FOURIER ANALYSIS USING REPRESENTATION THEORY

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Abstract. In this paper, it is our goal to develop the concept of Fourier transforms by using Representation Theory. We begin by laying basic definitions that will help us understand the definition of a representation of a group. Then, we define representations and provide useful definitions and theorems. We study representations and key theorems about representations such as Schur's Lemma. In addition, we develop the notions of irreducible represenations, *-representations, and equivalence classes of representations. After doing so, we develop the concept of matrix realizations of irreducible represenations. This concept will help us come up with a few theorems that lead up to our study of the Fourier transform. We will develop the definition of a Fourier transform and provide a few observations about the Fourier transform.

Contents

1. Essential Definitions

Definition 1.1. An *internal law of composition* on a set R is a product map

$$P : R \times R \to R$$

Definition 1.2. A *group* G, is a set with an internal law of composition such that:
(i) P is associative. i.e. $P(x, P(y, z)) = P(P(x, y), z)$
(ii) $\exists$ an identity, $e, \ni$ if $x \in G$, then $P(x, e) = P(e, x) = x$
(iii) $\exists$ inverses $\forall x \in G$, denoted by x^{-1}, $\ni P(x, x^{-1}) = P(x^{-1}, x) = e$.

Let it be noted that we shorthand $P(x, y)$ as xy.

Definition 1.3. Given a group G, we call the group *abelian* if $xy = yx \; \forall x, y \in G$. (In other words, the map is commutative.)

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Definition 1.4. If $\#(G)$ is finite, then we call $\#(G)$ the *order* of G and denote it by $o(G)$.

Definition 1.5. A is a *subgroup* of G if $\forall x, y \in A$, both xy and x^{-1} are in A.

Definition 1.6. A *bijection* is a map $\phi : A \to B$ such that the map is:
(i) *surjective*: $\forall b \in B \; \exists \; a \in A \ni \phi(a) = b$.
(ii) *injective*: $\forall b, y \in B$ and $a, x \in A \ni \phi(a) = b$ and $\phi(x) = y$, $b = y$ implies $a = x$.

Definition 1.7. A map $\phi : G \to H$, where G and H are groups, is called a *homomorphism* if $\phi(xy) = \phi(x)\phi(y) \forall x, y \in G$. If a homomorphism is a *bijection*, then we call it an *isomorphism*. If an isomorphism is from G to itself, then it is called an *automorphism*.

Note that this definition of a homomorphism implies that $\phi(x^{-1}) = (\phi(x))^{-1}$ and that $\phi(e) = e$. Also, Hom(G,H) will be the family of homomorphisms from G to H. Similarly, Aut(G) will be the automorphisms of G.

Definition 1.8. Given a group G and an element $g \in G$, we define an *inner automorphism* generated by g as $i_g(x) = gxg^{-1}$.

Definition 1.9. If N is a subgroup of G and $i_g(N) \subset N \; \forall g \in G$, then we call N a *normal subgroup* of G.

Definition 1.10. A *direct product* of two groups G and H is denoted by $G \times H$ and is a Cartesian product, $\{(g, h) \,| g \in G, h \in H\}$, equipped with the operations:

- $(g, h)(g^{'}, h^{'}) = (gg^{'}, hh^{'})$
- $e_G e_H = e_{G \times H}$
- $(g, h)^{-1} = (g^{-1}, h^{-1})$.

The following theorem is a simple theorem that illuminates several key points about the direct product.

Theorem 1.11. *Given groups G, K, let $\widetilde{G} = \{(g, e) | g \in G\}$, $\widetilde{K} = \{(e, k) | k \in K\}$. Both $\widetilde{G}$ and $\widetilde{K}$ are contained in $G \times K$. The following are true:*
(i) $\widetilde{G}$ and $\widetilde{K}$ are normal subgroups of $G \times K$.
(ii) $\widetilde{G} \cap \widetilde{K} = \{e_{G \times K}\}$.
(iii)$\widetilde{G}$ and $\widetilde{K}$ generate $G \times K$, i.e. if $x \in G \times K$ then $x = \widetilde{g}\widetilde{k}$ for some $\widetilde{g} \in \widetilde{G}$ and $\widetilde{k} \in \widetilde{K}$.
(iv) If H is a group and $\widetilde{G}$ and $\widetilde{K}$ obey (i)-(iii), where $G \times K$ is replaced by H, we can conclude that H is isomorphic to $\widetilde{G} \times \widetilde{K}$.

Proof. (i) Let $x \in G \times K$, where $x = (g', k')$. Then $i_x((g, e)) = (g'gg'^{-1}, e)$ and we know that $g'gg'^{-1} \in G$ since $g \in G, g' \in G$. So $i_x(\widetilde{G}) \subset \widetilde{G} \; \forall x \in G \times K$.
The proof is similar for $\widetilde{K}$.
(ii) In one direction, it is clear that $\widetilde{G} \cap \widetilde{K} \supset \{e_{G \times K}\}$
To show the other direction, suppose $\widetilde{G} \cap \widetilde{K} \subset \{e_{G \times K}\}$ is not true. Then $\exists$ some (g, e) or $(e, k) \ni$ it is in both $\widetilde{G}$ and $\widetilde{K}$. Then, this implies that there exists more than one identity in G or K, which is impossible.
(iii) Let $x = (g, k)$ for some $g \in G$ and $k \in K$. Then $(g, e)(e, k) = (g, k)$ for

$(g, e) \in \widetilde{G}, (e, k) \in \widetilde{K}$.
(iv) First note that if $\widetilde{g}\widetilde{k} = gk$, where $\widetilde{g}, g \in \widetilde{G}$ and $\widetilde{k}, k \in \widetilde{K}$, then $g^{-1}\widetilde{g} = k\widetilde{k}^{-1} \in \widetilde{G} \cap \widetilde{K} \Rightarrow g = \widetilde{g}, k = \widetilde{k}$.
Furthermore, since H obeys (iii), we see that $\tau : \widetilde{G} \times \widetilde{K} \to H$, where $(\widetilde{g}, \widetilde{k}) \mapsto \widetilde{g}\widetilde{k}$, is a bijection.
To show that τ preserves products, we first see that given $g \in \widetilde{G}$ and $k \in \widetilde{K}$, $gkg^{-1}k^{-1} = (gkg^{-1})k^{-1} \in \widetilde{K}$.

Similarly, $g(kg^{-1}k^{-1}) \in \widetilde{G}$. Both of the last two statement are true due to (i) and imply that $gk = kg$, i.e. we have commutativity by (ii).

$\tau((g, k), (\widetilde{g}, \widetilde{k})) = \tau(g\widetilde{g}, k\widetilde{k}) = (g\widetilde{g})(k\widetilde{k}) = (gk)(\widetilde{g}\widetilde{k}) = \tau((g, k))\tau((\widetilde{g}, \widetilde{k}))$. Thus τ preserves products and is an isomorphism. $\square$

In the next definition, we assume the reader knows the definition of a vector space.

Definition 1.12. A *linear map* $T : V \to V$, where V is a *vector space*, is linear if it preserves vector addition and scalar multiplication. That is, $T(v+w) = T(v)+T(w)$ and $T(a \cdot v) = aT(v)$, $\forall v, w \in V$ and $\forall$ scalars a.

In the next definition, the reader should know that a *basis* for a vector space is a set of linearly independent vectors such that any vector in the vector space can be written as a linear combination of vectors in the basis.

Definition 1.13. The *dimension* of V, a vector space, is the number of elements of a *basis* of the vector space.

Just for the reader's knowledge, any two bases of a vector space have the same cardinality, or number of vector elements. Also, if a linear map T has an inverse, then it is called invertible. The set of all invertible linear maps is denoted by $GL(V)$.

2. Group Representations

Definition 2.1. Given a group G, a *group representation* of that group is an element of $\mathrm{Hom}(G, GL(V))$ for some vector space V.

The *degree* of a representation is simply the dimension of V.

Definition 2.2. An *inner product* is a map $\langle \cdot, \cdot \rangle$: $V \times V \to \mathbb{C} \ni$:
(i) $\langle v, \cdot \rangle$ is a linear map if v is fixed.
(ii) $\langle v, w \rangle = \overline{\langle w, v \rangle}$
(iii)$\langle v, v \rangle \geq 0$ and $\langle v, v \rangle = 0$ iff $v = 0$.

Definition 2.3. A *unitary* representation preserves inner product. In notation, this means that $\langle U(g)v, U(g)w \rangle = \langle v, w \rangle \, \forall g \in G, v, w \in V$.

Theorem 2.4. *Given* $U : G \to GL(V)$*, a representation of a finite group* G*,* $\exists$ *inner product* $\langle \cdot, \cdot \rangle \ni$ *each* $U(g)$ *is unitary.*

Proof. Suppose we have inner product $\langle \ , \ \rangle'$ on our vector space V. Let us construct another inner product $\langle v, w \rangle = \frac{1}{o(G)} \sum_{g \in G} \langle U(g)v, U(g)w \rangle'$
Using our new inner product on V,

$$\begin{aligned}\langle U(i)v, U(i)w\rangle &= \frac{1}{o(G)}\sum_{g\in G}\langle U(g)U(i)v, U(g)U(i)w\rangle' \\ &= \frac{1}{o(G)}\sum_{g\in G}\langle U(gi)v, U(gi)w\rangle' \\ &= \frac{1}{o(G)}\sum_{g\in G}\langle U(g)v, U(g)w\rangle'\end{aligned}$$

since $\{gi|g \in G\} = G$.

And, $\frac{1}{o(G)}\sum_{g\in G}\langle U(g)v, U(g)w\rangle' = \langle v, w\rangle$ □

Note that we need our group G to be finite.

Definition 2.5. A *unitary representation* is a homomorphism from a finite group G to $\mathbf{U(V)}$, the group of unitary operators on V.

From here onwards, when we use the term representation, we mean unitary representation. This is not such a big deal because, by the theorem above, any representation can be made unitary by the appropriate choice of inner product.

Definition 2.6. Given $U : G \to \mathbf{U(V)}, T : G \to \mathbf{U(W)}$, where G is finite, U is *unitarily equivalent* to T iff $\exists$ unitary $S : V \to W \ni\ T(g) = SU(g)S^{-1}\ \forall g \in G$.

Definition 2.7. Given vector spaces V and W, we define the *direct sum* $V \oplus W$ to be the Cartesian product of the two vector spaces with operations carried out coordinate-wise. (i.e. $(v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2)$ and inner product is defined by $\langle (v_1, w_1), (v_2, w_2)\rangle = \langle v_1, v_2\rangle + \langle w_1, w_2\rangle$). Given two linear maps $A \in Hom(V)$ and $B \in Hom(W)$, we define $A \oplus B \in Hom(V \oplus W)$ by $(A \oplus B)(v, w) = (Av, Bw)$.

Theorem 2.8. Given representations T, U of G on vector spaces V, W, respectively, $U \oplus T$ is also a representation of G where $(U \oplus T)(g) = U(g) \oplus T(g)$.

Proof. First we need to show that if $U \in \mathbf{U(V)}$ and $T \in \mathbf{U(W)}$, then $U \oplus T \in \mathbf{U}(V \oplus W)$. This is easily verified by a simple calculation:

$$\begin{aligned}(U \oplus T)((v_1, w_2) + (v_2, w_2)) &= (U \oplus T)(v_1 + v_2, w_1 + w_2) \\ &= (U(v_1 + v_2), T(w_1 + w_2)) \\ &= (U(v_1) + U(v_2), T(w_1) + T(w_2)) \\ &= (U(v_1), T(w_1)) + (U(v_2), T(w_2)).\end{aligned}$$

Thus, $U \oplus T$ is a linear map. So if U, T are representations of G on V, W, respectively, then $(U \oplus T)(g) = U(g) \oplus T(g)$. □

Definition 2.9. Given a group G, a vector space V, and a representation U of G on V, we call a subspace $W \subset V$ *invariant* if $\forall w \in W$ and $g \in G$, $U(g)w \subset W$.

The next theorem is an essential one:

Theorem 2.10. Given an invariant subspace $W \subset V$, $W^\perp = \{v \in V | \langle v, w\rangle = 0 \ \forall w \in W\}$ is also an invariant subspace. If we restrict a representation U to W and $W^\perp$, and call these restrictions U_1 and U_2, respectively, then U_1 and U_2 are representations. U is equivalent to $U_1 \oplus U_2$. Conversely, $\forall\, U_1 \oplus U_2$, defined in the same way as above for some representation U, $\{(w, 0)\}$ is an invariant subspace of V.

Proof. Given $w \in W$, $w' \in W^\perp$, and $g \in G$, recall that $U(g^{-1})w \in W$ since W is invariant. Then $\langle U(g)w', w\rangle = \langle w', U^*(g)w\rangle = \langle w', U(g^{-1})w\rangle = 0$.
$\Rightarrow U(g)w' \in W^\perp \Rightarrow W^\perp$ is also invariant. This implies that we can decompose V into a direct sum of W and $W^\perp$ and write

$$U(g) = \begin{pmatrix} U_1(g) & 0 \\ 0 & U_2(g) \end{pmatrix}$$

So $U \cong U_1 \oplus U_2$. The converse is easily proven:
Let $(y, 0) \in \{(w, 0) | w \in W\}$, where W is an invariant subspace. Then $U_1(y) = z \in W$. $(U_1 \oplus U_2)(g)(y, 0) = (U_1(g) \oplus U_2(g))(y, 0) = (U_1(g)(y), U_2(g)(0)) = (z, 0)$, as required. □

Definition 2.11. Given a representation U of G on V, we call it irreducible iff 0 and V are the only invariant subspaces of V. An irreducible representation is called an irrep.

Theorem 2.12. *U is irreducible iff it is impossible to write it as a direct sum of nontrivial representations.*

Proof. This immediately follows from Theorem 2.10. □

Theorem 2.13. *Any representation U can be written as a direct sum of irreps.*

Proof. This theorem is proved by induction. Consider a representation with degree 1. Then, the representation is obviously irreducible by merit of being a one-dimensional representation.
Now suppose this theorem is true for all representations $\ni deg(U) < n$. Consider $U \ni deg(U) = n$. Suppose U is not irreducible; since otherwise we are done. Then, write $U = U_1 \oplus U_2$, where both U_1 and U_2 have degree less than n. Thus, both U_1 and U_2 are sums of irreps, and hence, we see that so is U. By the principle of induction, we are done. □

Definition 2.14. The set of equivalence classes of irreps, where each class contains all unitarily equivalent irreps, is called the *dual object*. It is denoted by $\widehat{G}$.

Definition 2.15. Let G be a finite group. The *group algebra* is the complex vector space $A(G)$ of functions on G equipped with a product, called the *convolution*, and the *conjugate* map.
Convolution: $a, b \in A(G), (a * b)(g) = \sum_{h\in G} a(gh^{-1})b(h)$
Conjugate: $a^*(g) = \overline{a(g^{-1})}$

The two above definitions will be used throughout the paper.

Definition 2.16. Given a representation U of a finite group G, we define a $^*-$ *representation* by $U_A(a) = \sum_{g\in G} a(g)U(g) \ \forall a \in A(G)$. It is a representation on

$A(G)$ that obeys the following properties:
(i) $U_A(a+b) = U_A(a) + U_A(b)\ \forall a, b \in A(G)$
(ii) $U_A(a * b) = U_A(a)U_A(b)\ \forall a, b \in A(G)$
(iii)$U_A(a^*) = U_A(a)^*\ \forall a \in A(G)$
(iv) $U_A(\delta_e) = I$, where δ_e is the function that takes on value 0 at $g \neq e$ and 1 at $g = e$, and I is the indentity.

Theorem 2.17. *Given any *-representation, U_A, $\exists$ a unitary representation U of $G \ni U_A(a) = \sum_{g \in G} a(g)U(g)$.*

Proof. Let $U(g) = U_A(\delta_g)$. By (i), it is clear that $U_A(a) = \sum a(g)U(g)$. (ii) implies $U(g)U(h) = U(gh)$ since the convolution of δ_g and δ_h is δ_{gh}. Since $\delta_g^* = \delta_{g^{-1}}$, by (iii), we have that $U(g)^* = U(g^{-1})$, which along with (iv) implies $I = U(g)U(g)^* = U(g)^*U(g)$. Thus, U is unitary. $\square$

The previous theorem tells us that there exists an injective relation between representations of G and *-representations of $A(G)$. Let $L_A(a)b := a * b$.
If we equip $A(G)$ with inner product $\langle a, b\rangle = \frac{1}{o(G)} \sum_{g \in G} \overline{a(g)}b(g)$, then $\langle a, b * c\rangle = \langle b^* * a, c\rangle$.
Note that if $(L_g a)(h) = a(g^{-1}h)$ then L_A is the induced map on $A(G)$ since $(\delta_g * a)(h) = \sum_i \delta_g(hi^{-1})a(i) = a(g^{-1}h)$.
Thus, $L_g\delta_h = \delta_g * \delta_h = \delta_{gh}$, which implies that L_g is unitary. This gives us the following theorem:

Theorem 2.18. *$L_g a(h) \equiv a(g^{-1}h)$ on $A(G)$, with inner product as defined above, is a unitary representation of G.*

Definition 2.19. The representation described in the previous theorem is called the *$left\ regular\ representation$.*

The next theorem, $Schur's\ Lemma$, is an important lemma in the study of group representations, but will seem similar to previous statements made in the paper.

Theorem 2.20. *Let U_A be an irrep of $A(G)$ on V. If $\exists\ T \in Hom(V) \ni TU_A(a) = U_A(a)T\ \forall a \in A(G)$, then $T = cI$, where c is a constant. (Note that the statement $TU_A(a) = U_A(a)T$ is equivalent to $TU(g) = U(g)T\ \forall g \in G$).*

Proof. Let λ be an eigenvalue of T. Consider $v \ni (T - \lambda)v = 0$.
Then $(T - \lambda)U_A(a)v = U_A(a)(T - \lambda)v = 0$ so the subspace $\{v|(T - \lambda)v = 0\}$ is invariant. Since λ is an eigenvalue, the invariant subspace is not $\{0\}$. Thus, since U_A is an irrep, the invariant space must be V. i.e. $T = \lambda I$. $\square$

The second form of Schur's Lemma is as follows.

Theorem 2.21. *Let S, U be irreps of a finite group G on vector spaces V, W, respectively. Let $T : V \to W$ be a map $\ni TU(g) = S(g)T\ \forall g \in G$. Then either $T = 0$ or U and S are unitarily equivalent and T is unique up to a constant.*

Proof. Suppose $TU(g) = S(g)T\ \forall g \in G$. Replace g with g^{-1} and take adjoints of both sides. We get that $T^*S(g) = U(g)T^*\ \forall g \in G$.
Then $(T^*T)U(g) = U(g)(T^*T)$ and $(TT^*)S(g) = S(g)(TT^*)$.
By the first form of Schur's Lemma, we see that $TT^* = c_T I$ and $T^*T = c_T I$.
Note that $TT^* = T^*T$ because both are constant diagonal matrices and have the

same determinant, and thus, are equivalent.
Then either $c_T = 0$ (which implies that $T = 0$) or $RU(g) = S(g)R$ where $R = c_T^{-\frac{1}{2}}T$ is unitary, which implies that S and U are unitarily equivalent.
If the second case holds, then given two maps T and Q that obey $TU(g) = S(g)T \; \forall g \in G$, then we have that $TQ^* = c_1 I$ and thus, $T = TQ^*Qc_Q^{-1} = c_1 c_Q^{-1} Q$. Thus, T is unique up to a constant. $\square$

Note that if our group is abelian, then every irrep of the group has degree 1, i.e. our vector space must have dimension 1.

3. Tensor Products

A quick foray into the concept of tensor products is in order before talking about matrix realizations.

Definition 3.1. A *Hilbert space*, H, is a vector space that is equipped with an inner product $\ni$ if we define the norm as $|f| = \sqrt{\langle f, f\rangle}$, then H is a complete metric space.

Now we can define a tensor product.

Definition 3.2. Given two Hilbert spaces, X and Y, their tensor product is the vector space of bi-antilinear maps with a certain inner product. The tensor product is denoted as $X \otimes Y$.
This means that if $f \in X \otimes Y$, then $f : X \times Y \to \mathbb{C}$ is a map $\ni$:
$f(x, \alpha y \beta w) = \overline{\alpha} f(x, y) + \overline{\beta} f(x, w)$
where $x \in X$, $w, y \in Y$, and $\alpha, \beta \in \mathbb{C}$.
$f(\alpha x \beta w, y) = \overline{\alpha} f(x, y) + \overline{\beta} f(w, y)$
where $x, w \in X$, $y \in Y$, and $\alpha, \beta \in \mathbb{C}$.

Note that $X \otimes Y$ is a vector space.
Given $x \in X$ and $y \in Y$, we define the map $x \otimes y \in X \otimes Y$ as the following:
$(x \otimes y)(r, s) = \langle r, x\rangle \langle s, y\rangle$.
Thus, we have the map $(x, y) \mapsto (x \otimes y)$ as a bilinear map of $X \times Y$ to $X \otimes Y$ and we see that the set of such maps span the vector space $X \otimes Y$. Next is a quick fact with outlined proof about tensor products.

Theorem 3.3. *$dim(X \otimes Y) = dim(X)dim(Y)$.*

Proof. If $\{x_i\}_{i=1}^n$ is a basis for X and $\{y_j\}_{j=1}^m$ is a basis for Y, then $\{x_i \otimes y_j\}_{i=1,j=1}^{n,m}$ is a basis for $X \otimes Y$. Thus, $dim(X \otimes Y) = dim(X)dim(Y)$. $\square$

Now we can give $X \otimes Y$ a unique inner product such that, given $x \otimes y$ and $F \in X \otimes Y$, $\langle x \otimes y, F\rangle = F(x, y)$. Under this inner product, we have the following obvious theorem:

Theorem 3.4. *If $\{x_i\}_{i=1}^n$ and $\{y_j\}_{j=1}^m$ are orthonormal bases of X and Y, respectively, then $\{x_i \otimes y_j\}_{i=1,j=1}^{n,m}$ is an orthonormal basis for $X \otimes Y$.*

Now, let us consider $A \in Hom(X)$ and $B \in Hom(Y)$. Then, $A \otimes B$ is defined as follows:
For $C \in X \otimes Y, x \in X, y \in Y$, we have $(A \otimes B)(C)(x, y) = C(A^*x, B^*y)$.
From this definition, we see that $(A \otimes B)(x \otimes y) = Ax \otimes By$. Thus, given representations U, V of a group G on vector spaces X, Y, define $U \otimes V$ as a representation

of G on $X \otimes Y$ by the following:
$(U \otimes V)(g) = U(g) \otimes V(g)$. This leads to the following definition:

Definition 3.5. The *tensor product representation* is the product of representations U, V and is defined by $(U \otimes V)(g) = U(g) \otimes V(g)$.

Now, the important concept that stems from tensor product representations is the fact that if $U^{(\alpha)}, V^{(\beta)}$ are irreducible representations, then $U^{(\alpha)} \otimes V^{(\beta)}$ is not irreducible. However, it is a direct sum of irreducibles. Given $\alpha, \beta \in \widehat{G}$,
$U^{(\alpha)} \otimes V^{(\beta)} \cong \oplus_{\gamma \in \widehat{G}} n^{\gamma}_{\alpha\beta} U^{(\gamma)}$.
The $n^{\gamma}_{\alpha\beta}$ are integers that signify the number of times the specific irrep occurs in the direct sum. These integers are called $Clebsch - Gordan\ integers$. Also, recall that $U^{(\alpha)}$ is a member of the equivalence class of $\alpha \in \widehat{G}$.
Taking matrix realizations of our irreducible representations and using the facts explored previously, we have the following theorem. (Given an *irrep* $\alpha \in \widehat{G}$, we represent α by $D^{(\alpha)}(g)$. $D^{(\alpha)}(g)$ is a $d_\alpha \times d_\alpha$ function matrix, where d_α is the degree of the irrep. The entries of the matrix are $D^{(\alpha)}_{ij}(g)$. This will be restated and explained in the next section.)

Theorem 3.6. *$D^{(\alpha)}_{ij}(g) D^{(\beta)}_{kl}(g) = \sum_{m,p,q} c^{\alpha\beta m}_{ij;kl;pq} D^{(\gamma_m)}_{pq}(g)$, where $c^{\alpha\beta m}_{ij;kl;pq}$ are constants. The sum occurs over $m = 1, ..., M$ where given any m, $\exists$ a $\gamma_m \in \widehat{G}$ associated with that value of m; and $p, q = 1, ..., d_{\gamma_m}$.*

This theorem essentially states that the set of linear combinations of matrix realizations of irreps is an algebra, i.e. closed under the product map.

4. Matrix Realizations of Representations

It is necessary to introduce the concept of functions and representations as matrices. To do this, we provide some key theorems regarding the group algebra. Also, in this section we explore the properties of matrix realizations while developing the concept of the Fourier transform.

Recall that $\alpha \in \widehat{G}$ is an equivalence class of irreducible representations and that d_α is the degree of the irrep. Given an *irrep* $\alpha \in \widehat{G}$, we represent α by $D^{(\alpha)}(g)$, which is a $d_\alpha \times d_\alpha$ matrix. The entries of the matrix are $D^{(\alpha)}_{ij}(g)$.

Each $D^{(\alpha)}_{ij}(g)$ is an element of $A(G)$. Given a basis $\{v_1, ..., v_{d_\alpha}\}$ of the vector space V for our representation $U^{(\alpha)}$, the element $D^{(\alpha)}_{ij}(g)$ is determined by

$$D^{(\alpha)}_{ij}(g) = \left\langle U^{(\alpha)}(g) v_i, v_j \right\rangle$$

Thus, the matrix realization $D^{(\alpha)}(g)$ is a $d_\alpha \times d_\alpha$ matrix. We write $D^{(\alpha)}(g)$ rather than $D^{(\alpha)}$ to remind the reader that the entries of the matrix are not fixed.

A good definition before reaching our main theorems:

Definition 4.1. Let S be a representation of G on V and let U be a representation of G on W. We call the two representations S and U *intertwined* if $\exists\ T : V \to W \ni TU(g) = S(g)T\ \forall g \in G$. T is called their *intertwining map.*

Theorem 4.2. *The following is true:* $\frac{1}{o(G)}\sum_{g\in G}\overline{D_{ij}^{(\alpha)}(g)}D_{kl}^{(\beta)}(g) = \frac{1}{d_\alpha}\delta_{\alpha\beta}\delta_{ik}\delta_{jl}$ *where* $\delta_{\alpha\beta}$ *is unity if* $\alpha=\beta$ *and zero otherwise.*

Proof. Given $\mathbb{C}^{d_\alpha}$ as the representation space for the representation U^α (the represenation space is the vector space given for the representation), let
$A:\mathbb{C}^{d_\alpha}\to\mathbb{C}^{d_\alpha}$
and define $\widetilde{A}=\frac{1}{o(G)}\sum_{g\in G}U^{(\beta)}(g)AU^{(\alpha)}(g)^{-1}$. We do this to produce a map that intertwines $U^{(\alpha)}$ and $U^{(\beta)}$ as we see below.
Then,

$$\begin{aligned}
U^{(\beta)}(h)\widetilde{A} &= \frac{1}{o(G)}\sum_{g\in G}U^{(\beta)}(h)U^{(\beta)}(g)AU^{(\alpha)}(g)^{-1}\\
&= \frac{1}{o(G)}\sum_{g\in G}U^{(\beta)}(hg)AU^{(\alpha)}(g)^{-1}\\
&= \frac{1}{o(G)}\sum_{j\in G}U^{(\beta)}(j)AU^{(\alpha)}(h^{-1}j)^{-1}\\
&= \frac{1}{o(G)}\sum_{j\in G}U^{(\beta)}(j)AU^{(\alpha)}(j)^{-1}U^{(\alpha)}(h)\\
&= \widetilde{A}U^{(\alpha)}(h).
\end{aligned}$$

$\Rightarrow \widetilde{A}$ intertwines $U^{(\alpha)}$ and $U^{(\beta)}$. Thus, by Schur's Lemma (Second Form), if $\alpha\neq\beta$ then $\widetilde{A}=0$. If $\alpha=\beta$, then $\widetilde{A}=cI$. Thus, $Tr(\widetilde{A})=c\cdot d_\alpha$, so constant $c=\frac{1}{d_\alpha}Tr(\widetilde{A})=\frac{1}{d_\alpha}Tr(A)$.
This implies that $\widetilde{A}=\frac{1}{d_\alpha}Tr(A)\delta_{\alpha\beta}I$. Recall that A is simply a map represented as a matrix. Since we produce an intertwining map $\widetilde{A}$ with any choice of A, we can now specifically choose our A so that the expression $\frac{1}{o(G)}\sum_{g\in G}U^{(\beta)}(g)AU^{(\alpha)}(g)^{-1}$ becomes $\frac{1}{o(G)}\sum_{g\in G}\overline{D_{ij}^{(\alpha)}(g)}D_{kl}^{(\beta)}(g)$. To achieve this, we take our matrix realization of A to have only a single nonzero element, $A_{yz}=\delta_{yk}\delta_{zj}$. (Thus, when we multiply A with our representations, we have just one non-zero entry in the resulting matrix.)
So if $\alpha=\beta$, then $Tr(A)=\delta_{kj}$.
Since $\widetilde{A}=\frac{1}{d_\alpha}Tr(A)\delta_{\alpha\beta}I$, then
$\frac{1}{o(G)}\sum_{g\in G}\overline{D_{ij}^{(\alpha)}(g)}D_{kl}^{(\beta)}(g)=\frac{1}{d_\alpha}\delta_{\alpha\beta}\delta_{ik}\delta_{jl}$. □

Theorem 4.3. Given a complex vector space of functions, A, on a finite set $X \ni$:
(i) If $a,b\in A$, then $ab\in A$ where ab is the pointwise product of the two functions.
(ii)If $x,x'\in X$, then $\exists f_{xx'}\in A\ni f_{xx'}(x)=1$ and $f_{xx'}(x')=0$.
Then we conclude that A is the set of all functions on X.

Proof. Recall the function δ_x, where $\delta_x(x')=1$ if $x=x'$ and 0 otherwise. $\forall x,x'$, let $f_{xx'}$ be the function that obeys (ii) for that specific x,x'. So, $\delta_x=\prod_{x'\neq x}f_{xx'}$ which implies that $\delta_x\in A$ because by (i), the product of two functions in A also lies in A. Because the functions δ_x form a basis for the set of functions on X, A is all functions on X. □

The combination of these last two theorems shows that $D^{(\alpha)}_{ij}$ spans $A(G)$. We now will find a orthonormal basis for $A(G)$. However, we must first present a corollary to Theorem(2.18).

Corollary 4.4. *The functions $D^{(\alpha)}_{ij}$ seperate points, i.e. $\forall g, h \in G, \exists$ a function f of the form:*
$f = \sum_{n=1}^{m} C_{nij} D^{(\alpha_n)}_{i_n j_n}$ *such that* $f(g) = 1, f(h) = 0$.

Proof. Let L be a left regular representation. Let $f(g') = o(G)\langle \delta_g, L(g')\delta_e \rangle$. Since $L(g')\delta_e = \delta_{g'e} = \delta_{g'}$, we have $f(g) = 1$ and $f(h) = 0$. Since L is a representation, as we have shown earlier in the paper, we can write it as a direct sum of irreducible representations. This is equivalent to stating:

$(L(x))_{gh} = \sum_{n;i,j} M^{(\alpha_n)}_{gi} D^{(\alpha_j)}_{ij}(x) \overline{M}^{(\alpha_n)}_{hj}$, where the $M^{(\alpha_n)}_{gi}$ are constants and each $(L(x))_{gh}$ is an entry of the matrix realization of L. This yields the summation form of f that we wanted. □

Theorem 4.5. $\left\{\sqrt{d_\alpha} D^{\alpha}_{ij}(g)\right\}_{\alpha \in \widehat{G}; i,j=1,\ldots,d_\alpha}$ *is an orthonormal basis of $A(G)$ equipped with the inner product $\langle a, b\rangle = \frac{1}{o(G)} \sum_g \overline{a(g)} b(g)$.*
An important side note is that $\sum_\alpha d_\alpha^2 = o(G)$.

Proof. Set $A = \left\{\sum_{i,j=1,\ldots,d_\alpha} c^{(\alpha)}_{ij} D^{(\alpha)}_{ij}(g) | c^{(\alpha)}_{ij} \in \mathbb{C}\right\}$.
A is clearly constructed to be a vector space. The elements of $\left\{\sqrt{d_\alpha} D^{\alpha}_{ij}(g)\right\}$ are orthnormal. Now, to show that $A = A(G)$, we need to use the previous Theorem(4.3). A satisfies the two conditions of the hypothesis of Theorem(4.3):
(i) is satisfied by Theorem(3.6).
(ii) follows directly from the above corollary.
Thus, $A = A(G)$ and $\left\{\sqrt{d_\alpha} D^{(\alpha)}_{ij}(g)\right\}$ is a basis.

□

Theorem 4.6. *Define $X^{(\alpha)}_{ij}(g) = \frac{d_\alpha}{o(G)} D^{(\alpha)}_{ij}(g)$. Then, $X^{(\alpha)}_{ij} * X^{(\beta)}_{kl} = \delta_{\alpha\beta}\delta_{jk} X^{(\alpha)}_{il}$.*

Proof.

$$\begin{aligned}
X^{(\alpha)}_{ij} * X^{(\beta)}_{kl}(g) &= d_\alpha d_\beta \frac{1}{o(G)^2} \sum_{h \in G} D^{(\alpha)}_{ij}(gh^{-1}) D^{(\beta)}_{kl}(h) \\
&= d_\alpha d_\beta \frac{1}{o(G)^2} \sum_{h \in G; n=1,\ldots d_\alpha} D^{(\alpha)}_{in}(g) D^{(\alpha)}_{nj}(h^{-1}) D^{(\beta)}_{kl}(h) \\
&= d_\alpha d_\beta \frac{1}{o(G)^2} \sum_{h \in G; n=1,\ldots d_\alpha} D^{(\alpha)}_{in}(g) \overline{D^{(\alpha)}_{jn}(h)} D^{(\beta)}_{kl}(h) \\
&= \delta_{\alpha\beta} d_\beta \frac{1}{o(G)} \delta_{jk} D^{(\alpha)}_{il}(g).
\end{aligned}$$

□

Next is an important theorem for this paper. This theorem introduces the concept of the Fourier transform.

Theorem 4.7. A(G) is isomorphic to a direct sum of matrix algebras $\ni$ any inner product on $A(G)$ is a multiple of the following inner product on matrix algebras: $\langle M, N\rangle = tr(M^* N)$, also called the Hilbert-Schmidt inner product. To put it in

notation and more directly,
$A(G) \cong \oplus_{a\in\widehat{G}} Hom(\mathbb{C}^{d_\alpha})$ through a map from $A(G) \to \oplus_{a\in\widehat{G}} Hom(\mathbb{C}^{d_\alpha})$. Given $f \in A(G)$ and $\widehat{f} \in \oplus_{a\in\widehat{G}} Hom(\mathbb{C}^{d_\alpha})$, our map transforms f into

$$\widehat{f}_{\alpha;i,j} = \frac{o(G)}{d_\alpha} \sum_{g\in G} \overline{X}_{ij}^{(\alpha)}(g) f(g) = \sum_{g\in G} \overline{D_{ij}^{(\alpha)}(g)} f(g).$$

Proof. $f(x) = \sum_{\alpha;i,j} \widehat{f}_{\alpha;i,j} X_{ij}^{(\alpha)}(g)$ by our choice of $\widehat{f}$.
Given norm of $\widehat{f}$ as $\sum_\alpha \frac{o(G)^2}{d_\alpha} \sum_{i,j} \left|\widehat{f}_{\alpha;i,j}\right|^2$, we see that $\widehat{\ }$ is unitary because $\left\{\frac{o(G)}{\sqrt{d_\alpha}} X_{ij}^{(\alpha)}\right\}$ is orthonormal, and hence, since $\widehat{f}_{\alpha;i,j} = \frac{o(G)}{d_\alpha} \sum_{g\in G} \overline{X}_{ij}^{(\alpha)}(g) f(g)$, the norm is preserved.
The next thing to show is that $\widehat{f * g} = \widehat{f}\widehat{g}$. This is equivalent to showing that $\widehat{f * g}_{\alpha;i,j} = \sum_{k=1}^{d_\alpha} \widehat{f}_{\alpha;ik} \widehat{g}_{\alpha;kj}$, which follows from Theorem (4.6).
By this theorem,
$X_{ij}^{(\alpha)} * X_{kl}^{(\beta)} = \delta_{\alpha\beta}\delta_{jk} X_{il}^{(\alpha)}$, so given that
$\widehat{f * g}_{\alpha;i,j} = \frac{o(G)}{d_\alpha} \sum_{h\in G} \overline{X_{ij}^{(\alpha)}(h)} (f * g(h))$, we can break our $X_{ij}^{(\alpha)}$ into a convolution of $X_{ik}^{(\alpha)}$ and $X_{kj}^{(\alpha)}$. From there, it follows that $\widehat{f * g}_{\alpha;i,j} = \sum_{k=1}^{d_\alpha} \widehat{f}_{\alpha;ik} \widehat{g}_{\alpha;kj}$. □

5. Fourier Analysis

Before defining the Fourier transform, the reader should know that $L^p(G)$ is the set of functions on G with norm $\|f\|_p^p = \frac{1}{o(G)} \sum_{g\in G} |f(g)|^p$. Pick an irrep $D^{(\alpha)}$ of G on vector space X_α, where $\alpha \in \widehat{G}$. Now, let $C(\widehat{G})$ to be a function from $\widehat{G} \to Hom(X_\alpha)$. This means that if $T \in C(\widehat{G})$, then T is a sequence $\{T_\alpha\}_{\alpha\in\widehat{G}} \ni T_\alpha \in Hom(X_\alpha)$. Thus, if we equip $T \in C(\widehat{G})$ with the norm $\|T\|_p^p = \sum_{\alpha\in\widehat{G}} d_\alpha Tr(|T_\alpha|^p)$, then $C(\widehat{G})$ is equivalent to $L^p(G)$.

Now for the definition of the Fourier transform:

Definition 5.1. The Fourier transforms are maps $^\star : A(G) \to C(\widehat{G})$ and $^\ddagger : C(\widehat{G}) \to A(G)$, which are defined as follows:
$(f^\star)_\alpha = \frac{1}{o(G)} \sum_{g\in G} f(g) D^{(\alpha)}(g)$ and
$(T^\ddagger)(g) = \sum_\alpha d_\alpha Tr(D^{(\alpha)}(g)^* T_\alpha)$.

To finish this paper, I will present a theorem that provides observations about our two transformation maps.

Theorem 5.2. *The following are true:*
(i) $^\star, ^\ddagger$ *are adjoint maps, i.e.* $(T^\ddagger, f)_{L^2(G)} = (T, f^\star)_{L^2(\widehat{G})}$.
(ii) $^\star, ^\ddagger$ *are inverses.*
(iii) $\|f^\star\|_{L^2(G)} = \|f\|_{L^2(\widehat{G})}$.

Proof. (i) The inner product $(f,h)_{L^2(G)} = \frac{1}{o(G)} \sum_{g\in G} \overline{f(g)} h(g)$. The inner product $(f,h)_{L^2(\widehat{G})} = \sum_{\alpha\in\widehat{G}} d_\alpha Tr(h_\alpha f_\alpha)$.

$$\begin{aligned}
(T^\ddagger, f)_{L^2(G)} &= \frac{1}{o(G)} \sum_{g\in G} \sum_{\alpha\in\widehat{G}} d_\alpha \overline{Tr(D^\alpha(g)^* T_\alpha)} f(g) \\
&= \frac{1}{o(G)} \sum_{g\in G} d_\alpha \sum_{\alpha\in\widehat{G}} Tr(T_\alpha^* D^\alpha(g) f(g)) \\
&= \sum_{\alpha\in\widehat{G}} d_\alpha Tr(T_\alpha^* f_\alpha^\star) \\
&= (T, f^\star)_{L^2(\widehat{G})}
\end{aligned}$$

We prove (iii) before (ii).
(iii) This is essentially Theorem(4.5).
To prove (ii), we realize that (iii) tells us that * is an isometry $\Rightarrow$ * is unitary because we are dealing with a finite dimensional space. So, since $^\star,^\ddagger$ are adjoint maps and * is unitary, ‡ is a two-sided inverse. $\square$

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References

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