# POINCARÉ DUALITY 

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#### Abstract

In this paper, we define the notion of orientation on manifolds using homology and prove the Poincaré Duality theorem that links homology and cohomology. We will give a couple examples of application at the end of the paper. The reader is expected to be familiar with Homology and Cohomology.


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## 1. Orientation for Manifolds

In this paper, unless otherwise stated, all homology will have $\mathbb{Z}$ coefficients. We begin by defining the notion of orientation on manifolds using homology. To do that, we start with a proposition.

Proposition 1.1. Let $M$ be an n-dimensional manifold. Then for all $x \in M$, we have

$$
H_{k}(M, M \backslash\{x\})= \begin{cases}\mathbb{Z}, & \text { if } k=n \\ 0, & \text { if } k \neq n .\end{cases}
$$

Proof. Let $U$ be an open neighborhood of $x$. Then we have:

$$
\begin{aligned}
H_{k}(M, M \backslash\{x\}) & \cong H_{k}(U, U \backslash\{x\}) \\
& \cong H_{k}\left(\mathbb{R}^{n}, \mathbb{R}^{n} \backslash\{x\}\right) \\
& \cong \tilde{H}_{k-1}\left(\mathbb{R}^{n} \backslash\{x\}\right) \\
& \cong \tilde{H}_{k-1}\left(S^{n-1}\right) \\
& \cong\left\{\begin{array}{cc}
\mathbb{Z}, & \text { if } k=n \\
0, & \text { if } k \neq n .
\end{array}\right.
\end{aligned}
$$

The first equality follows from excision, and the third equality from long exact sequence.

For the rest of the paper, $M$ will always denote a $n$-dimensional manifold.
Definition 1.2. The choice of one of the generators for $H_{n}(M, M \backslash\{x\}) \cong \mathbb{Z}$ is called a local orientation of $M$ at $x$.

[^0]Now we prove some results to define global orientation. Let $x \in A \subseteq M$, where $A$ is any subset of $M$, and $p_{x}^{A}:(M, M \backslash A) \rightarrow(M, M \backslash\{x\})$ be the inclusion of pairs.

Lemma 1.3. Suppose $U$ is an open neighborhood of $x$.
(1) Then there exists an open neighborhood $W$ of $x$ such that $x \in W \subseteq U$, and $\left(p_{y}^{W}\right)_{*}: H_{*}(M, M \backslash W) \rightarrow H_{*}(M, M \backslash\{y\})$ is an isomorphism for all $y \in W$.
(2) Let $\alpha \in H_{n}(M, M \backslash U)$. Let $W$ be the neighborhood found in part (1). If $\beta \in H_{n}(M, M \backslash W)$ satisfies $\left(p_{y_{0}}^{W}\right)_{*}(\beta)=\left(p_{y_{0}}^{U}\right)_{*}(\alpha)$ for some $y_{0} \in W$, then $\left(p_{z}^{W}\right)_{*}(\beta)=\left(p_{z}^{U}\right)_{*}(\alpha)$ for all $z \in W$.

Proof. Within $U$, find smaller neighborhoods $W$ and $V$ of $x$ such that $x \in W \subseteq V$ with $V \backslash W \cong S^{n-1}$. Then for all $y \in W$, we obtain the following commutative diagram:


The two vertical isomorphisms are due to excision, and the horizontal isomorphism is from homotopy equivalence. The first claim follows from the diagram. Now, if we have $\left(p_{y_{0}}^{W}\right)_{*}(\beta)=\left(p_{y_{0}}^{U}\right)_{*}(\alpha)$ for some $y_{0}$, then we have $p_{*}(\alpha)=\beta$. Since the diagram holds for every $y$, the second claim follows.

Theorem 1.4. Let $K \subseteq M$ be compact, and $x \in K$. Then
(1) $H_{q}(M, M \backslash K)=0$ if $q>n$.
(2) If $\alpha \in H_{n}(M, M \backslash K)$ satisfies $\left(p_{x}^{K}\right)_{*}(\alpha)=0$, then $\alpha=0$.

Proof. We prove the theorem in various cases, building from simplest to the most general.
Case $1 M=\mathbb{R}^{n}$, and $K$ is compact and convex. Then we have $\mathbb{R}^{n} \backslash K \simeq \mathbb{R}^{n} \backslash\{x\}$, so our claim is immediate.
Case $2 M=\mathbb{R}^{n}$, and $K=K_{1} \cup K_{2}$, where the theorem holds for $K_{1}, K_{2}$ and $K_{1} \cap K_{2}$. Note that we have $\left(M \backslash K_{1}\right) \cap\left(M \underline{K}_{2}\right)=M \backslash K$ and $\left(M \backslash K_{1}\right) \cup$ $\left(M \underline{K}_{2}\right)=M \backslash\left(K_{1} \cap K_{2}\right)$, and so we apply Mayer Vietoris to get:
$0 \rightarrow H_{n}(M, M \backslash K) \rightarrow H_{n}\left(M, M \backslash K_{1}\right) \oplus H_{n}\left(M, M \backslash K_{2}\right) \rightarrow H_{n}\left(M, M \backslash\left(K_{1} \cap K_{2}\right)\right) \rightarrow \cdots$
where we get the first 0 from the fact that $H_{n+1}\left(M, M \backslash\left(K_{1} \cap K_{2}\right)\right)=0$. The first part of the claim then follows. For the second part, let $x \in K_{1}$ and consider the following diagram:


Since the diagram commutes, $\left(p_{x}^{K}\right)_{*}\left(\left(p_{K_{1}}\right)(\alpha)\right)=\left(p_{x}^{K_{1}}\right)_{*}(\alpha)=0$. So, by our hypothesis on $K_{1}$, we get $\left(p_{K_{1}}\right)_{*}(\alpha)=0$. A similar argument gives $\left(p_{K_{2}}\right)_{*}(\alpha)=0$. Then by exactness, we get $\alpha=0$.
Case $3 M=\mathbb{R}^{n}$, and $K=K_{1} \cup \cdots \cup K_{q}$, with each $K_{i}$ convex and compact. This case follows from induction on the previous two cases.
Case $4 M=\mathbb{R}^{n}$, and $K$ is just compact. By exactness, we get $H_{k}\left(\mathbb{R}^{n}, \mathbb{R}^{n} \backslash K\right) \cong$ $H_{k-1}\left(\mathbb{R}^{n} \backslash K\right)$. Now, let $y \in H_{k-1}\left(\mathbb{R}^{n} \backslash K\right)$, and $L_{y}$ be a compact set with $i$ : $L_{y} \hookrightarrow \mathbb{R}^{n} \backslash K$ and $y=i_{*}\left(y^{\prime}\right)$ for some $y^{\prime} \in H_{k-1}\left(L_{y}\right)$ [2, pg.156]. Then given a subset $A$ such that $L_{y} \subseteq A \subseteq \mathbb{R}^{n} \backslash K$, we have the following commutative diagram:


Let $a_{y}$ be the image of $y$ in $H_{k-1}\left(\mathbb{R}^{n} \backslash A\right)$. Then we have $y=\left(i^{\prime}\right)_{*}\left(a_{y}\right)$. For convenience of notation, let $y$ and $a_{y}$ be the respective images in $H_{k}\left(\mathbb{R}^{n}, \mathbb{R}^{n} \backslash K\right)$ and $H_{k}\left(\mathbb{R}^{n}, \mathbb{R}^{n} \backslash A\right)$. Now, cover $K$ using balls whose closures are disjoint from $L_{y}$, and choose a finite subcover. Let $A_{y}$ be the union of the closures of the finite subcover. Then by the previous case, the theorem holds for $A_{y}$. If $k>n$, then we get $a_{y}=0$, and hence $y=0$. So the first part holds. For the second claim, suppose $\left(p_{x}^{K}\right)(y)=0$ for any $x \in K$. It suffices to prove that $\left(p_{x}^{A_{y}}\right)\left(a_{y}\right)=0$ for any $x \in A_{y}$, since we can apply the theorem to $A_{y}$ to get that $a_{y}=0$ which will give us $y=0$. By definition, we have $A_{y}=B_{1} \cup \cdots \cup B_{m}$ where each $B_{i}$ is a closed $n$-ball such that $B_{i} \cap K \neq \emptyset$. Then suppose $x \in B_{i} \subseteq A_{y}$ and $z \in B_{i} \cap K$. We have the following commutative diagram:


By hypothesis we have $\left(p_{z}^{K}\right)_{*}(z)=0$, so we get $\left(p_{B_{i}}^{A_{y}}\right)_{*}\left(\left(p_{z}^{B_{i}}\right)_{*}\left(a_{y}\right)\right)=0$, which implies $\left(p_{B_{i}}^{A_{y}}\right)_{*}\left(a_{y}\right)=0$. Hence $\left(p_{x}^{A_{y}}\right)_{*}\left(a_{y}\right)=\left(p_{x}^{A_{y}}\right)_{*}\left(\left(p_{B_{i}}^{A_{y}}\right)_{*}\left(a_{y}\right)\right)=0$. Thus our claim holds.
Case $5 K \subseteq U \subseteq M$, where $U$ is an open coordinate neighborhood. This follows immediately from the previous case, since by excision we have $H_{k}(M, M \backslash K) \cong$ $H_{k}(U, U \backslash K)$.
Case 6 Finally the general case. Let $K=K_{1} \cup \cdots \cup K_{m}$ with each $K_{i}$ contained in an open coordinate neighborhood, as in the previous case. Then the claim follows by induction and cases 2 and 5 . Thus our claim holds.

Theorem 1.5. For each $x \in M$, let $g_{x}$ be a generator of $H_{n}(M, M \backslash\{x\})$. Suppose that the generators are compatible, that is, for all $x \in M$, there is an open neighborhood $U_{x}$ and $\alpha_{U_{x}} \in H_{n}\left(M, M \backslash U_{x}\right)$ such that $\left(p_{y}^{U_{x}}\right)_{*}\left(\alpha_{U_{x}}\right)=\alpha_{y}$ for all $y \in U_{x}$. Then for any compact $B \subseteq M$, there exists a unique $\alpha_{B} \in H_{n}(M, M \backslash B)$ such that $\left(p_{b}^{B}\right)_{*}\left(\alpha_{B}\right)=\alpha_{b}$ for all $b \in B$.

Proof. We use induction to prove existence. First suppose $B \subseteq U_{x}$ for some $x$. We can then set $\alpha_{B}=p_{*}\left(\alpha_{U_{x}}\right)$, where $p_{*}: H_{n}\left(M, M \backslash U_{x}\right) \rightarrow H_{n}(M, M \backslash B)$. Now suppose $B=B_{1} \cap B_{2}$, with $\alpha_{B_{1}}$ and $\alpha_{B_{2}}$ known. Then we get a Mayer Vietoris sequence

$$
\begin{aligned}
& \cdots \rightarrow H_{n+1}\left(M, M \backslash\left(B_{1} \cap B_{2}\right)\right) \rightarrow H_{n}(M, M \backslash B) \rightarrow \\
& \quad H_{n}\left(M, M \backslash B_{1}\right) \oplus H_{n}\left(M, M \backslash B_{2}\right) \rightarrow H_{n}\left(M, M \backslash\left(B_{1} \cap B_{2}\right)\right) \rightarrow \cdots
\end{aligned}
$$

with the maps $\left(p_{B_{1}}\right)_{*} \oplus\left(p_{B_{2}}\right)_{*}: H_{n}(M, M \backslash B) \rightarrow H_{n}\left(M, M \backslash B_{1}\right) \oplus H_{n}\left(M, M \backslash B_{2}\right)$ and $p_{*}^{\prime}-p_{*}^{\prime \prime}: H_{n}\left(M, M \backslash B_{1}\right) \oplus H_{n}\left(M, M \backslash B_{2}\right) \rightarrow H_{n}\left(M, M \backslash\left(B_{1} \cap B_{2}\right)\right)$. Now, for any $x \in\left(B_{1} \cap B_{2}\right)$, we have

$$
\left(p_{x}^{B_{1} \cap B_{2}}\right)_{*}\left(p_{*}^{\prime}-p_{*}^{\prime \prime}\right)\left(\alpha_{B_{1}}, \alpha_{B_{2}}\right)=\left(p_{x}^{B_{1}}\right)_{*}\left(\alpha_{B_{1}}\right)-\left(p_{x}^{B_{2}}\right)_{*}\left(\alpha_{B_{2}}\right)=0
$$

So, by the previous theorem we have $\left(p_{*}^{\prime}-p_{*}^{\prime \prime}\right)\left(\alpha_{B_{1}}, \alpha_{B_{2}}\right)=0$. So, from the exact sequence, there exists $\alpha_{B} \in H_{n}(M, M \backslash B)$ such that $\left(p_{B_{1}}\right)_{*}\left(\alpha_{B}\right)=\alpha_{B_{1}}$ and $\left(p_{B_{2}}\right)_{*}\left(\alpha_{B}\right)=\alpha_{B_{2}}$. Then $\alpha_{B}$ is our desired element. Our claim for existence now follows by induction, by letting $B=B_{1} \cup \cdots \cup B_{k}$, where the closure of each of $B_{i}$ is contained in some $U_{x}$. Uniqueness follows from the previous theorem.

Definition 1.6. Let $M$ be compact. Then $M$ is orientable if there exists an element $\mu \in H_{n}(M)$ such that $\left(j_{x}^{M}\right)_{*}(\mu)$ is the local orientation for $M$ at $x$ for each $x \in M$. We say that $\mu$ is the (global) orientation for $M$.

Another equivalent definition can be stated as follows.
Definition 1.7. Suppose $M$ a $n$-dimensional manifold. Then an orientation of $M$ is a set of elements $\left\{\mu_{K} \in H_{n}(M, M \backslash K) \mid K\right.$ compact subset of $\left.M\right\}$ such that $\left(j_{x}^{K}\right)_{*}\left(\mu_{K}\right)$ is a local orientation for $M$ at $x$ for all $x \in K$, and if $x \in K_{1} \cap K_{2}$, then $\left(p_{x}^{K_{1}}\right)_{*}\left(\mu_{K_{1}}\right)=\left(p_{x}^{K_{2}}\right)_{*}\left(\mu_{K_{2}}\right)$.

So the previous theorem states that if $M$ has a compatible set of local orientations at each point, then it is orientable, which is pretty natural.

Now, Poincaé Duality is a theorem that applies to orientable manifolds. But not every manifold is orientable with $\mathbb{Z}$ coefficients. However, every manifold is orientable with $\mathbb{Z} / 2 \mathbb{Z}$ coefficients.

Example 1.8. Consider $\mathbb{R} P^{2}$, which is a 2-dimensional manifold. Then we have:

$$
H_{k}\left(\mathbb{R} P^{2} ; \mathbb{Z}\right)=\left\{\begin{array}{cc}
\mathbb{Z}, & \text { if } k=0 \\
\mathbb{Z} / 2 \mathbb{Z}, & \text { if } k=1 \\
0 & \text { if } \mathrm{k}=2 \\
\cdot &
\end{array}\right.
$$

Since $H_{2}\left(\mathbb{R} P^{2}\right)=0, \mathbb{R} P^{2}$ is not orientable with $\mathbb{Z}$ coefficients. However, since $H_{k}\left(\mathbb{R} P^{2} ; \mathbb{Z} / 2 \mathbb{Z}\right)=\mathbb{Z} / 2 \mathbb{Z}$ for $k=0,1,2, \mathbb{R} P^{2}$ is orientable with $\mathbb{Z} / 2 \mathbb{Z}$ coefficients, as discussed before.

Also, any manifold $M$ has an orientable 2-sheeted cover. Let $\tilde{M}=\left\{\mu_{x} \mid x \in M\right\}$, where $\mu_{x}$ is a local orientation of $M$ at $x$. We topologize this set to make the map $\tilde{M} \rightarrow M$, defined by $\mu_{x} \rightarrow x$, into a covering map. Let $B \subseteq \mathbb{R}^{n} \subseteq M$ be a ball of finite radius and let $\mu_{B} \in H_{n}(M, M \backslash B)$ be a generator. Let $U\left(\mu_{B}\right)$ be the set of all $\mu_{x} \in \tilde{M}$ such that $x \in B$ and $\mu_{x}=\left(p_{x}^{B}\right)_{*}\left(\mu_{B}\right)$. It is an exercise to check that $U\left(\mu_{B}\right)$ is a basis for a topology on $\tilde{M}$, and that with this topology, the 2-to-1 projection $\tilde{M} \rightarrow M$ is a covering map.

Now that we're through with orientation, we move on to proving the theorem.

## 2. Poincaré Duality

Let $M$ be a oriented manifold with $\left\{\mu_{K}\right\}$ its orientation. If $M$ is compact, let $\mu_{K}=\mu$.

We begin this section by stating Poincaré Duality.
Theorem 2.1. Suppose $M$ is a compact orientable manifold. Let $D: H^{i}(M) \rightarrow$ $H_{n-i}(M)$ be defined by $D(\alpha)=\mu \frown \alpha$ for each $i$. Then $D$ is an isomorphism for all $i$.

In order to understand the map $D$, we first define cap product.
Definition 2.2. Let $X$ be a space, and $R$ a coefficient ring. Then we define a $\mathbb{Z}$-bilinear map $\frown: C_{k}(X) \times C^{l}(X)$ for $l \leq k$ by

$$
\sigma \frown \varphi=\varphi\left(\sigma \mid\left[v_{0}, \cdots, v_{l}\right]\right) \sigma \mid\left[v_{l}, \cdots, v_{k}\right]
$$

where $\sigma: \Delta^{k} \rightarrow X$ and $\varphi \in C^{l}(X)$. This map is defined to be cap product.
We leave it as an exercise to check that cap product in the cochain groups induce a cap product map in homology and cohomology. One needs to check that we get

$$
\partial(\sigma \frown \varphi)=(-1)^{l}(\partial \sigma \frown \varphi-\sigma \frown \delta \varphi) .
$$

Definition 2.3. With the above calculations, we get an induced cap product map

$$
\frown: H_{k}(X) \times H^{l}(X) \rightarrow H_{k-l}(X)
$$

We similarly get relative forms,

$$
\frown: H_{k}(X, A) \times H^{l}(X) \rightarrow H_{k-l}(X, A)
$$

and

$$
\frown: H_{k}(X, A) \times H^{l}(X, A) \rightarrow H_{k-l}(X)
$$

Now that we understand the maps, we prove a number of lemmas that we need to prove the theorem. First we begin with defining directed limit groups.

Definition 2.4. (1) We say that a set $J$ is a directed set if for any $a, b \in J$ there exists $c \in J$ with $a \leq c$ and $b \leq c$.
(2) A directed system of groups is a collection of abelian groups $\left\{G_{s} \mid s \in J\right\}$, with $J$ a directed set, where for each pair $s \leq t$, there exists a homomorphism $f_{s t}: G_{s} \rightarrow G_{t}$ such that $f_{s} s$ is the identity for all $s \in J$, and if $s \leq t \leq u$, then $f_{t u} \circ f_{s t}=f_{s u}$.
Definition 2.5. Let $\left\{G_{s} \mid s \in J\right\}$ be a directed system of groups. Then the direct limit group $\lim G_{s}$ is the quotient $\bigoplus_{s} G_{s} /\left\langle a-f_{s t}(a) \mid a \in G_{s}\right\rangle$, where we are viewing $G_{s}$ as a subgroup of $\bigoplus_{s} G_{s}$.

Let $\left\{G_{s} \mid s \in J\right\}$ be a directed system of groups, and define an equivalence relation $\sim$ on the set $\coprod_{s \in J} G_{s}$ by letting $a \sim b$ if there exists $t$ such that $f_{s t}(a)=f_{t u}(b)$ where $a \in G_{s}$ and $b \in G_{u}$. Now, if $[a]$ and $[b]$ are two equivalence classes, then they have representatives $a^{\prime}$ and $b^{\prime}$ in $G_{t}$, respectively. We define an abelian group structure on the equivalence classes by defining $[a]+[b]=\left[a^{\prime}+b^{\prime}\right]$. Now, define a map that sends $[a]$ to the coset of $a$ in $\lim G_{s}$. It is an exercise to check that the construction here is well-defined and that the defined map is an isomorphism. Thus we have two equivalent definitions of a directed limit.

Lemma 2.6. Suppose $X=\bigcup_{\alpha \in I} X_{\alpha}$, where $\left\{X_{\alpha} \mid \alpha \in I\right\}$ is a directed set of subspaces, and every compact subset $K \subseteq X$ is contained in some $X_{\alpha}$. Then the natural map $h: \underset{\longrightarrow}{\lim } H_{i}\left(X_{\alpha} ; G\right) \rightarrow H_{i}(X ; \bar{G})$ is an isomorphism for all $i$ and $G$.

Now we define cohomology with compact supports, which will be what we use to prove Poincaré Duality. First note that given a space $X$, the set of compact subsets of $X$ form a directed set under inclusion.

Definition 2.7. Let $X$ be a space, and $K \subseteq X$ be a compact subset. Then for a fixed $i$ and abelian group $G$ and for each $K$, we get a group $H^{i}(X, X \backslash K ; G)$ and for each inclusion $K \subseteq L$, we associate a natural homomorphism $h_{K L}^{i}$ : $H^{i}(X, X \backslash K ; G) \rightarrow H^{i}(X, X \backslash L ; G)$. Then we define the cohomology group with compact support of $X$ as $H_{c}^{i}(X ; G)=\lim _{K} H^{i}(X, X \backslash K: G)$ where the limit is taken over compact subsets $K \subseteq X$.

There is an alternate definition for cohomology with compact support that builds from $C_{c}^{i}(X ; G)$ which is a subset of $C^{i}(X ; G)$ that consists of cochains that is zero on all chains in $X \backslash K$. The two definitions are equivalent, and we'll be using the given definition to prove Poincaré Duality. Now we prove a handy lemma. We start with an exercise and a lemma that we will need later.

Lemma 2.8 (Five Lemma). Consider the following diagram of abelian groups:


If the two rows are exact and $\alpha, \beta, \delta, \varepsilon$ are isomorphisms, then $\gamma$ is also an isomorphism.

Proof. One needs to prove:
(1) $\gamma$ is surjective if $\beta$ and $\delta$ are surjective and $\varepsilon$ injective.
(2) $\gamma$ is injective if $\beta$ and $\delta$ are injective and $\alpha$ surjective.

The proof is straightforward diagram chasing, and is left to the reader to finish.
Exercise 2.9. Suppose $\left\{C_{\alpha}, f_{\alpha \beta}\right\}$ is a directed system of chain complexes, with $f_{\alpha \beta}: C_{\alpha} \rightarrow C_{\beta}$ chain maps. Then $H_{n}\left(\underset{\longrightarrow}{\lim C_{\alpha}}\right)=\lim _{n}\left(C_{\alpha}\right)$. In particular, a direct limit of exact sequences is exact.

Lemma 2.10. Suppse $M$ is a union of two open sets $U$ an $V$. Then the following diagram commutes.


Proof. Let $K \subseteq U$ and $L \subseteq V$ be compact sets. Then consider the following diagram:


The upper and lower rows are obtained by Mayer-Vietoris. The isomorphisms come from excision. We want to show that this diagram commutes. First consider the following square:


Let $\sigma \in H^{k}(U \cap V,(U \cap V) \backslash(K \cap L))$, and let $\tilde{f} \in H^{k}(U, U \backslash(K \cap L))$ such that $\left(p_{U \cap L}^{U}\right)^{*}(\tilde{f})=f$. Let $\mu_{K \cap L}^{U}$ be the restriction of $\mu_{K}$ to $K \cap L$, that is,

$$
\mu_{K \cap L}^{U}=\left(p_{K \cap L}^{K}\right)_{*}\left(\mu_{K}\right)
$$

Then by the compatibility of orientations, we have $\left(p_{U \cap V}^{U}\right)_{*}\left(\mu_{K \cap L}^{U \cap L}\right)=\left(\mu_{K \cap L}^{U}\right)$. So we have:

$$
\begin{aligned}
\left(p_{U \cap V}^{U}\right)_{*}\left(\mu_{K \cap L} \frown f\right) & =\left(p_{U \cap V}^{U}\right)_{*}\left(\mu_{K \cap L} \frown\left(p_{U \cap L}^{U}\right)^{*}(\tilde{f})\right) \\
& =\left(p_{U \cap V}^{U}\right)_{*}\left(\mu_{K \cap L}^{U \cap L}\right) \frown \tilde{f} \\
& =\left(\mu_{K \cap L}^{U}\right) \frown \tilde{f} \\
& =\left(p_{K \cap L}^{K}\right)_{*}\left(\mu_{K}\right) \frown \tilde{f} \\
& =\mu_{K} \frown\left(p_{K \cap L}^{K}\right)^{*}(\tilde{f})
\end{aligned}
$$

To get the third and last equality, we are using the fact that if $\varphi:(X, A) \rightarrow(Y, B)$ is a map of pairs, $g \in C^{l}(Y, B)$ and $x \in C_{k+l}(X, A)$, then

$$
(\varphi)_{*}\left(x \frown\left(\varphi^{*}\right)(g)\right)=\left(\varphi_{*}\right)(x) \frown g
$$

in $C_{k}(Y)$. So, the lower square commutes. The same proof works for the case where we interchage ( $U, U \backslash K$ ) with $(V, V \backslash L)$, and so we get the commutativity of the first square in our original diagram. The commutativity of the next square is proved with the same argument. Now we prove the commutativity of the last square:


Since this will be long, it will be its own lemma.
Lemma 2.11. Suppose $Y \subseteq X$, with $Y=Y_{1} \cup Y_{2}, X=X_{1} \cup X_{2}$, with each $X_{i}$ and $Y_{i}$ open in $X$, and further suppose that $X_{1} \cup Y_{1}=X_{2} \cup Y_{2}=X$. Let $A=X_{1} \cap X_{2}$, $B=Y_{1} \cap Y_{2}$, and $[v] \in H_{n}(X, B)$. Then the following diagram commutes for all $k \leq n$ :

where $\Delta^{*}$ and $\Delta_{*}$ are homomorphisms obtained from Mayer Vietoris in homology and cohomology, and $\left[v^{\prime}\right]$ is defined by:

$$
H_{n}(X, B) \longrightarrow H_{n}(X, Y) \longleftarrow \cong H_{n}(A, A \cap Y)
$$

That is, $\left[v^{\prime}\right]$ is the element in $H_{n}(A, A \cap Y)$ that is the isomorphic copy of an element in $H_{n}(X, Y)$ that is the image of $[v]$.

Proof. By the definition of $\Delta^{*}$ and $\Delta_{*}$, we get the following diagram, which we want to show commutes:


The isomorphisms come from excision. Note that the set $\mathcal{A}=\left\{X_{1} \cap Y_{2}, X_{2} \cap Y_{1}, A\right\}$ cover $X$. So, $C_{*}^{\mathcal{A}}(X, B) \rightarrow S_{*}(X, B)$ induces an isomorphism in homology. Thus there exists a representative $\tilde{u}$ of $[v]$ in $C_{n}^{\mathcal{A}}(X, B)$, where its preimage $u$ in $S_{n}^{\mathcal{A}}(X)$ can be written as $u=u_{1}+u_{2}+u^{\prime}$, with $u_{1} \in C_{n}\left(X_{1} \cap Y_{2}\right), u_{2} \in C_{n}\left(X_{2} \cap Y_{1}\right)$, and $u^{\prime} \in S_{n}(A)$ where $\partial u \in C_{n-1}(B)$. Since $u_{1}$ and $u_{2}$ are in $C_{n}(Y)$, their images in $C_{n}(X, Y)$ vanish, meaning that the image of $[v]$ under $H_{n}(X, B) \rightarrow H_{n}(X, Y)$ is represented by $u^{\prime} \bmod C_{n}(Y)$. Hence $\left[v^{\prime}\right]=\left[u^{\prime}\right] \bmod C_{n}(Y)$. Thus the image of $[f] \in H^{k-1}(X, B)$ through the left/bottom of the square is

$$
\Delta_{*}(u \frown f)=\Delta_{*}\left(u_{1} \frown f\right)+\Delta_{*}\left(u_{2} \frown f\right)+\Delta_{*}\left(u^{\prime} \frown f\right)
$$

Now, since we have $u_{2} \in C_{n}\left(X_{2} \cup Y_{1}\right) \subseteq C_{n}\left(X_{2}\right)$ and $u^{\prime} \in C_{n}(A) \subseteq C_{n}\left(X_{2}\right)$, this implies that $\left(u_{2} \frown f\right) \in C_{n-k+1}\left(X_{2}\right)$ and $\left(u^{\prime} \frown f\right) \in C_{n-k+1}\left(X_{2}\right)$. Hence $\left(u_{2} \frown f\right)$ and $\left(u^{\prime} \frown f\right)$ both vanish under $C_{n-k+1}(X) \rightarrow C_{n-k+1}\left(X, X_{2}\right)$, which is part of $\partial$. Thus $\partial(u \frown f)=\partial\left(u_{1} \frown f\right)$. Furthermore, since $\left(u_{1} \frown f\right) \in C_{*}\left(X_{1}\right)$, its image under the excision isomorphism is its reduction $\bmod C_{*}(A)$. Hence we can just use $\left(u_{1} \frown f\right)$ as the pre-image of the reduction to compute the homomorphism $\partial$. Thus we get

$$
\Delta_{*}\left(u_{1} \frown f\right)=\partial\left(u_{1} \frown f\right)=(-1)^{k}\left(\partial u_{1} \frown f-u_{1} \frown \delta f\right)=(-1)^{k+1}\left(\partial u_{1} \frown f\right)
$$

where the last equality follows since $f$ is a cocycle. So, the image of $[f]$ in the left/bottom is $(-1)^{k+1}\left(\partial u_{1} \frown f\right)$.

Now we compute the image the other way. The image of $[f]$ under $H^{k-1}(X, B) \rightarrow$ $H^{k-1}\left(Y_{2}, B\right)$ is represented by the restriction of $f$ to $C_{k-1}\left(Y_{2}\right)$. The image under the excision isomorphism is represented by a cocylce $f^{\prime} \in C^{k-1}\left(Y, Y_{1}\right)$ whose restriction to $Y_{2}$ is homologous to $\left.f\right|_{C_{k-1}\left(Y_{2}\right)}$ in $C^{k-1}\left(Y_{2}, B\right)$. Thus there exists $g \in C^{k-2}\left(Y_{2}, B\right)$ such that $\left.f^{\prime}\right|_{C_{k-1}\left(Y_{2}\right)}=\left.f\right|_{C_{k-1}\left(Y_{2}\right)}+\delta g$. We eliminate the $g$ with the following process:
$g \in C^{k-2}\left(Y_{2}, B\right)$ is defined on $C_{k-2}(Y)$. Extend it to $g^{\prime} \in C_{k-2}(Y)$, by defining it to be zero on all generators of $C_{k-2}(Y)$ that are outside of $C_{k-2}\left(Y_{2}\right)$. Let

$$
f^{\prime \prime}=f^{\prime}-\delta g^{\prime} \in C^{k-1}(Y)
$$

Then $f^{\prime \prime}$ is still a cocycle, $\left[f^{\prime \prime}\right]=\left[f^{\prime}\right]$, and $\left.f^{\prime \prime}\right|_{C_{k-1}\left(Y_{2}\right)}=\left.f\right|_{C_{k-1}\left(Y_{2}\right)}$. Extend $f^{\prime \prime}$ again to an element $\tilde{f} \in C^{k-1}(X)$, by setting it to be zero on generators outside $C_{k-1}(Y)$. Thus $\tilde{f}$ is a preimage of $f^{\prime \prime}$ under the surjection $C^{k-1}\left(X, Y_{1}\right) \rightarrow C^{k-1}\left(Y, Y_{2}\right)$, and so we can use $\tilde{f}$ to compute $\delta^{*}\left[f^{\prime \prime}\right]=[\delta \tilde{f}]$. Hence we get $\Delta^{*}[f]=[\delta \tilde{f}]$, and so the right/top image of $[f]$ is

$$
\left[v^{\prime}\right] \frown[\delta \tilde{f}]=\left[u^{\prime} \frown \delta \tilde{f}\right] .
$$

Now, since $u^{\prime} \in C_{*}(A), u^{\prime} \frown \tilde{f} \in C_{*}(A)$, so we get $\left[\partial\left(u^{\prime} \frown \tilde{f}\right)\right]=0$. But we have $\partial\left(u^{\prime} \frown \tilde{f}\right)=(-1)^{k}\left(\partial u^{\prime} \frown \tilde{f}-u^{\prime} \frown \delta \tilde{f}\right)$, and hence $\left[u^{\prime} \frown \delta \tilde{f}\right]=(-1)^{k}\left[\partial u^{\prime} \frown \tilde{f}\right]$.

Finally, we have to show that $\left[\partial u^{\prime} \frown \tilde{f}\right]=-\left[\partial u_{1} \frown f\right]$. From previous construction, we have

$$
\partial u^{\prime} \frown \tilde{f}=\partial u \frown \tilde{f}-\partial u_{1} \frown \tilde{f}-\partial u_{2} \frown \tilde{f}
$$

with $\partial u \in C_{n-1}\left(Y_{1}\right), \partial u_{2} \in C_{n}\left(X_{2} \cap Y_{1}\right) \subseteq C_{n}\left(Y_{1}\right)$, which implies that $\partial u_{2} \in$ $C_{n-1}\left(Y_{2}\right)$. Similarly, we have $\partial u_{1} \in C_{n-1}\left(Y_{2}\right)$. But we have $\left.\tilde{f}\right|_{C_{*}(Y)}=\left.f^{\prime \prime}\right|_{C_{*}(Y)}$ and $\left.\tilde{f}\right|_{C_{*}\left(Y_{2}\right)}=\left.f^{\prime \prime}\right|_{C_{*}\left(Y_{2}\right)}=f_{C_{*}\left(Y_{2}\right)}$, which implies that $\partial u \frown \tilde{f}=\partial u \frown f^{\prime \prime}$, $\partial u_{2} \frown \tilde{f}=\partial u_{2} \frown f^{\prime \prime}$, and $\partial u_{1} \frown \tilde{f}=\partial u_{1} \frown f^{\prime \prime}$. Now, since $\left.f^{\prime \prime}\right|_{Y_{1}}=0$, the two
terms in the above equation are 0 , and so we get

$$
\left[\partial u^{\prime} \frown \tilde{f}\right]=-\left[\partial u_{1} \frown f\right]
$$

Thus our claim holds.
Now, back to the original proof. To prove the commutativity of the last square, use the lemma with $X=M, X_{1}=U, X_{2}=V, Y=M \backslash(K \cap L), Y=M \backslash K$, and $[v]=\mu_{K \cup L}$. Thus the last square commutes. Now, consider passing to the limit over compact sets $K \subseteq U$ and $L \subseteq V$. Then any compact subset of $U \cap V$ is contained in $K \cap L$ for some $K \subseteq U$ and $L \subseteq V$ and similarly for $U \cup V$, and hence when we pass through the limit, we get:

which is the desired diagram. The exactness of the first row follows from exercise 2.8 , since a direct limit of an exact sequence is exact.

Finally, on to the proof of the theorem. For convenience we'll state it again here.
Theorem 2.12. Suppose $M$ is an orientable manifold. Let $D: H^{i}(M) \rightarrow H_{n-i}(M)$ be defined by $D(\alpha)=\mu \frown \alpha$ for each $i$. Then $D$ is an isomorphism for all $i$.

Proof. We prove in the same manner as Theorem 1.4, building from simple to more general cases.
Case $1 M=\mathbb{R}^{n}$. We will prove the theorem for $M=\operatorname{int}\left(\Delta^{n}\right)$, and the desired result follows by homotopy equivalence. The map $D_{M}$ can be identified with the map $D_{M}^{\prime}: H^{k}\left(\Delta^{n}, \partial \Delta^{n}\right) \rightarrow H_{n-k}\left(\Delta^{n}\right)$ defined by

$$
D_{M}^{\prime}(\alpha)=\left[\Delta^{n}\right] \frown \alpha
$$

where [ $\Delta^{n}$ ] is defined by the identity map of $\Delta^{n}$. Note that the only nontrivial case is when $k=n$ since in all other cases both the homology and cohomology groups are 0 . In the case $k=n$, the generator of $H^{n}\left(\Delta^{n}, \partial \Delta^{n}\right) \cong$ $\operatorname{Hom}\left(H_{n}\left(\Delta^{n}, \partial \Delta^{n}\right), \mathbb{Z}\right)$ is represented by a cocycle $\varphi$ that is 1 on $\Delta^{n}$, and hence we get:

$$
\left[\Delta^{n}\right] \frown \varphi=\varphi\left(\left[\Delta^{n}\right] \mid\left[v_{0}, \cdots, v_{n}\right]\right)\left[\Delta^{n}\right]\left|\left[v_{n}\right]=\left[\Delta^{n}\right]\right|\left[v_{n}\right]
$$

where $\left[\Delta^{n}\right] \mid\left[v_{n}\right]$ is the last vertex of $\Delta^{n}$, which is a generator of $H_{0}\left(\Delta^{n}\right)$. So, since $D_{M}^{\prime}$ takes a generator to a generator, $D_{M}^{\prime}$ is an isomorphism, and so our claim holds.
Case $2 M=U \cap V, U$ and $V$ open subsets of $M$, with the theorem known for $U, V$, and $U \cap V$. This follows immediately from lemma 2.9 and the five lemma.
Case $3 M=\bigcup_{\alpha} U_{\alpha}$, where $\left\{U_{\alpha}\right\}$ is a nested family of open sets, with the theorem known for each $U_{\alpha}$. By excision, $H_{c}^{k}\left(U_{\alpha}\right)$ is the limit of the groups $H^{k}(M, M \backslash K)$, for $K$ compact subsets of $U_{\alpha}$. Thus we get natural maps $H_{c}^{k}\left(U_{\alpha}\right) \rightarrow H_{c}^{k}\left(U_{\alpha+1}\right)$, since the latter is a limit over a larger collection of $K$ 's. Then since the compact sets of $M$ and $U_{i}$ 's coincide, we have $H_{c}^{k}(M) \cong$ $\lim _{\longrightarrow} H_{c}^{k}\left(U_{\alpha}\right)$. Also, by exercise 2.6 , we have $H_{n-k}(M) \cong \underline{\lim } H_{n-k}\left(U_{\alpha}\right)$. So,
the map $D_{M}$ is a limit of the isomorphisms $D_{U_{\alpha}}$, and hence is an isomorphism.
Case $4 M$ is an open subset of $\mathbb{R}^{n}$. If $V$ is a convex open subset of $M$, then since $V$ is homeomorphic to $\mathbb{R}^{n}$, the theorem holds for $V$ by Case 1. If $V, W$ are convex open subsets of $\mathbb{R}^{n}$, then so is $V \cap W$, and so by Case 2 the theorem holds for $V \cup W$. So, if $V=V_{1} \cup \cdots V_{m}$, with each $V_{i}$ convex open, then the theorem holds by induction. Now, write $M=\bigcup_{i}=1^{\infty} V_{i}$ by letting $\left\{V_{i}\right\}$ be the set of balls contained in $M$ with rational radius, centered around points with rational coordinates. Then $\left\{V_{i}\right\}$ is countable. Let $W_{j}=\bigcup_{i=1}^{j} V_{i}$. Then the theorem holds for $W_{j}$ fr all $j$ by the above. Since $\left\{W_{j}\right\}$ are nested, and $M=\bigcup_{j=1}^{\infty} W_{j}$, the theorem holds for $M$ by Case 3.
Case 5 General case. Let $\mathcal{U}$ be the collection of open sets $U \subseteq M$ such that the theorem holds. Then this set is partially ordered by inclusion, and by Case 3 the union of every totally ordered subcollection is again in $\mathcal{U}$. By Zorn's Lemma, there exists a maximal set $N$ for which the theorem holds. Suppose for contradiction that $N \neq M$. Then let $x \in M \backslash N$ and $V$ a neighborhood of $x$ that is homeomorphic to $\mathbb{R}^{n}$. Then by Case 4 , the theorem holds for $V$ and $U \cap V$, and so by Case 2 the theorem holds for $U \cup V$, which is a contradiction. Hence $N=M$, and the theorem holds in general.

Now, a couple corollaries to see an application of Poincaré Duality.
Corollary 2.13. A closed manifold of odd dimension has Euler characteristic 0 .
Proof. First suppose that $M$ is an orientable closed $n$-manifold. Then by Poincaré Duality, we have $H_{i}(M) \cong H^{n-i}(M)$, and so we get rank $H_{i}(M)=\operatorname{rank} H^{n-i}(M)$. Moreover, by the Universal Coefficient Theorem, we get rank $H^{n-i}(M)=\mathrm{rank}$ $H_{n-i}(M)$. So this implies that we get $\operatorname{rank} H_{i}(M)=\operatorname{rank} H_{n-i}(M)$. So, $\chi(M)=$ $\sum_{i=0}^{n}(-1)^{i}$ rank $H_{i}(M)=0$, since the sum cancels in pairs. So our claim holds if $M$ is orientable. Now suppose $M$ is non-orientable. Then let $\tilde{M}$ be an orientable 2 sheeted cover of $M$. Then by the same argument, $\chi(\tilde{M})=0$, and so $\chi(M)=\frac{0}{2}=0$. Thus our claim holds.

Corollary 2.14. $\mathbb{R} P^{2}$ is not a boundary of any 3 dimensional compact manifold.
Proof. Suppose for contradiction that $\mathbb{R} P^{2}$ is a boundary of $M$, a three dimensional manifold. Now let $M_{1}, M_{2}$ be two copies of $M$ and glue them together along their boundaries. Then we get

$$
\begin{aligned}
\chi(M) & =\chi\left(M_{1} \cup_{\mathbb{R} P^{2}} M_{2}\right) \\
& =\chi\left(M_{1}\right)+\chi\left(M_{2}\right)-\chi\left(\mathbb{R} P^{2}\right) \\
& =2 \chi(M)-\chi\left(\mathbb{R} P^{2}\right) \\
& =2 \chi(M)-1
\end{aligned}
$$

But this implies that $\chi(M)=\frac{1}{2}$, which is a contradiction. Note that this proof works if we substitute any compact even dimensional manifold with an odd Euler characteristic for $\mathbb{R} P^{2}$.

## References

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