# ELEMENTARY TOPOI: SETS, GENERALIZED 

CHRISTOPHER HENDERSON


#### Abstract

An elementary topos is a nice way to generalize the notion of sets using categorical language. If we restrict our world to categories which satisfy a few simple requirements we can discuss and prove properties of sets without ever using the word "set." This paper will give a short background of category theory in order to prove some interesting properties about topoi.


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## 1. Categorical Basics

Definition 1.1. A category, $\mathcal{C}$ is made up of:

- Objects: $A, B, C, \ldots$
- Morphisms: $f, g, h, \ldots$

This is a generalization of "functions" between objects. If $f$ is a morphism in a category $\mathcal{C}$ then there are objects $A, B$ in $\mathcal{C}$ such that $f$ is a morphism from $A$ to $B$, as below. We call $A=\operatorname{dom}(f)$ (the "domain") and $B=\operatorname{cod}(f)$ (the "codomain").

$$
A \xrightarrow{f} B
$$

[^0]If we have morphisms $f: A \rightarrow B$ and $g: B \rightarrow C$ then there exists an arrow $h: A \rightarrow C$ such that the following diagram commutes:


Here we are only asserting some rule of composition.
We denote $h=g \circ f$. This rule of composition must satisfy associativity, so that for $f: A \rightarrow B, g: B \rightarrow C, h: C \rightarrow D$, we have that:

$$
h \circ(g \circ f)=(h \circ g) \circ f
$$

Lastly, for every object $A$ in $\mathcal{C}$, there is a morphism $i d_{A}: A \rightarrow A$ such that for any other morphisms $g: A \rightarrow B$ and $h: B \rightarrow A$, we have that:

$$
g=g \circ i d_{A}, \quad h=i d_{A} \circ h
$$

Remark 1.2. We will sometimes, for convenience, write $f \circ g$ as $f g$ and we will rarely use parentheses. Also, we may refer to morphisms as "arrows."

Definition 1.3. The dual of a category $\mathcal{C}$ is a category $\mathcal{C}^{o p}$, which has all the same objects in $\mathcal{C}$, but with all morphisms reversed. Explicitly, for $f: A \rightarrow B$, $f^{o p}: B \rightarrow A$ is a morphism in $\mathcal{C}^{o p}$.

Definition 1.4. Two objects $A, B \in \mathcal{C}$ are isomorphic if there exist morphisms $f: A \rightarrow B, g: B \rightarrow A$ such that $f g=i d_{B}$ and $g f=i d_{A}$.

Definition 1.5. An arrow $m: C \rightarrow D$ is monic if for any pair $B \underset{g}{\stackrel{f}{\longrightarrow}} C$ such that $m f=m g$, then $f=g$. An arrow $e: A \rightarrow B$ is epi if for any parallel arrows $f, g$ with $f e=g e$ then $f=g$. We denote a monic by $C{ }^{m}>D$ and an epi by $A \xrightarrow{e} B$.

A monic is just the generalization of a one-to-one function and an epi is the generalization of an onto function. It should be noted that it is not in general true that in categories where objects have underlying sets (like Set, Top, Grp, etc.) a monic is one-to-one and an epi is surjective.
Example 1.6. Define a category with the two sets $\{a, b\}$ and $\{c\}$ as objects and with morphisms $i d_{\{a, b\}}, i d_{\{c\}}, f, g, g f$ where $f:\{a, b\} \rightarrow\{c\}$ and $g:\{c\} \rightarrow\{a, b\}$. Clearly $f(a)=c=f(b)$, but define $g(c)=a$. Clearly $f$ is not one-to-one, but it is indeed monic.

Lemma 1.7. If $m: A \rightarrow B, n: B \rightarrow C$, with nm monic, then $m$ must be monic.
Proof. Suppose there are $g, h: D \rightarrow A$ such that $m g$ equals $m h$. Then composotion on the left gives that $n m g$ equals $n m h$. Since $n m$ is monic, $g$ equals $h$.

Definition 1.8. A terminal object of $\mathcal{C}$, denoted 1, is an object such that for any object $A$ in $\mathcal{C}$, there is exactly one arrow $!_{A}: A \rightarrow \mathbf{1}$.

Definition 1.9. An initial object of $\mathcal{C}$, denoted $\mathbf{0}$, is an object which is a terminal object in $\mathcal{C}^{o p}$.

Note that our definition of initial object is just a fancy way of describing an object that has exactly one morphism $!_{A}: \mathbf{0} \rightarrow A$ for every object $A$. In the category where objects are all sets and morphisms are all functions between sets, a terminal object is any set with one element and an initial object is the empty set. In other categories, initial and terminal objects are simply generalizations of this idea.
Proposition 1.10. In a category $\mathcal{C}$, all terminal objects are isomorphic. Similarly, all initial objects are isomorphic.
Proof. Let $\mathbf{1}, \mathbf{1}^{\prime}$ be two terminal objects. Then there are unique arrows $f: \mathbf{1} \rightarrow \mathbf{1}^{\prime}$, $g: \mathbf{1}^{\prime} \rightarrow \mathbf{1}$. Note that $f g: \mathbf{1}^{\prime} \rightarrow \mathbf{1}^{\prime}$. But $i d_{\mathbf{1}^{\prime}}: \mathbf{1}^{\prime} \rightarrow \mathbf{1}^{\prime}$. Thus by uniqueness $f g=i d_{\mathbf{1}^{\prime}}$. Similarly, $g f=i d_{\mathbf{1}}$.

The result for initial objects follows from the fact that an initial object is a terminal object in the dual category.

Definition 1.11. A diagram $A<{ }_{\leftarrow}^{\pi_{1}} P \xrightarrow{\pi_{2}} B$ is a product of $A$ and $B$ if for any arrows $\phi_{1}: C \rightarrow A, \phi_{2}: C \rightarrow B$, there is a unique arrow $h: C \rightarrow P$ making the following diagram commute:


We usually denote the product $P$ as $A \times B$.
Exercise 1.12. The following hold in any category $\mathcal{C}$ with products and terminal objects:

$$
A \times B \cong B \times A, \quad A \times \mathbf{1} \cong A
$$

Remark 1.13. We should note that it follows from Exercise 1.12 and duality that in any category with coproducts (denoted by " + ") and initial objects:

$$
A+B \cong B+A, \quad A+\mathbf{0} \cong A
$$

Definition 1.14. The pullback of a diagram: $A \xrightarrow{a} C \stackrel{b}{\longleftrightarrow} B$ is an object $P$ with morphisms $p_{1}: P \rightarrow A, p_{2}: P \rightarrow B$ such that $b p_{2}=a p_{1}$ and such that for any other object $P^{\prime}$ with morphisms $p_{1}^{\prime}: P^{\prime} \rightarrow A, p_{2}^{\prime}: P^{\prime} \rightarrow B$ such that $b p_{2}^{\prime}=a p_{1}^{\prime}$, then there is a unique arrow $h: P^{\prime} \rightarrow P$ such that the diagram below commutes:


We usually denote the pullback $P$ as $A \times{ }_{C} B$.
In the category Set, described above, the pullback of the diagram above would be isomorphic to the set $\{(x, y): x \in A, y \in B, a(x)=b(y)\}$. The pullback is just a generalization of this notion.

Definition 1.15. An equalizer of morphisms $A \underset{g}{\stackrel{f}{\longrightarrow}} B$ is a morphism $E \xrightarrow{e} A$ such that $f e=g e$ and such that for any morphism $U \xrightarrow{u} A$ with $f u=g u$, there is a unique morphism $U \xrightarrow{\gamma} E$ such that the following diagram commutes:


In the category Set, an equalizer is simply the set $\{a \in A: f(a)=g(a)\}$.
Proposition 1.16. An equalizer is monic.
Proof. Let $m: A \rightarrow B$ be the equalizer of arrows $x, y: B \rightarrow C$. Suppose there are arrows $f, g: D \rightarrow A$ such that $m g=m f$. Then $y m g=y m f=x m f=x m g$. Thus there is a unique arrow $u: D \rightarrow A$ such that the following commutes:


But both $f$ and $g$ make that diagram commute. So, by uniqueness, we get that $u=f$ and $u=g$. Thus $f=g$, and so $m$ must be monic.

The preceding (terminal objects, pullbacks, equalizers, and products) are all examples of limits. While we will not define what a limit is in this paper, it suffices to simply think of these four examples. When we discuss a "colimit," one may simply consider initial objects, pushouts, coequalizers, and coproducts (which are respectively terminal objects, pullbacks, equalizers, and products in the dual category).

The reader should consider a category $\mathcal{C}$ to have all finite limits if it has a terminal object and all pullbacks. It wouldn't be fruitful to define limits and to give a proof that having all finite limits is equivalent to having a terminal object and all pullbacks, so we continue without doing so. The reader unfamiliar with categorical language can skip the "proof" of the following theorem.

Theorem 1.17. All limits (and colimits) are unique up to isomorphism.
Proof. This is a consequence of Proposition 1.9 and the fact that limits are defined as terminal objects in the category of cones and colimits are terminal objects in the category of co-cones.

Definition 1.18. If $\mathcal{C}, \mathcal{D}$ are categories then a (covariant) functor $F: C \rightarrow D$ satisfies the following:

- If $A$ is an object of $\mathcal{C}$ then $F(A)$ is an object in $\mathcal{D}$
- If $f: A \rightarrow B$ is a morphism is $\mathcal{D}$ then $F(f): F(A) \rightarrow F(B)$ is a morphism D
- $F\left(i d_{A}\right)=i d_{F(A)}$
- $F(f \circ g)=F(f) \circ F(g)$

Definition 1.19. If $\mathcal{C}, \mathcal{D}$ are categories with functors $\mathcal{C} \underset{G}{F} \mathcal{D}$, then a natural transformation, $\alpha: F \rightarrow G$ is a family of morphisms $\alpha_{C}: F C \rightarrow G C$ for every $C$ which is an object of $\mathcal{C}$, such that for any $f: A \rightarrow B$ in $\mathcal{C}$ the following diagram commutes:


Natural transformations are in fact morphisms between functors, and they serve in defining an important type of functor, which we will define now:
Definition 1.20. If $\mathcal{C}, \mathcal{D}$ are categories then functors $C \underset{G}{\stackrel{F}{\leftrightarrows}} D$ form an adjunction if for any $C \in \mathcal{C}, D \in \mathcal{D}$, there an isomorphism

$$
\phi: \operatorname{Hom}_{\mathcal{D}}(F C, D) \cong \operatorname{Hom}_{\mathcal{C}}(C, G D)
$$

which is natural in both $C$ and $D$. Here, $\operatorname{Hom}_{\mathcal{C}}\left(O_{1}, O_{2}\right)$ denotes the set of morphisms in category $\mathcal{C}$ from $O_{1}$ to $O_{2}$.

In this case we say that $F$ is the left adjoint of $G$, or equivalently that $G$ is the right adjoint of $F$.

What we mean by "natural in $C$ " is that given $h: C^{\prime} \rightarrow C$, and denoting composition on the right by $h^{*}$, the following diagram commutes:


In other words, given $f: F C \rightarrow D$ we get that:

$$
\phi_{C, D}(f) h=\phi_{C^{\prime}, D}(f(F h))
$$

What we mean by "natural in $D$ " is that given a morphism $g: D \rightarrow D^{\prime}$, and denoting composition on the left by $g_{*}$, the following diagram commutes:


In other words, given $f: F C \rightarrow D$ we get that:

$$
G g \phi_{C, D}(f)=\phi_{C, D^{\prime}}(g f)
$$

One important property of adjunctions is that functors which are right adjoints preserve limits and functors which are left adjoints preserve colimits. By this we mean that a functor $F: \mathcal{C} \rightarrow \mathcal{D}$ preserves limits if it takes limits in $\mathcal{C}$ to limits in $\mathcal{D}$. While we state this without proof, one can find proofs of this fact in any book on Category Theory.

## 2. Subobject Classifiers

In our attempts to generalize the notion of a set, we need to generalize two important concepts: subsets and characteristic functions. A key property of sets is that a subset $S \subset A$ is uniquely determined by a characteristic function $\phi_{S}$. Thus, if we know the characteristic map of $S$ then we can recover $S$.

Remark 2.1. The reader should consider a category $\mathcal{C}$ to have all finite limits if it has a terminal object and all pullbacks.

Definition 2.2. A subobject of an object $A$ is a monic $S \stackrel{m}{\longrightarrow} A$.
Remark 2.3. We consider two subobjects, $n, m$ to be equivalent if there is an isomorphism $\phi$ such that $n=m \phi$. A subobject is actually an equivalence class of monics, but, for the purposes of this paper, we ignore this point of rigor. For an object $A$, we denote the set of subobjects of $A$ by $\operatorname{Sub}_{\mathcal{C}}(A)$.

One can easily define a functor using hom-sets; namely, $\operatorname{Hom}(-, B): \mathcal{C}^{o p} \rightarrow$ Set is a functor which takes $X \in \mathcal{C}$ to $\operatorname{Hom}(X, B)$ and which takes $f: X \rightarrow Y$ to $f^{*}: \operatorname{Hom}(Y, B) \rightarrow \operatorname{Hom}(X, B)$, recalling that $f^{*}$ is composition of $f$ on the right. Similarly we can create a functor $\operatorname{Sub}(-): \mathcal{C}^{o p} \rightarrow$ Set, which takes an object $A$ to the set $\operatorname{Sub}(A)$ and which takes a morphism $f: B \rightarrow A$ to $\operatorname{Sub}(f): \operatorname{Sub}(A) \rightarrow$ $\operatorname{Sub}(B)$, defined by the pullback square below.


It is a quick exercise to show that this is, in fact, a functor since it preserves equivalence classes.

Definition 2.4. Two subobjects $A, B$ of $S$ in a $\mathcal{C}$ are disjoint subobjects if the diagram below is a pullback.


Definition 2.5. In a category $\mathcal{C}$ with finite limits, a subobject classifier is an object $\Omega$ and a monic true : $\mathbf{1} \rightarrow \Omega$ such that for any subobject, $m: S \rightarrow A$ there is a unique arrow $\phi: A \rightarrow \Omega$ making the following square a pullback:


As it turns out, not all categories have subobject classifiers, but we will soon restrict ourselves to only considering those that do.
Example 2.6. In the category Set, the subobject classifier is true : $\{*\} \rightarrow\{0,1\}$, with $\operatorname{true}(*)=1$.

Example 2.7. The slice category $\mathcal{C} / B$ of a category $\mathcal{C}$ over an object $B$, has morphisms with codomain $B$ as objects and commutative triangles as morphisms. Given a category $\mathcal{C}$ with finite limits and a subobject classifier $\Omega$, we can define a subobject classifier for $\mathcal{C} / B$. Given the product below, we define true : id $d_{B} \rightarrow \omega_{1}$ to be the unique arrow satisfying:


Noting that $i d_{B}$ is the terminal object in $\mathcal{C} / B$, it is an easy exercise to show that true : $i d_{B} \rightarrow \omega_{1}$ is the subobject classifier for $\mathcal{C} / B$.

Proposition 2.8. Any two subobject classifiers $1>{ }^{t}>\Omega, 1>{ }^{t^{\prime}}>\Omega^{\prime}$ are isomorphic.

Proof. We then get two commutative squares, which, it is important to note, are pullbacks, as below, making the outer square a pullback (this is trivial to check):


But by the definition of a subobject classifier, there is a unique arrow which makes the diagram a pullback, and since the diagram below is a pullback, then $\alpha \circ \phi=i d_{\Omega}$ :


Similarly, we can show that $\phi \circ \alpha=i d_{\Omega^{\prime}}$, and thus $\Omega \cong \Omega^{\prime}$.
Proposition 2.9. A category $\mathcal{C}$ with finite limits and small hom-sets ${ }^{1}$ has a subobject classifier if and only if there is an object $\Omega$ and an isomorphism

$$
\theta_{X}: \operatorname{Sub}_{\mathcal{C}}(X) \cong \operatorname{Hom}_{\mathcal{C}}(X, \Omega)
$$

which is natural in $X$.
This should not be surprising since it is clearly true in the category Set. Namely, a subset of a given set $X$ defines a characteristic function $X — \phi \rightarrow\{0,1\} \cong \Omega$, just as a function, $\phi: X \rightarrow\{0,1\}$, defines a subset of $X$.

## 3. Exponentials

Observe that given two sets $A, B, \operatorname{Hom}(B, A)$ is itself a set. We will denote this hom-set, veiwed as an object in Set, by the exponential notation $A^{B}$. But more carefully, we should note that for any function $f: A \times B \rightarrow C$ we get a function

[^1]$g: A \rightarrow \operatorname{Hom}(B, C)$ given by $g(a)(-)=f(a,-)$. Thus, exponentials satisfy, in the category Set,
\[

$$
\begin{equation*}
\operatorname{Hom}(A \times B, C) \cong \operatorname{Hom}\left(A, C^{B}\right) \tag{3.1}
\end{equation*}
$$

\]

So if we work backwards, we can define the general notion of an exponent $A^{B}$. If we consider $-\times B: \mathcal{C} \rightarrow \mathcal{C}$ to be a functor (on a category $\mathcal{C}$ with finite products), then we can define $(-)^{B}: \mathcal{C} \rightarrow \mathcal{C}$ to be it's right adjoint (when such a functor exists).

Definition 3.2. Given an objects $A, B$, we define the exponent $A^{B}$ to be the right adjoint $(-)^{B}$ of $-\times B$ applied to the object $A$.

In fact, the above definition of exponentials works with categories, where we define $\mathcal{C}^{\mathcal{D}}$ to be the category of functors from $\mathcal{D}$ to $\mathcal{C}$. This allows us to consider the next example.

Example 3.3. Given functors $F, G: \mathcal{C}^{o p} \rightarrow$ Set the exponential functor is $F^{G}(C)=$ $\operatorname{Hom}(\operatorname{Hom}(-, C) \times G, F)$.

Proposition 3.4. In any category with a terminal object, finite products, and exponentials, the following hold:

$$
\begin{gather*}
\mathbf{1}^{X} \cong \mathbf{1} \quad X^{\mathbf{1}} \cong X  \tag{3.5}\\
(A \times B)^{C} \cong A^{C} \times B^{C} \quad A^{B \times C} \cong\left(A^{B}\right)^{C} \tag{3.6}
\end{gather*}
$$

Proof. We begin with the first of (3.5). $\operatorname{Hom}(Y \times X, \mathbf{1}) \cong \operatorname{Hom}\left(Y, \mathbf{1}^{X}\right)$. But $\operatorname{Hom}(Y \times X, \mathbf{1})=\left\{!_{Y \times X}\right\}$, and so there is exactly one morphism $Y \rightarrow 1^{X}$ for each object $Y$. This means that $\mathbf{1}^{X}$ is a terminal object, and thus isomorphic to $\mathbf{1}$.

For the second of (3.5), examine: $\operatorname{Hom}(Y, X) \cong \operatorname{Hom}(Y \times \mathbf{1}, X) \cong \operatorname{Hom}\left(Y, X^{\mathbf{1}}\right)$. Where the first isomorphism follows from $Y \cong Y \times \mathbf{1}$ and the second from the definition of exponential. This means that $X \cong X^{\mathbf{1}}$.

For the first of (3.6), examine:

$$
\begin{aligned}
\operatorname{Hom}\left(Y, A^{C} \times B^{C}\right) & \cong \operatorname{Hom}\left(Y, A^{C}\right) \times \operatorname{Hom}\left(Y, B^{C}\right) \\
& \cong \operatorname{Hom}(Y \times C, A) \times \operatorname{Hom}(Y \times C, B) \\
& \cong \operatorname{Hom}(Y \times C, A \times B) \\
& \cong \operatorname{Hom}\left(Y,(A \times B)^{C}\right)
\end{aligned}
$$

Giving us that $(A \times B)^{C} \cong A^{C} \times B^{C}$.
For the second of (3.5), examine:

$$
\begin{aligned}
\operatorname{Hom}\left(Y,\left(A^{B}\right)^{C}\right) & \cong \operatorname{Hom}\left(Y \times C, A^{B}\right) \\
& \cong \operatorname{Hom}(Y \times(C \times B), A) \\
& \cong \operatorname{Hom}(Y \times(B \times C), A) \\
& \cong \operatorname{Hom}\left(Y, A^{(B \times C)}\right)
\end{aligned}
$$

Giving us that $\left(A^{B}\right)^{C} \cong A^{(B \times C)}$.

## 4. Elementary Topoi

As it turns out, one can find categories which generalize the notion of a set simply by requiring a few of the structures already discussed, leading to the next definition:

Definition 4.1. An elementary topos (plural: topoi) is any category, $\mathcal{C}$, which has the following properties:
(1) $\mathcal{C}$ has all finite limits and colimits
(2) $\mathcal{C}$ has exponentials
(3) $\mathcal{C}$ has a subobject classifier

Example 4.2. The following categories are topoi:
(1) The category with one object and one (identity) arrow.
(2) Set.
(3) $\operatorname{Set}^{n}$, whose objects are $n$-tuples of sets and whose morphisms are $n$-tuples of functions.
(4) $\operatorname{Set}^{\text {Cop }}$.
(5) The slice category $\mathcal{C} / B$, where $\mathcal{C}$ is a topos and $B \in \mathcal{C}$. This will be discussed in more detail later, as it provides a nice "backdoor" to proving some useful properties.
(6) $G$-Sets, whose objects are $G$-sets with $G$-actions and whose morphisms are functions between $G$-sets, which respect the $G$-action.

There is a second, equivalent, manner in which to define a topos, which we will state now because it brings to light the concept of the transpose of a morphism.

Definition 4.3. A topos is a category $\mathcal{C}$ with:
(1) Pullbacks,
(2) A terminal object denoted 1,
(3) An object $\Omega$ and a monic arrow true : $1 \rightarrow \Omega$ such that for any monic $m: S \rightarrow B$, there is a unique arrow $\phi: B \rightarrow \Omega$ making the following diagram a pullback:


We often write $\phi$ as $\operatorname{char}(S)$ or $\operatorname{char}(m)$.
(4) For an object $B$, there is an object $P B$ and an arrow $\epsilon_{B}: B \times P B \rightarrow \Omega$ such that for every $f: B \times A \rightarrow \Omega$ there is a unique arrow $g: A \rightarrow P B$ such that the following commutes:


We call $g$ the P-transpose of $f$, and we call $f$ the P -transpose of $g$. We denote this $f=\hat{g}$.

Remark 4.4. Note that by examining the similarities between the definition of the exponential as a right adjoint and the definition of the "power object" $P B$ of an object $B$, we see that we can consider $P B$ to be $\Omega^{B}$.

Lemma 4.5. In a topos, every monic is an equalizer and every arrow which is both monic and epi is an isomorphism.

Proof. Since we are working in a topos, the most obvious place to look for a pair of morphisms for which a monic $A>{ }^{m}>B$ could be the equalizer would be in the definition of a subobject classifier. Thus we examine $A>\xrightarrow{m} B \underset{\operatorname{true}_{B}}{\stackrel{\operatorname{char}(m)}{\longrightarrow}} \Omega$, where $\operatorname{true}_{B}=\operatorname{trueo}_{B}$. It follows from the definition of a subobject classifier and the definition of a terminal object that $\operatorname{true}_{B} \circ m=\operatorname{char}(m) \circ m$ and that $m$ is in fact the equalizer of these arrows.


Now, suppose $m$ is both a monic and an epi. Then there is some pair $B=f \rightarrow C$ such that $m$ is the equalizer of such a pair. But since $m$ is an epi and $f m=g m$, then $f=g$. Thus since $f \circ i d_{B}=g \circ i d_{B}$, there is some unique map $u: B \rightarrow A$ such that $m u=i d_{B}$. This also gives us that $m u m=i d_{B} \circ m=m=m \circ i d_{A}$, and since $m$ is monic then $u m=i d_{A}$. Thus, $m$ is an isomorphism.

Remark 4.6. Observe that, Lemma 4.5 along with Proposition 1.14 gives that in a topos an arrow is an equalizer if and only it is monic. In addition, one can quickly derive that any isomorphism is a monic and an epi as a consequence of the existence of an inverse. Thus, an arrow is monic and epi if and only if it is an isomorphism.

## 5. Lattices and Heyting Algebras

Definition 5.1. A lattice is a partially ordered set, when considered as a category, with $x \rightarrow y$ iff $x \leq y$, that has all finite limits and colimits.

In the usual definition of a lattice, there are two important operations $\wedge, \vee$ : $L \times L \rightarrow L$ defined equationally as

$$
\begin{gathered}
x \wedge x=x=x \vee x \quad 1 \wedge x=x \quad 0 \vee x=x \\
x \wedge(y \vee x)=x=(x \wedge y) \vee x
\end{gathered}
$$

We define $x \wedge y$ to be $x \times y$ and $x \vee y$ to be $x+y$. It is a quick exercise to show that the operations defined in this manner satisfy the equations above. For our purposes, we insist that all lattices be distributive; namely, that all $x, y, z$ in our lattice must satisfy $(x \wedge(y \vee z))=(x \wedge y) \vee(x \wedge z)$.

Intuitively we can view $\vee$ as "union" or "or" whereas $\wedge$ can be viewed as "intersection" or "and."

Proposition 5.2. In a lattice, $x \vee(y \wedge z)=(x \vee y) \wedge(x \vee z)$.

Proof.

$$
\begin{aligned}
(x \vee y) \wedge(x \vee z) & =[(x \vee y) \wedge x] \vee[(x \vee y) \wedge z] \\
& =x \vee[(x \wedge z) \vee(y \wedge z)] \\
& =[x \vee(x \wedge z)] \vee(y \wedge z) \\
& =x \vee(y \wedge z)
\end{aligned}
$$

Definition 5.3. A Heyting algebra is a lattice which, when viewed as a poset and thus a category, has exponentials.

Essentially what we have done is taken the notion of a lattice and restricted those we will consider to those which are also topoi when viewed as categories. Observe that when we apply our definition of exponential to our conception of a lattice, denoting $y^{x}$ by $(x \Rightarrow y)$, we get:

$$
z \leq(x \Rightarrow y) \text { if and only if } z \wedge x \leq y
$$

## Proposition 5.4. In a Heyting algebra the following hold:

$$
\begin{gather*}
(x \Rightarrow 1)=1 \quad(1 \Rightarrow x)=x \quad((y \wedge z) \Rightarrow x)=(z \Rightarrow(y \Rightarrow x))  \tag{5.5}\\
((x \vee z) \wedge y)=((x \wedge y) \vee(x \wedge z)) \quad(x \Rightarrow(y \wedge z))=((x \Rightarrow y) \wedge(x \Rightarrow z))  \tag{5.6}\\
z \leq(x \Rightarrow y) \text { if and only if } x \leq(z \Rightarrow y) \tag{5.7}
\end{gather*}
$$

Proof. (5.5) follows from Proposition 3.3, while (5.6) come from the definiton of $-\wedge y=-\times y$ as the left adjoint of $x \Rightarrow-=(-)^{x}$ (thus, the former preserving colimits and the latter preserving limits).

For (5.7): $z \leq(x \Rightarrow y)$ iff $z \wedge x \leq y$ iff $x \wedge z \leq y$ iff $x \leq(z \Rightarrow y)$
Proposition 5.8. In a Heyting algebra the following hold:

$$
\begin{gather*}
(x \Rightarrow x)=1  \tag{5.9}\\
x \wedge(x \Rightarrow y)=x \wedge y, y \wedge(x \Rightarrow y)=y  \tag{5.10}\\
x \Rightarrow(y \wedge z)=(x \Rightarrow y) \wedge(x \Rightarrow z) \tag{5.11}
\end{gather*}
$$

Thus, a Heyting algebra satisfies all those identities we would expect a structure to which we impart the language of logic to satisfy. In fact, we can even define an object to be, intuitively, the negation of $x$ as $\neg x=(x \Rightarrow 0)$. This in fact satisfies some of the identities we would expect:

$$
\begin{gathered}
x \leq \neg \neg x \\
x \leq y \text { implies } \neg y \leq \neg x \\
\neg x=\neg \neg \neg x \\
\neg \neg(x \wedge y)=\neg \neg x \wedge \neg \neg y
\end{gathered}
$$

Given the extra condition, $x=\neg \neg x$, a Heyting algebra is, in fact, a Boolean algebra.
As it turns out there are lots of familiar structures which are Heyting algebras. Some examples are $\operatorname{Sub}(X)$ for any object $X$ in a topos (we will prove this later), Set ${ }^{\mathbf{C o p}^{\mathrm{P}}}$, and Boolean Algebras.

We will take this idea further, in Section 8, by examining objects in topoi which have similar internal structures.

## 6. Factorizing Arrows

In Set, given a function $f: X \rightarrow Y$, we can easily decompose it as an onto function $e: X \rightarrow Z$ and a one-to-one function $m: Z \rightarrow Y$ as follows: let $e(x)=$ $f(x), Z=f(X)$, and $m(z)=z$. In fact, we will see that this decomposition can be generalized for all topoi.

Definition 6.1. A morphism $f: X \rightarrow Y$ factors through $m: Z \rightarrow Y$ if there is some morphism $t: X \rightarrow Z$ such that $f=m t$.

Definition 6.2. A morphism $m: X \rightarrow Y$ is the image of $f: W \rightarrow Y$ if $f=m e$ for some morphism $e: W \rightarrow X$ and if whenever $f$ factors through a monic $h, m$ factors through $h$.

Lemma 6.3. In a topos, every arrow $f$ has an image $m$ and factors as $f=m e$, where $e$ is an epi.

Proof. First we construct the factorization. Given $f: A \rightarrow B$, find $x, y: B \rightarrow P$ which are the pushout of $f$ with itself. Let $m: M \rightarrow B$ be the equalizer of this pair and using the definition of equalizer, we get a unique arrow $e: A \rightarrow M$ such that $f=m e$.

Now to show that $m$ is the image of $f$. Note that by Proposition 1.14, we know that $m$ is monic. Suppose there is some monic arrow $h$ such that factors through $h$ as in: $A \Longrightarrow A \rightarrow N$. Then using Lemma 4.5 we have that there are two arrows $x^{\prime}, y^{\prime}: B \rightarrow C$ whose equalizer is $h$. But then $x^{\prime} h=y^{\prime} h$ implies that:

$$
x^{\prime} f=\left(x^{\prime} h\right) g=\left(y^{\prime} h\right) g=y^{\prime} f
$$

By the definition of a pushout there is a unique arrow $u: P \rightarrow C$ such that $u x=x^{\prime}$ and $u y=y^{\prime}$. But this gives:

$$
x^{\prime} m=u x m=u y m=y^{\prime} m
$$

Then by definition of $h$ as an equalizer of $x^{\prime}, y^{\prime}$, there is a unique arrow $v: M \rightarrow N$ such that $h v=m$. Thus $m$ factors through $h$, i.e. if $f$ factors through $h$ then so does $m$. This completes the proof that $m$ is the image of $f$.

Now we must show that $e$ is epi. We apply our factorization above to $e$ to get that $e=m^{\prime} e^{\prime}$ where $A-e^{\prime}>M^{\prime} \succ m^{\prime} \rightarrow M$. Since $f=m m^{\prime} e^{\prime}$ and the composition of two monics is monic, then $m$ must factor through $\mathrm{mm}^{\prime}$ by some unique arrow $u$ : $M \rightarrow M^{\prime}$. Thus $m=m m^{\prime} u$, and since $m$ is monic then $i d_{M}=m^{\prime} u$. Also, $m^{\prime} u m^{\prime}=$ $i d_{M} m^{\prime}=m^{\prime}=m^{\prime} i d_{M^{\prime}}$ gives that $u m^{\prime}=i d_{M^{\prime}}$. Thus $m^{\prime}$ is an isomorphism. This means that if $s, t: M^{\prime} \rightarrow C$ is the pushout of $m^{\prime}$, i.e. $s m^{\prime}=t m^{\prime}$, then:

$$
s m^{\prime}\left(m^{\prime}\right)^{-1}=t m^{\prime}\left(m^{\prime}\right)^{-1} \Rightarrow s=t
$$

Now suppose there are arrows $g, h: M \rightarrow P^{\prime}$ such that $g e=h e$. By the definition of a pushout, this gives a unique arrow $v: C \rightarrow P$ as below:


Thus, $g=v s=h$. So $e$ is epi.

Proposition 6.4. If $f=m e$ and $f^{\prime}=m^{\prime} e^{\prime}$ with $m, m^{\prime}$ monic, and $e, e^{\prime}$ epi, then any map $(r, t)$ from $f$ to $f^{\prime}$ defines a unique map of $m$, e to $m^{\prime}, e^{\prime}$ as below:


By quick inspection we see that Proposition 6.4 gives us that for any arrow, factorization is unique up to isomorphism.

## 7. Slice Categories as Topoi

We first assert, without proof, the following theorem which will allow us to apply our earlier machinery to slice categories.

Theorem 7.1. If $\mathcal{C}$ is a topos and $B$ is an object in $\mathcal{C}$, then $\mathcal{C} / B$ is a topos.
Slice categories are important topoi for two reasons: they provide a nice backdoor to prove some useful facts about topoi (like Corollaries 7.3, 7.4, 7.5, and 7.7 and Proposition 7.9) and they provide a nice example of a topoi whose objects don't have an underlying set.

Theorem 7.2. For any arrow $k: B \rightarrow A$ in a topos $\mathcal{C}$ we can create a "change-of-base" functor $k^{*}: \mathcal{C} / A \rightarrow \mathcal{C} / B$ which has both a right and a left adjoint.

Remark 7.3. Our functor $k^{*}$ is merely taking the pullback $f^{\prime}: B \times{ }_{A} X \rightarrow B$ of an arrow $f: X \rightarrow A$.

Proof. For the left adjoint $\Sigma_{k}$, we simply use composition with $k$. To check that this is a left adjoint we need to verify $\operatorname{Hom}\left(\Sigma_{k} h, g\right) \cong \operatorname{Hom}\left(h, k^{*} g\right)$ for $h: H \rightarrow B$ and $g: G \rightarrow A$. However, it is easy to see that a map $\gamma: H \rightarrow G$ is a map from $\Sigma_{k} h=k h$ to $g$ if and only if it gives a unique map from $h$ to $k * g$, as is evident by
considering the following diagrams:


For a right adjoint we first notice that pullback along two morphisms gives us the notion of product in slice categories. Thus, if we define the right adjoint $k^{*}$ to be $(-)^{k}$ we complete the proof as:

$$
\begin{aligned}
\operatorname{Hom}\left(k^{*} g, h\right) & \cong \operatorname{Hom}(k \times g, h) \\
& \cong \operatorname{Hom}\left(g, h^{k}\right)
\end{aligned}
$$

Corollary 7.4. In a topos, the pullback of an epi is epi.
Proof. Take an epi $e: X \rightarrow A$ and notice that $e$ is epi if and only if the square below is a pushout:


This is also a pushout in the slice category. Now take a morphism $k: B \rightarrow A$ and note that $k^{*} e$ is the pullback of $e$. But $k^{*}$ is a left adjoint (i.e. has a right adjoint), so it preserves all colimits. Thus, the square below must be a pushout:


In other words, $k^{*} e$, the pullback of $e$, is epi.

Corollary 7.5. In a topos, any arrow $k: A \rightarrow \mathbf{0}$ is an isomorphism.
Remark 7.6. This should seem pretty natural since, in Set, $\mathbf{0}$ is the empty set and the only functions which have the empty set as codomain, have it also as domain.

Proof. We begin by noticing that the unique arrow $!_{0}: \mathbf{0} \rightarrow \mathbf{0}$ is both initial and terminal in $\mathcal{C} / \mathbf{0}$. Using Lemma 7.2 , we know that the pullback of $!_{0}$ must be both initial and final in the category $\mathcal{C} / A$. Also, $!_{A}: \mathbf{0} \rightarrow B$ is the unique (up to isomorphism) initial object in $\mathcal{C} / A$.

This gives that the square below is a pullback and, since $i d_{\mathbf{0}}$ is both monic and epi, then so is $g$ (by Corollary 7.3). Thus, $g$ is an isomorphism by Lemma 4.5.


Since $k g=i d_{\mathbf{0}}$ then we get that:

$$
k=k g\left(g^{-1}\right)=i d_{\mathbf{0}}\left(g^{-1}\right)=g^{-1}
$$

Corollary 7.7. Every arrow $k: 0 \rightarrow B$ is monic.
Proof. Suppose there is $g: A \rightarrow \mathbf{0}$, then $A \cong \mathbf{0}$. Thus $A$ is an initial object and so $g$ is the unique map from $A$ to $\mathbf{0}$.

While in Set this seems obvious, it took a considerable amount of machinery to prove in general. The work, however, was worthwhile as we can now prove the following:

Theorem 7.8. For any object $A$ in a topos, $\operatorname{Sub}(A)$ is a Heyting Algebra.
Proof. Let the reader be aware that we will abuse notation in this proof; when we write $S \in \operatorname{Sub}(A)$, we actually refer to $S>\stackrel{s}{\longrightarrow} A$.

We first need to show that $\operatorname{Sub}(A)$ is in fact a lattice. To do this we will explicitly constuct some of the important features of a lattice. For an initial object, we apply Corollary 7.7 to get $\mathbf{0}$ as a subobject of $A$. For a terminal object we simply use $A$. To define $\wedge$, which we will denote here by $\cap$, we simply take the pullback below:


If we factor $f: S+T \rightarrow A$ as $S+T \xrightarrow{e} S \cup T \gg \xrightarrow{m} A$, we get $\vee$, which we denote here by $\cup$. The reader can quickly check that the properties asked of the maps $\wedge, \vee$ in the definition of a lattice hold with our definition of $\cup, \cap$.

Thus, $\operatorname{Sub}(A)$ is a lattice as we have already taken care of finite limits since we have a terminal object, $A$, and pullbacks $S \cap T$. Now we need to show that it is indeed a Heyting Algebra. In order to do this we need only construct exponentials. We assert without proof that $\operatorname{Sub}_{\mathcal{C}}(A) \cong \operatorname{Sub}_{\mathcal{C} / A}(\mathbf{1})$ (however, this is quite immediate), and thus we shall prove that in any topos $\operatorname{Sub}(\mathbf{1})$ has exponentials. Take two subobjects $S, T$ of $\mathbf{1}$ and we define $S^{T}=\theta^{-1}\left(\theta(S)^{\theta(T)}\right.$ ) where
$\theta: \operatorname{Sub}(1) \cong \operatorname{Hom}(1, \Omega)$. To check that this is valid:

$$
\begin{aligned}
\operatorname{Hom}(U \times T, S) & \cong \operatorname{Hom}(\theta(U) \times \theta(T), \theta(S)) \\
& \cong \operatorname{Hom}\left(\theta(U), \theta(S)^{\theta(T)}\right) \\
& \cong \operatorname{Hom}\left(\theta^{-1}(\theta(U)), \theta^{-1}\left(\theta(S)^{\theta(T)}\right)\right) \\
& =\operatorname{Hom}\left(U, S^{T}\right)
\end{aligned}
$$

Thus $\operatorname{Sub}_{\mathcal{C} / A}(\mathbf{1}) \cong \operatorname{Sub}_{\mathcal{C}}(A)$ has exponentials and is a Heyting algebra.
One interesting consequence of this is that in any topos $\operatorname{Hom}(A, \Omega)$ is a Heyting algebra since it is isomorphic to $\operatorname{Sub}(A)$. More specifically, $\operatorname{Hom}(\mathbf{1}, \Omega)$ is a Heyting algebra. Intuitively, we can think of any morphism $\mathbf{1} \rightarrow \Omega$ as specifying an element of $\Omega$ and so we get that $\Omega$, in some sense, has the structure of a Heyting algebra.

Proposition 7.9. If $S$ and $T$ are disjoint subobjects of $B$, then $S+T \cong S \cup T$.
Proof. By definition of the coproduct, we get an arrow $f: S+T \rightarrow B$ such that the following commutes:


Moreover, by examination, we see that this is also a coproduct in $\mathcal{C} / B$. Note that the diagrams below are pullbacks (the left by hypothesis, the right because $t$ is monic):


Then, since $T \cong T+\mathbf{0}$, the following diagram on the left (in $\mathcal{C} / B$ ) becomes the diagram on the right (in $\mathcal{C} / T$ ) because, by Lemma 7.2, we know that pullback along $t$ preserves coproducts:


Thus the following is a pullback:


This means that the pullback of $t$ along $f$ is $i_{2}$. Similarly we get that the pullback of $s$ along $f$ is $i_{1}$. Now, since pullback along $f$ preserves coproducts, we know that the following diagram is a coproduct in $\mathcal{C} /(S+T)$ :


This means that the pullback of $f$ along itself is the identity. But this means that $f$ is monic and thus, $S+T$ is a subobject of $B$. Thus the image of $f$ is itself, finishing the proof, as $S \cup T=\operatorname{cod}($ image of $f)=\operatorname{cod}(f)=S+T$.

Since the concept of coproduct is loosely that of disjoint union, the preceding should be a reassuring result.

Proposition 7.10. In a topos, if $f: A \rightarrow C$ and $g: B \rightarrow D$ are epi, then $f \times g: A \times B \rightarrow C \times D$ is epi.

Remark 7.11. $f \times g$ is the unique map which satisfies:


Proof. By examination, we see that the following are pullbacks:


Thus, by Lemma $7.2, f \times i d_{B}$ and $i d_{C} \times g$ are epi. Then the composition $\left(i d_{C} \times\right.$ $g) \circ\left(f \times i d_{B}\right)=f \times g$ is epi.

## 8. Lattice Objects and Heyting Algebra Objects

Definition 8.1. A lattice object is an object $L$ in a topos, along with arrows $\Lambda, \bigvee: L \times L \rightarrow L$ and $\top, \perp: \mathbf{1} \rightarrow L$, such that the following two diagrams commute:

where the diagonal arrow $\Delta$ and the twist arrow $\tau$ are defined as the unique arrows satisfying:


Note that the large top diagram is simply generalizing the rule that $x=(x \wedge y) \vee$ $x=x \wedge(y \vee x)$ in categorical language, and the large bottom diagram is generalizing the rule that $x \vee \perp=x$ and $x \wedge \top=x$. We intuitively think of $\top$ as representing true and $\perp$ as representing false.

Definition 8.2. A Heyting algebra object is a lattice object $H$ in a topos with an additional operation $\Rightarrow: H \times H \rightarrow H$ which satisfies the diagrams given by the identities in Propostion 5.8.

Example 8.3. In the category Set, given an object (i.e. a set), $X$, the power set of $X$ is a Heyting algebra object. The maps $\wedge, \vee$ correspond to $\cap, \cup$ respectively. The maps $\top, \perp$ correspond to $* \mapsto X$ and $* \mapsto\}$ respectively. $\Rightarrow$ is given by $Y \Rightarrow Z=Y \cap Z^{c}$.

Motivated by the observation that $x \wedge y=x$ if and only if $x \leq y$ we define an object $\leq_{L}$ as the equalizer:

$$
\leq_{L^{\succ}} \stackrel{e}{\longrightarrow} L \times L \stackrel{\wedge}{\stackrel{\wedge}{\pi_{1}}} L
$$

From this we can also characterize the rules for reflexivity, antisymmetry and transitivity in our categorical language.

Characterization 8.4. (Reflexivity) The diagonal factors through $\leq_{L}$ as below:


Characterization 8.5. (Antisymmetry) Define $\geq_{L}$ as the monic $\leq_{L^{\succ}} \stackrel{e}{\longrightarrow} L \times L \stackrel{\tau}{\longrightarrow} L \times L$ and take the pullback:


Then antisymmetry is that the arrow $\geq_{L} \cap \leq_{L} \rightarrow L \times L$ factors as:

$$
\geq_{L} \cap \leq_{l} \longrightarrow L \succ \quad{ }^{\Delta} L \times L
$$

Characterization 8.6. (Transitivity) [Mac Lane/Moerdijk] Define $C$ to be pullback:


Then transitivity is that the arrow $\left\langle\pi_{1} e v, \pi_{2} e u\right\rangle: C \rightarrow L \times L$ factors through $e: \leq_{L} \rightarrow L \times L$.

Characterization 8.7. (Transitivity) [Henderson] Define $P$ to be the pullback:


Then transitivity is that the arrow $\left(\pi_{1} \phi_{1} \times \phi_{2}\right) p: P \rightarrow L \times L$ factors through $e: \leq_{L} \rightarrow L \times L$, where we let $p=(1 \times e) p_{2}=(e \times 1) p_{1}$, the projections

$$
L \longleftarrow \stackrel{\pi_{1}}{\longleftarrow}(L \times L) \longleftarrow \stackrel{\phi_{1}}{\leftrightarrows}(L \times L) \times L \xrightarrow{\phi_{2}} L
$$

We include Characterization 8.7 because we feel that it is more intuitive than Characterization 8.6. We include a proof that the two characterizations are equivalent because, although unimportant mathematically, the author is proud of his characterization
Theorem [Henderson] 8.8. Both characterizations (8.6 and 8.7) of transitivity are the same (i.e. $P \cong C$ and $\left\langle\pi_{1} e v, \pi_{2} e u\right\rangle \cong\left(\pi_{1} \phi_{1} \times \phi_{2}\right) p$, as defined above).
Proof. Given projections $\phi_{1}, \phi_{2}, \pi_{1}, \pi_{2}$ we can choose projections $\alpha_{1}, \alpha_{2}$ as follows. Let $\alpha_{1}=\pi_{1} \phi_{1}$ and let $\alpha_{2}$ be the unique arrow which makes the diagram below commute:


Now, what we hope is that the $\left\langle e v, \pi_{2} e u\right\rangle=\left\langle\pi_{1} e v, e u\right\rangle$, and to prove this we examine the following diagram:


By inspection, the diagram without the arrow $\gamma$ commutes. Thus if we let $\gamma=$ $\left\langle e v, \pi_{2} e u\right\rangle$, let us check that the whole diagram commutes, which will give us that $\gamma$ is in fact $\left\langle\pi_{1} e v, e u\right\rangle$ as well. This amounts only to checking that $\alpha_{2} \gamma=e u$. We know already that $\pi_{2} \alpha_{2} \gamma=\phi_{2} \gamma=\pi_{2} \mathrm{eu}$, so we need only check that $\pi_{1} \alpha_{2} \gamma=\pi_{1} \mathrm{eu}$ :

$$
\begin{array}{rlrl}
\pi_{1} \alpha_{2} \gamma & =\pi_{2} \phi_{1} \gamma & & \text { by choice of } \alpha_{2} \\
& =\pi_{2} e v & & \\
& =\pi_{1} e u \quad & \text { by Mac Lane's construction }
\end{array}
$$

Thus observing that

$$
(1 \times e)\left\langle\pi_{1} e v, u\right\rangle=\left\langle\pi_{1} e v, e u\right\rangle=\gamma=\left\langle e v, \pi_{2} e u\right\rangle=(e \times 1)\left\langle v, \pi_{2} e u\right\rangle
$$

the diagram below commutes, giving us a unique arrow $l$ :


Now we claim that the arrow $l$ is monic. We will show that $\left\langle\pi_{1} e v, u\right\rangle$ is monic, and thus that $l$ is monic by Lemma 1.7, since $p_{2} l=\left\langle\pi_{1} e v, u\right\rangle$. Suppose there are arrows $a, b: A \rightarrow C$ such that $\left\langle\pi_{1} e v, u\right\rangle a=\left\langle\pi_{1} e v, u\right\rangle b$. Let $\beta_{1}, \beta_{2}$ be the projections for $L \times \leq_{L}$. Then from the following diagram, we get that $\pi_{1} e v a=\pi_{1} e v b$ and $u b=u a$ :


Consider the commutative diagram below that we obtain from the above equations. Since putting $b$ in place of $q$ makes the diagram commute, if we show that putting $a$ in place of $q$ makes the diagram commute, the uniqueness of $q$ will then imply that $a=b$.


The diagram, above, will commute with $a$ in place of $q$ if and only if $v a=v b$. So we continue by examining the product below (left). Since the diagram on the left
commutes, then the diagram on the right commutes with both $e v a$ and $e v b$ as the center downward arrow.


Thus, by uniqueness, we get that $e v a=e v b$. But since $e$ is an equalizer by construction, it is monic. This gives that $v b=v a$. As we noted before, this suffices to show that $a=b$. Thus $\left\langle\pi_{1} e v, u\right\rangle$ is monic, as claimed.

Finally, we will show that $l$ is in fact an isomorphism, completing the proof. We will do this by showing that $C$ is in fact also a pullback of the same diagram that $P$ is a pullback of. For this, it only remains to show that if there is an object $C^{\prime}$ and maps $p_{1}^{\prime}, p_{2}^{\prime}$ such that the diagram below commutes, then there is a unique arrow from $C^{\prime}$ to $C$ which makes the diagram commute. So, suppose there are two maps $a$ and $b$ which make the following commute:


Since $P$ is a pullback, there is a unique arrow $C^{\prime} \rightarrow P$ making the diagram commute. Thus $l a=l b$. However, we know that $l$ is monic, so this implies that $b=a$.

Thus, $C$ is a pullback of the diagram, making $l$ an isomorphism $l: C \cong P$.
Now observe that:

$$
\left(\pi_{1} \phi_{1} \times \phi_{2}\right) p l=\left(\pi_{1} \phi_{1} \times \phi_{2}\right)(e \times 1)\left\langle v, \pi_{2} e u\right\rangle=\left(\pi_{1} \phi_{1} \times \phi_{2}\right)\left\langle e v, \pi_{2} e u\right\rangle
$$

Then the diagram below gives us that $\left(\pi_{1} \phi_{1} \times \phi_{2}\right)\left\langle e v, \pi_{2} e u\right\rangle$ is in fact equal to $\left\langle\pi_{1} e v, \pi_{2} e u\right\rangle$ :


Thus we get that $l:\left(\pi_{1} \phi_{1} \times \phi_{2}\right) p \cong\left\langle\pi_{1} e v, \pi_{2} e u\right\rangle$, as desired.

## 9. Well-Pointed Topoi

In order to get one step closer to full generalization of sets, we add one last requirement. Though it is beyond the scope of this paper, it is interesting to note that a topos with this last requirement allows one to give an alternative foundation to classical mathematics.

Definition 9.1. A topos $\mathcal{C}$ is generated by a collection $\mathcal{G}$ of objects of $\mathcal{C}$ if for all $f, g: A \rightarrow B$ such that $f \neq g$, there exists a morphism $u: G \rightarrow A$ such that $f u \neq g u$.

Definition 9.2. A topos $\mathcal{C}$ is well-pointed if it is generated by 1.
One nice property of a well-pointed topos is that terminal objects have only one "proper" subobject. This is exactly what we would hope for, since in the category Set, a terminal object is a one-point set and it's only proper subset is the empty set.

Lemma 9.3. The only subobjects of $\mathbf{1}$ in a well-pointed topos are $\mathbf{1}$ and $\mathbf{0}$.
Proof. $\mathbf{0}$ is a subobject of $\mathbf{1}$ by Corollary 7.7. Thus, take $U \xrightarrow{u} \mathbf{1}$ and assume $U \neq \mathbf{0}$. We can get an arrow $s: \mathbf{1} \rightarrow U$ as the map which we get from the definition of well-pointed applied to $\operatorname{char}\left(i d_{U}\right) \neq \operatorname{char}(!)$, where $!: \mathbf{0} \rightarrow U$. Then since $i d_{\mathbf{1}}$ is the unique arrow from $\mathbf{1}$ to itself, we get that $i d_{\mathbf{1}}=u t$. This also gives us that $u t u=u i d_{U}$, but since $u$ is monic, this implies that $t u=i d_{U}$. This gives that

$$
u: U \cong \mathbf{1}
$$

We will now conclude by stating, without proof, some interesting theorems that provided the motivation for our study of elementary topoi. For the purposes of this paper, a Boolean topos is one in which for every object $E$ the Heyting algebra $\operatorname{Sub}(E)$ is also a Boolean algebra.
Theorem 9.4. A well-pointed topos is Boolean.
Theorem 9.5. If $\mathcal{C}$ is a topos which is generated by subobjects of 1 and which has the property that for each object $E, \operatorname{Sub}(E)$ is a complete Boolean algebra, then $\mathcal{C}$ satisfies the axiom of choice.

Theorem 9.6. There exists a Boolean topos satisfying the axiom of choice in which the continuum hypothesis fails.

## 10. Concluding Remarks

In this paper we have presented a lot constructions and results dealing with topoi, but the reader may be wondering "Where do we go from here/why do we care about topoi?" Throughout the paper we have repeated the phrase "This is a generalization of 〈blank 〉in the category of Set to all topoi," and for good reason. The notion of a topos is a very good generalization of a set, so good in fact that one can use topos theory to give a proof of the independence of the Axiom of Choice and the Continuum Hypothesis as we noted at the end of the last section.

We can even go so far as to create a foundation for classical mathematics alternative to the traditional Zermelo-Frænkel set theory axioms by expanding our notion of a topos to well-pointed topoi. One interesting property about this new
foundation is that the basic concept is that of a "function," or a morphism in a topos, rather than set membership.
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[2] Saunders Mac Lane. Categories for the Working Mathematician. Springer-Verlag, 1971.
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[^0]:    Date: August 5, 2009.

[^1]:    ${ }^{1}$ We say that a category $\mathcal{C}$ has small hom-sets if for all objects $A, B \in \mathcal{C}, \operatorname{Hom}(A, B)$ is a set.

