. Although the systems discussed here are occurring on differentiable manifolds, the neighborhoods around the fixed points in question are small enough to be in a single chart, so it is equivalent (and much simpler) to talk about open sets in $\mathbb{R}^n$.

3. Hyperbolicity

Definition 3.1. A linear diffeomorphism is *hyperbolic* if it has no eigenvalues with modulus equal to unity.

Since not all diffeomorphisms are linear, a method to view a non-linear diffeomorphism in a linear way is defined.

Definition 3.2. Given U, an open subset of $\mathbb{R}^n$ and $f : U \to \mathbb{R}^n$ a non-linear diffeomorphism with an isolated fixed point at $y \in U$, the *linearization* of f at y is

$$Df(y) = \left[\frac{\partial f_i}{\partial x_j}\right]_{i,j=1}^{n} \Big|_{x=y} \tag{3.3}$$

where $x_1, ..., x_n$ are coordinates on U.

With this definition, the concept of hyperbolicity can be generalized to apply to non-linear diffeomorphisms as well.

Definition 3.4. A fixed point y of f is *hyperbolic* if $Df(y)$ is a hyperbolic linear diffeomorphism.

Theorem 3.5. *If $A : M \to M$ is a hyperbolic linear diffeomorphism, then there are subspaces E^s and $E^u \subseteq \mathbb{R}^n$ invariant under A such that $A|E^s$ is a contraction, $A|E^u$ is an expansion, and $E^s \oplus E^u = \mathbb{R}^n$.*

Definition 3.6. A *contraction (expansion)* is a linear diffeomorphism $L : \mathbb{R}^n \to \mathbb{R}^n$ such that all of its eigenvalues have modulus less than (greater than) unity.

The subspaces E^s and E^u are the eigenspaces of A with eigenvalues of modulus less than and greater than unity respectively. They are referred to as the stable (E^s) and unstable (E^u) eigenspaces of A. If A is hyperbolic, then there are no eigenvalues of unit modulus, meaning that $E^s \oplus E^u = \mathbb{R}^n$.

For a linear transformation (or linearization) $A : \mathbb{R}^n \to \mathbb{R}^n$, there is a corresponding linear flow on $\mathbb{R}^n$, denoted

$$\phi_t(x) = exp(At)x \tag{3.7}$$

where $exp(At)$, the *exponential matrix*, is

$$exp(At) = \sum_{k=0}^{\infty} \frac{(At)^k}{k!} \tag{3.8}$$

Definition 3.9. The linear flow $exp(At)x$ is *hyperbolic* if A has no eigenvalues with real part zero.

Proposition 3.10. *A linear flow $exp(At)x$ is hyperbolic iff the diffeomorphisms that make up the flow $exp(At)$ are hyperbolic.*

Proof. ($\Leftarrow$) If $exp(At)x$ is hyperbolic, by definition 3.9, A has no eigenvalues with real part zero. This implies that the matrix $exp(At)$ has no eigenvalues with unit modulus, meaning that by definition 3.1, the diffeomorphisms $exp(At)$ are hyperbolic.
($\Rightarrow$) If the diffeomorphisms $exp(At)$ are hyperbolic, by definition 3.1, the matrix has no eigenvalues of unit modulus. Subsequently, the matrix A has no eigenvalues with zero as the real part. Therefore, the flow $exp(At)x$ is hyperbolic by definition 3.9. $\square$

If $exp(At)x$ is hyperbolic then $Ax = 0$ has only the trivial solution, making A non-singular. This implies that the only fixed point of the flow is the origin. When this happens, both the fixed point and the vector field Ax are hyperbolic as well.

If A is a hyperbolic linear diffeomorphism with stable and unstable eigenspaces E^s_A and E^u_A respectively, define $A_i = A|E^i_A, i = s, u$. Then A_i is called *orientation-preserving* if $Det(A_i) > 0$. A_i is *orientation-reversing* if $Det(A_i) < 0$.

Theorem 3.11. *Let $A, B : \mathbb{R}^n \to \mathbb{R}^n$ be hyperbolic linear diffeomorphisms. Then A and B are topologically conjugate if and only if*

- *$dimE^s_A = dimE^s_B$*
- *for $i = s, u$, A_i and B_i are both either orientation-preserving or both orientation-reversing.*

Theorem 3.12. *If x^* is a hyperbolic fixed point of diffeomorphism $f : U \to \mathbb{R}^n$, then there is a neighborhood $N \subseteq U$ of x^* and a neighborhood $N' \subseteq \mathbb{R}^n$ containing the origin such that $f|N$ is topologically conjugate to $Df(x^*)|N'$.*

Theorem 3.13. *If $\dot{x} = Ax$ defines a hyperbolic linear flow on $\mathbb{R}^n$ with $dim E^s = n_s$, then $\dot{x} = Ax$ is topologically equivalent to the system*

$$\begin{aligned} \dot{x}_s &= -x_s \quad x_s \in \mathbb{R}^{n_s} \\ \dot{x}_u &= x_u \quad\; x_u \in \mathbb{R}^{n_u} \end{aligned}$$

where $n_u = n - n_s$.

This theorem implies that two hyperbolic, linear flows $exp(At)x$ and $exp(Bt)x$ are topologically equivalent if A and B have the same number of eigenvalues with positive (or negative) real part because each will be topologically equivalent to 3.16. Theorem 3.13 indicates that $exp(At)|E^i, i = s, u$ is orientation-preserving for all t, meaning there is only one planar saddle-type flow (up to topological equivalence). However, theorems 3.11 and 3.12 imply that there are 4 topologically distinct saddle-type diffeomorphisms each depending on whether A_s and A_u are each orientation-preserving or orientation-reversing. The general implication is that there are $n + 1$ distinct topological types of hyperbolic linear flows in $\mathbb{R}^n$ and $4n$ distinct topological types of hyperbolic linear diffeomorphisms in $\mathbb{R}^n$.

Theorem 3.14. *If x^* is a hyperbolic fixed point of $\dot{x} = X(x)$ with flow $\phi_t : U \subseteq \mathbb{R}^n \to \mathbb{R}^n$, then there is a neighborhood N of x^* on which ϕ is topologically conjugate to the linear flow $exp(DX(x^*)t)x$.*

This theorem allows for a generalized version of theorem 3.13 that provides a classification of hyperbolic fixed points.

Theorem 3.15. *If x^* is a hyperbolic fixed point of $\dot{x} = X(x)$ with flow $\phi_t : U \subseteq \mathbb{R}^n \to \mathbb{R}^n$, then there is a neighborhood N of x^* on which ϕ_t is topologically equivalent to the flow of the linear differential equation*

$$\begin{aligned} \dot{x}_s &= -x_s \quad x_s \in \mathbb{R}^{n_s} \\ \dot{x}_u &= x_u \quad\; x_u \in \mathbb{R}^{n_u} \end{aligned}$$

where $n_u = n - n_s$. The number n_s is the dimension of the stable eigenspace of $exp(DX(x^*)t)$.

4. Structural Stability

Ideally, when using mathematics to model the real world, it would be best for the model to not change too much even if the quantities involved are slightly changed (or perturbed). In a space of dynamical systems, a concept of 'closeness' can be developed to determine if a small perturbation results in a significantly different system. Like hyperbolicity, *structural stability* is a property of a dynamical system, but not necessarily typical of all systems in the space. First, a few definitions are needed to reach the definition of structural stability. Let $L(\mathbb{R}^n)$ be the set of real linear transformation of $\mathbb{R}^n$ to itself. These transformations are written as $n \times n$ matrices. The *norm* of such a matrix $A = [a_{ij}]$ is defined to be $\|A\| = \sum_{i,j=1}^{n} |a_{ij}|$. A very small area around A called an *ε-neighborhood* and is defined to be $N_\varepsilon(A) = \{B \in L(\mathbb{R}^n ||\|B - A\| < \varepsilon\}$. Every $B \in N_\varepsilon$ is said to be ε-close to A.

Definition 4.1. A linear flow, $exp(At) : \mathbb{R}^n \to \mathbb{R}^n$, is said to be *structurally stable* in $L(\mathbb{R}^n)$ if there exists an ε-neighborhood of A, $N_\varepsilon(A) \subseteq L(\mathbb{R}^n)$, such that for every $B \in N_\varepsilon(A), exp(Bt)$ is topologically equivalent to $exp(At)$. Similarly, a diffeomorphism A is said to be structurally stable if for every $B \in N_\varepsilon(A), B$ is topologically conjugate to A.

Theorem 4.2. *A linear flow or diffeomorphism on $\mathbb{R}^n$ is structurally stable in $L(\mathbb{R}^n)$ if and only if it is hyperbolic.*

Proof. ($\Leftarrow$) If $exp(At)$ is hyperbolic, all of the eigenvalues of matrix A have non-zero real part. Also, if a matrix B is ε-close to A, then its eigenvalues differ from those of A by a term $O(\varepsilon)$ (i.e. of order ε). Because of this fact, if ε is small enough, the eigenvalues of B must also have non-zero real part, and the eigenvalues corresponding to those of matrix A with negative (positive) real parts will also have negative (positive) real parts. This means that A and B will both have stable (unstable) eigenspace of dimension $n_s(n_u)$. Now, by theorem 3.13, $exp(At)$ and $exp(Bt)$ are both equivalent to the same flow, making them equivalent to each other. Therefore A is structurally stable.
($\Rightarrow$) On the other hand, if the flow $exp(At)$ is *not* hyperbolic, then the matrix A has at least one eigenvalue with zero real part. Then a matrix $B = A + \varepsilon I$ is hyperbolic for most $\varepsilon \neq 0$, and with ε small enough, B can be made arbitrarily close to A. This indicates that $exp(At)$ is not structurally stable. Therefore, if a linear flow is structurally stable, it must be hyperbolic. The proof for a diffeomorphism follows similar lines. □

The next step is to discuss structural stability of non-linear dynamical systems. For an open set U in $\mathbb{R}^n$, let $Vec^1(U)$ be the set of C^1-vector fields on U. The size of a vector field $\mathbf{X} \in Vec^1(U)$ is given by its C^1-*norm*, $\|\mathbf{X}\|_1$,

$$\|\mathbf{X}\|_1 = \sup_{x \in U} \left\{ \sum_{i=1}^{n} |X^i(x)| + \sum_{i,j=1}^{n} \left| \frac{\partial X^i(x)}{\partial x_j} \right| \right\}.$$

With this definition, if $\mathbf{X}(x) = (X^1(x), ..., X^n(x))^T$, then $\|\mathbf{X}\|_1$ is small when $X^i(x)$ and $\partial X^i(x)/\partial x_j$, $i, j = 1, ..., n$, are small for all $x \in U$. An ε-neighborhood of X in $Vec^1(U)$ can be defined as

$$N_\varepsilon(X) = \{Y \in Vec^1(U) | \|X - Y\|_1 < \varepsilon\}.$$

A vector field $Y \in Vec^1(U)$ is called specifically $\varepsilon - C^1$-close to X to make clear that not only the values are close, but the values of the first-order partial derivatives are also close.

Theorem 4.3. *If $X \in Vec^1(U)$ has a hyperbolic singularity x^*, then there is a neighborhood V of x^* in U and a neighborhood N of X in $Vec^1(U)$ such that each $Y \in N$ has a unique hyperbolic singularity $y^* \in V$. In addition, the stable (unstable) eigenspace of the linearized flow $exp(DY(y^*)t)$ is of the same dimension as that of $exp(DX(x^*)t)$.*

Looking at this theorem with theorem 3.14, it is clear that X and Y are topologically equivalent on neighborhoods around their respective hyperbolic singularities. This theorem also gives that the dimensions of the stable eigenspaces of $exp(DY(y^*)t)$ and $exp(DX(x^*)t)$ are equal, meaning that they are topologically equivalent by theorem 3.13. Using theorem 3.14 again to assert that there is a neighborhood of the hyperbolic singularities on which flows $\dot{x} = X(x)$ and $\dot{y} = Y(y)$ are topologically conjugate to $exp(DX(x^*)t)$ and $exp(DY(y^*)t)$, U_{x^*} and U_{y^*} can be defined to be the respective neighborhoods on which theorem 3.14 holds. Then a local C^0-equivalence comes from

$$\phi_t|U_{x^*}\ exp(DX(x^*)t)\ exp(DY(y^*)t)\ \psi_t|U_{y^*}, \tag{4.4}$$

where ϕ_t is the flow on X and ψ_t is the flow on Y. In other words, $\phi_t : U \to \mathbb{R}^n$ is *locally structurally stable*, since for every $Y \in N$, there is a neighborhood U_{y^*} on which the flow of Y is C^0-equivalent to the flow on X. The equation 4.4 can also be interpreted to say that the *topological type* of the fixed point x^* is structurally stable. Hyperbolic fixed points are maintained with small C^1-perturbations for diffeomorphisms as well as flows. Let $Diff^1(U)$ is the set of C^1-diffeomorphisms $f : U \subseteq \mathbb{R}^n \to \mathbb{R}^n$, with the C^1-norm.

Theorem 4.5. *If x^* is a hyperbolic fixed point of a diffeomorphism $f : U \to \mathbb{R}^n$, then there exists a neighborhoods $V \in U$ and $N \in Diff^1(U)$ such that every $g \in N$ has a unique hyperbolic fixed point $y^* \in V$ with the same topological type as x^*.*

5. Bifurcations

With the established definitions and theorems, the topic of bifurcations can now be approached. Dynamical systems often involve more than just a single differential equation or diffeomorphism, but rather a group or *family* of them. A bifurcation refers to when a topological change occurs in a neighborhood, creating distinct topological types in the family.

Definition 5.1. Let $X : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$ be a C^r-family of vector fields with m parameters on $\mathbb{R}^n$, or $(\mu, x) \mapsto X(\mu, x), \mu \in \mathbb{R}^m, x \in \mathbb{R}^n$. The family X has a *bifurcation point* at μ^* if in every neighborhood of μ^* there exists values of μ such that the corresponding vector fields $X(\mu, \cdot) = X_\mu(\cdot)$ have topologically distinct behavior. A similar definition can be constructed using diffeomorphisms.

Bifurcations clearly do not occur in structurally stable members of the family. In systems with continuous time (flows), bifurcations occur when the real part of an eigenvalue of a fixed point is zero. They also occur in systems with discrete time (diffeomorphisms) when the function has an eigenvalue of modulus one. Essentially, non-hyperbolic fixed points of flows and diffeomorphisms are structurally unstable (as can be inferred from theorem 4.2) and can lead to bifurcations. Different bifurcation types are classified based on the actual value of eigenvalues of non-hyperbolic fixed points. The study of bifurcations, known as bifurcation theory is used to study structural stability in different regions of a dynamical system. The known information can be used to construct a map showing the properties within systems of the same bifurcation type. Such maps can be used to create models to analyze data gathered from dynamical systems such as human behavior, morphogenesis, or fluid dynamics. Understanding properties of bifurcations can ultimately help in understanding how systems in nature operate.

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References

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