# THE JORDAN-BROUWER SEPARATION THEOREM

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Abstract. The Classical Jordan Curve Theorem says that every simple closed curve in $\mathbb{R}^2$ divides the plane into two pieces, an "inside" and an "outside" of the curve. This paper will prove an considerable extension of this Theorem; that, in fact, every compact, connected hypersurface in $\mathbb{R}^n$ divides $\mathbb{R}^n$ into two connected open sets; an "inside", and an "outside", where the closure of the inside is also a compact manifold.

Contents



## 1. Preliminaries

**Definition 1.1.** Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ be open sets. A mapping $f : U \to V$ is called **smooth** if all of its partial derivatives exist and are continuous.

**Definition 1.2.** A map $f : X \to Y$ is called a **diffeomorphism** if $f$ carries $X$ homeomorphically onto $Y$ and if both $f$ and $f^{-1}$ are smooth.

**Definition 1.3.** A subset $X \subset \mathbb{R}^m$ is called a **smooth manifold of dimension n** if each $x \in X$ has a neighborhood $U \cap X$ that is diffeomorphic to an open subset $V$ of $\mathbb{R}^n$.

**Definition 1.4.** A subset $Y \subset \mathbb{R}^m$ is called a smooth manifold of dimension n **with boundary** if each $y \in Y$ has a neighborhood $U \cap Y$ that is diffeomorphic to an open subset $V$ of the half-space $H^m = \{(x_1, \dots, x_n) \in \mathbb{R}^n | x_n \geq 0\}$. The boundary $\partial Y$ is the set of points in $Y$ which correspond to the points of $\partial H^m$ under such a diffeomorphism.

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**Definition 1.5.** Suppose that $X$ is a manifold, $U \cap X$ is open, and $V$ is an open subset of $\mathbb{R}^n$. Any particular diffeomorphism $g : V \to U \cap X$ is called a **parametrization** of $U \cap X$, and the inverse diffeomorphsim $h : U \cap X \to V$ is called a **system of coordinates** on $U \cap X$.

**Definition 1.6.** Suppose $V$ is an open subset of $\mathbb{R}^n$. Let $g : V \to X \subset \mathbb{R}^m$ be a parametrization for a neighborhood $g(V)$ of a point $x \in X$, with $g(u) = x$. Think of $g$ as a mapping from $V$ to $\mathbb{R}^m$ so that the derivative $dg_u$ is defined:

$$dg_u : \mathbb{R}^n \to \mathbb{R}^m$$

We define the **tangent plane at x**, abbreivated $T_xX$, to be the image $dg_u(\mathbb{R}^n)$ of $dg_u$.

**Definition 1.7.** Consider two manifolds, $X \subset \mathbb{R}^k$ and $Y \subset \mathbb{R}^l$, and let $f$ be a smooth map $f : X \to Y$, with $f(x) = y$. Since $f$ is smooth, there exists an open set $U$ containing $x$ and a smooth map $F : U \to \mathbb{R}^l$ that coincides with $f$ on $U \cap X$. Define $df_x(v)$ to equal $dF_x(v)$ for all $v \in T_xX$ so that $df_x$ is a map from $T_xX$ to $T_yY$.

**Proposition 1.8.** *If $X$ is a manifold of dimension $n$, then $T_xX$ is an $n$-dimensional vector space.*

## 2. Immersions

**Definition 2.1.** Suppose that $X$ and $Y$ are manifolds with $\dim X < \dim Y$, and suppose that $f : X \to Y$ where $f(x) = y$. We call $f$ an **immersion at x** if $df_x : T_xX \to T_yY$ is injective. If $f$ is an immersion for every point $x \in X$, then $f$ is simply called an **immersion**.

**Definition 2.2.** The inclusion map of $\mathbb{R}^k$ into $\mathbb{R}^l$ where $l \geq k$ and where $(a_1, \ldots, a_k)$ maps to $(a_1, \ldots, a_k, 0, \ldots, 0)$ is called the **canonical immersion**.

**Theorem 2.3.** *(Local Immersion Theorem) If $f : X \to Y$ is an immersion at a point $x$ and $f(x) = y$, then there exist local coordinates around $x$ and $y$ such that*

$$f(x_1, \ldots, x_k) = (x_1, \ldots, x_k, 0, \ldots, 0).$$

*In other words, if $f$ is an immersion at $x$, then $f$ is a locally equivalent to the canonical immersion at $x$.*

*Proof.* To begin with, choose any local parametrization for $X$ and $Y$ centered at $x$ and $y$. This gives the following commutative diagram, with $\phi(x) = 0$ and $\psi(y) = 0$:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\phi"'] & Y \arrow[d, "\psi"] \\ U \arrow[r, "g"'] & V \end{tikzcd}$$

Now, $dg_0 : \mathbb{R}^k \to \mathbb{R}^l$ is injective, and via a change of basis in $\mathbb{R}^l$ we may assume it has an $l \times k$ matrix:

$$\begin{pmatrix} I_k \\ 0 \end{pmatrix}$$

where $I_k$ is the $k \times k$ identity matrix. Define a map $G : U \times \mathbb{R}^{l-k} \to \mathbb{R}^l$ by

$$G(x, z) = g(x) + (0, z).$$

Thus $G$ maps open sets of $\mathbb{R}^l$ to open sets of $\mathbb{R}^l$ and has a matrix for $dG_0$ of $I_l$. By the Inverse Function Theorem, $G$ is a local diffeomorphism of $\mathbb{R}^l$ centered at 0. Note that $g = G\circ$ (canonical immersion). Since $G$ and $\psi$ are local diffeomorphisms at 0, so too $\psi \circ G$ must be a local diffeomorphism at 0. We can use this $\psi \circ G$ as a new parametrization of $Y$ at $y$; thus, (shrinking $U$ and $V$ as necessary), this leads to a new commutative diagram:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\phi"'] & Y \arrow[d, "\psi\circ G"] \\ U \arrow[r, "\text{canonical immersion}"'] & V \end{tikzcd}$$

□

## 3. Regular Values

**Definition 3.1.** For a smooth map of manifolds $f : X \to Y$, a point $y \in Y$ is called a **regular value** for $f$ if $df_x : T_xX \to T_yY$ is surjective at every point $x$ such that $f(x) = y$. Any point $y \in Y$ that is not a regular value of $f$ is called a **critical value**. (See *Figure 1.*)

**Theorem 3.2.** *(The Preimage Theorem) If $f : X \to Y$ is a smooth map between manifolds with $\dim X = m$ and $\dim Y = n$ and where $m \geq n$, and if $y \in Y$ is a regular value, then the set $f^{-1}(y) \subseteq X$ is a smooth manifold of dimension $m - n$. (See Figure 1.)*

*Proof.* Let $x \in f^{-1}(y)$. Since $y$ is a regular value, the derivative $df_x$ is surjective, and therefore must map $T_xX$ onto $T_yY$. The null space $K \subset T_xX$ of $df_x$ will therefore be an $(m-n)$-dimensional vector space.

If $X \subset \mathbb{R}^k$, choose a linear map $L : \mathbb{R}^k \to \mathbb{R}^{m-n}$ that is nonsingular on this subspace $K \subset T_xX \subset \mathbb{R}^k$. Now define

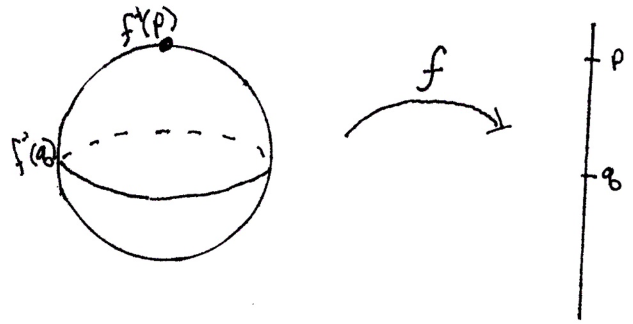
$$F : X \to Y \times \mathbb{R}^{m-n}$$

by $F(\zeta) = (f(\zeta), L(\zeta))$. The derivative $dF_x$ is therefore given by the formula

$$dF_x(v) = (df_x(v), L(v)).$$

It is clear that $dF_x$ is nonsingular. Hence $F$ maps some neighborhood $U$ of $x$ diffeomorphically onto a neighborhood $V$ of $(y, L(x))$. Note that the image of the $f^{-1}(y) \subset X$ under $F$ is the hyperplane $y \times \mathbb{R}^{m-n}$. In fact $F$ maps $f^{-1}(y) \cap U$ diffeomorphically onto $(y \times \mathbb{R}^{m-n}) \cap V$. This proves that $f^{-1}(y)$ is a smooth manifold of dimension $m - n$. □

*Figure 1*




*The height function $f$ from the sphere $S^2$ to $[-1, 1]$. Here, $q$ is a regular value, while $p$ is a critical value. Note also $f^{-1}(q)$ is a submanifold of dimension $2 - 1 = 1$, the circle $S^1$.*

**Definition 3.3.** Suppose that $g_1, \ldots, g_l$ are smooth, real-valued functions on a manifold $X$ of dimension $k \geq l$, we say they are **independent** at $x$ if the $l$ functionals $d(g_1)_x, \ldots, d(g_l)_x$ are linearly independent on $T_xX$.

*Remark* 3.4. Note that if we define a function $g = (g_1, \ldots, g_l) : X \to \mathbb{R}^l$, and define the set $Z = g^{-1}(0) \subset X$, then for any point $x \in Z$ we have $dg_x : T_xX \to \mathbb{R}^l$ as surjective if and only if the $l$ functionals $d(g_1)_x, \ldots, d(g_l)_x$ are linearly independent on $T_xX$. Therefore, if the $l$ functions, $g_1, \ldots, g_l$ are independent at every point $x \in Z$ then 0 is a regular value of $g$ and thus $Z$ is a submanifold of $X$.

**Proposition 3.5.** *If the smooth, real-valued functions $g_1, \ldots, g_l$ on $X$ are independent at each point where they all vanish, then the set $Z$ of common zeros is a submanifold of $X$ with dimenstion equal to* $\dim X - l$.

**Proposition 3.6.** *If $y$ is a regular value of a smooth map $f : X \to Y$, then the preimage submanifold $f^{-1}(y)$ can be cut out by independent functions.*

**Proposition 3.7.** *Every submanifold of $X$ is locally cut out by independent functions.*

**Lemma 3.8.** *Let $Z$ be the preimage of a regular value $y \in Y$ under the smooth map $f : X \to Y$. Then the kernel of the derivative $df_x : T_xX \to T_yY$ at any point $x \in Z$ is precisely the tangent space to $Z$, $T_xZ$.*

*Proof.* Because $f$ is constant on $Z$, $df_x = 0$ on $T_xZ$. But because $y$ is a regular value, $df_x : T_xX \to T_yY$ must be surjective, so the dimension of the kernel of $df_x$ is given by:

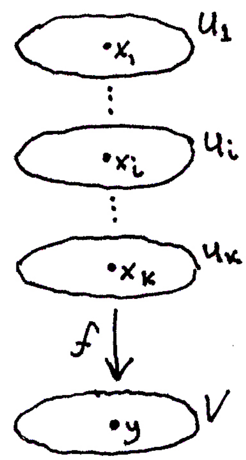
$$\dim T_xX - \dim T_yY = \dim X - \dim Y = \dim Z.$$

Thus, $T_xZ$ is a subspace of the kernel which also has the same dimension as the kernel. Therefore, $T_xZ$ must be the kernel. $\square$

**Theorem 3.9.** *(Stack of Records Theorem) Suppose $f : X \to Y$ is a smooth map, with $X$ compact,* $\dim X = \dim Y$*, and let $y \in Y$ be a regular value. Then $f^{-1}(y)$ is a finite set $\{x_1, \ldots, x_k\}$, and there are neighborhoods $U_i$ of $x_i$ and $V$ of $y$ such that $U_i \cap U_j = \emptyset$ for $i \neq j$ and $f^{-1}(V) = U_1 \cup \ldots \cup U_k$. Furthermore, $f$ maps each $U_i$ diffeomorphically onto $V$. (See Figure 2.)*

*Proof.* Since $y$ is a regular value, $f^{-1}(y)$ must be a manifold of dimension $\dim X - \dim Y = 0$. Since $X$ is compact, and thus bounded, $f^{-1}(y)$ must be finite, else it would contain an accumulation point, and contradict its status as a manifold of dimension zero. Now, $f$ is a local diffeomorphism at each $x_i$, so there exists neighborhoods $U_i'$ of $x$ and $V_i$ of $y$ so that $f : U_i' \to V_i$ is a diffeomorphism. Now, we may simply shrink the $U_i'$'s so that $U_i' \cap U_j' = \emptyset$ for $i \neq j$. Let $V' = V_1 \cap \ldots \cap V_k$ and $U_i'' = U_i' \cap f^{-1}(V')$. Clearly, $f : U_i'' \to V'$ is a diffeomorphism. Lastly, $Z = f(X \setminus \cup U_i'')$ is closed in $Y$ and does not contain $y$. Thus, $V = V' - Z$ and $U_i = U_i'' \cap f^{-1}(V)$ fullfil the requirements. $\square$

*Figure 2*




**Proposition 3.10.** *(Sard's Theorem) If $f : X \to Y$ is any smooth map of manifolds, then the image of the set of critical points in $X$ is a set of measure zero..*

**Proposition 3.11.** *Any compact, smooth, connected 1-dimensional manifold is diffeomorphic either to the circle $S^1$ or the closed unit interval* $[0, 1]$.

## 4. Tranversality

**Definition 4.1.** Suppose that $f : X \to Y$ is a map, and that $Z$ is a submanifold of $Y$. We say that $f$ is **transversal** to $Z$, abbreviated $f \pitchfork Z$, if

$$\text{Image}(df_x) + T_y Z = T_y Y.$$

**Theorem 4.2.** *If the smooth map $f : X \to Y$ is transversal to a submanifold $Z \subset Y$, then the preimage $f^{-1}(Z)$ is a submanifold of $X$. Furthermore, the codimension of $f^{-1}(Z)$ in $X$ equals the codimension of $Z$ in $Y$. (See Figure 3.)*

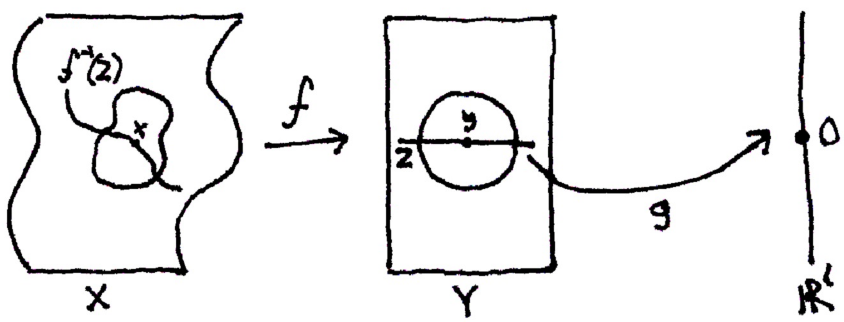
*Proof.* Whether $f^{-1}(Z)$ is a submanifold of $X$ is a local question; that is, it is a manifold if and only if every point $x \in f^{-1}(Z)$ has a neighborhood $U$ such that $U \cap f^{-1}(Z)$ is a manifold. Now then, suppose that $f(x) = y \subset Z$. By Proposition 3.7, in a neighborhood of $y$, we may describe $Z$ as the zero set of a collection of independent functions, $g_1, \ldots, g_l$ where $l$ is the codimension of $Z$ in $Y$. Thus, we can also use these functions in a neighborhood of $x$ to describe $f^{-1}(Z)$ as the zero set of the functions $g_1 \circ f, \dots, g_l \circ f$. Define $g = (g_1, \dots, g_l)$ defined in a neighborhood of $y$. We now will consider the map $g \circ f : W \to \mathbb{R}^l$. By the chain rule, the derivative of $g \circ f$ is given by:

$$d(g \circ f)_x = dg_y \circ df_x$$

It is here that we invoke the transversality of $f$ with respect to $Z \subset Y$. By transversality, Image$(df_x) + T_yZ = T_yY$. Thus, since $dg_y : T_yY \to \mathbb{R}^l$ is surjective and, by Lemma 3.8 has a kernel equal to $T_yZ$, and since Image$(df_x)$ and $T_yZ$ span $T_yY$, this implies that $d(g \circ f)_x$ is also surjective. Thus, for any $x \in f^{-1}(Z) = (g \circ f)^{-1}(0)$, we have shown that $d(g \circ f)_x$ is surjective; by definition, 0 is a regular value for $g \circ f$, and thus $f^{-1}(Z) = (g \circ f)^{-1}(0)$ is a manifold.

Furthermore, we have locally described $f^{-1}(Z)$ as the zero set of a collection of independent functions $g_1 \circ f, \dots, g_l \circ f$; thus the codimension of $f^{-1}(Z)$ is $l$, equal to what we specified as the codimension of $Z$ in $Y$. □

*Figure 3*




**Definition 4.3.** By applying the definition of transversality to the case of inclusion maps, we may extend the language of our new notion. Consider the inclusion map $i$ of one submanifold $X \subset Y$ with another, $Z \subset Y$. When we write $x \in i^{-1}(Z)$, this simply means that $x \in X \cap Z$. Additionally, $di_x : T_xX \to T_xY$ is the inclusion map of $T_xX$ into $T_xY$. Thus $i \pitchfork Z$ if and only if, for every $x \in X \cap Z$,

$$T_xX + T_xZ = T_xY.$$

When such a case is true, we add to our definition of transversality and say that $X$ is transversal to $Z$, abbrevieated $X \pitchfork Z$. Notice that this relation is symmetric, so that $X \pitchfork Z$ is the same as $Z \pitchfork X$.

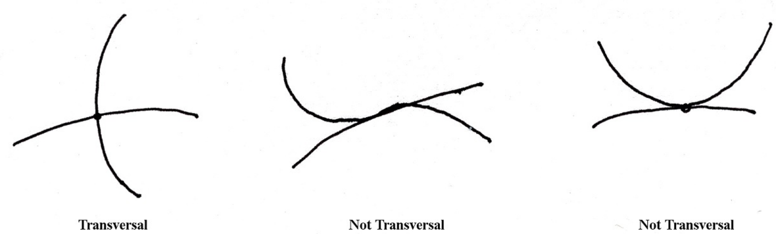
**Theorem 4.4.** *The intersection of two transversal submanifolds of $Y$ is again a submanifold. Furthermore,*

$$\text{codim}(X \cap Z) = \text{codim}X + \text{codim}Z.$$

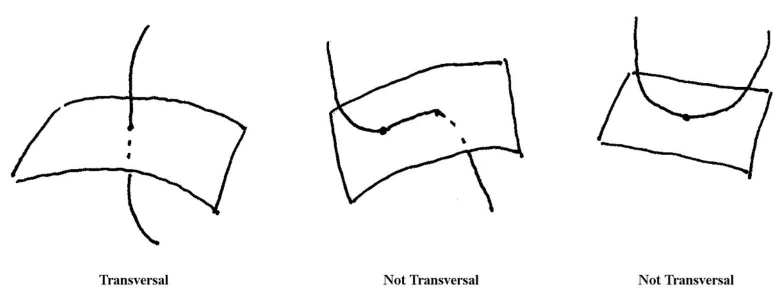
*Proof.* Pick a point $x \in X \cap Z$. Then around $x$, the submanifold $X$ is cut out of $Y$ by $l = \text{codim}X$ independent functions. Likewise, around $x$, the submanifold $Z$ is cut out of $Y$ by $k = \text{codim}Z$ independent functions. Taken together, these $l + k$ functions are independent due to the transversality of $X$ and $Z$. Thus, $l + k = \text{codim}X + \text{codim}Z$ is precisely equal to $\text{codim}(X \cap Z)$. □

*Figure 4*

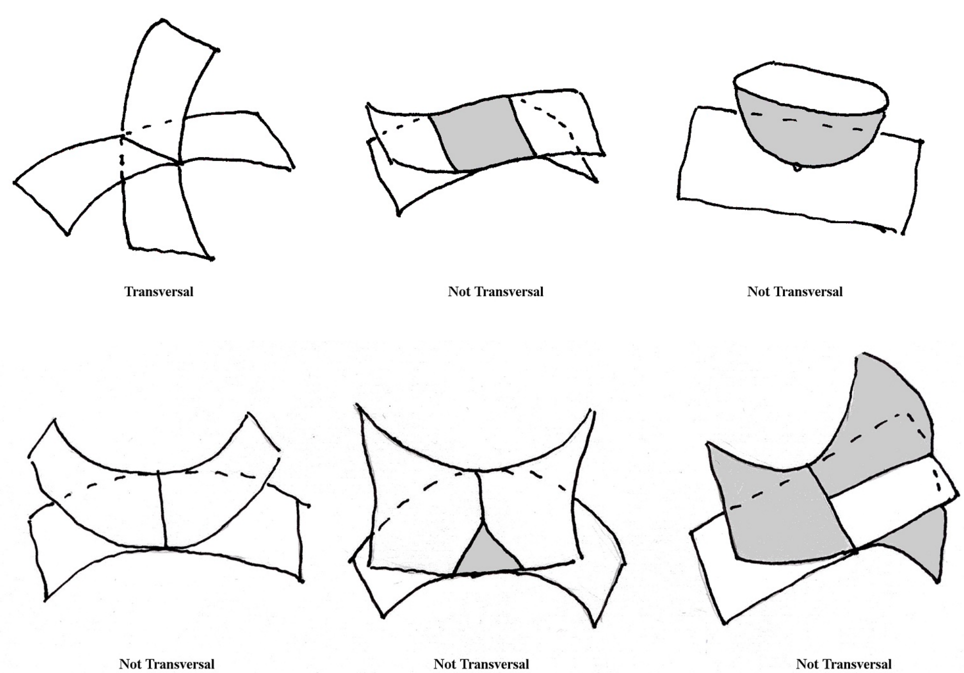
*Curves in* $\mathbb{R}^2$.




*Curves and Surfaces in* $\mathbb{R}^3$.




*Surfaces in* $\mathbb{R}^3$.



**Lemma 4.5.** *Let $f : X \to Y$ be a map transversal to a submanifold $Z$ in $Y$, and let $W = f^{-1}(Z)$ be the resulting submanifold of $X$. Furthermore, let $x \in W$, and let $f(x) = y \in Z$. Then $T_xW$ is the preimage of $T_yZ$ under the linear map $df_x : T_xX \to T_yY$. In other words, the tangent space to the preimage of $Z$ is the preimage of the tangent space of $Z$.*

*Proof.* The begining of this proof utilizes the tools used to prove Theorem 4.2. Again, by Proposition 3.7, in a neighborhood of $y$, we may describe $Z$ as the zero set of a collection of independent functions, $g_1, \ldots, g_l$ where $l$ is the codimension of $Z$ in $Y$. Define $g = (g_1, \ldots, g_l)$ defined in a neighborhood of $y$. Now, by Lemma 3.8, because 0 is a regular value for $g$, the kernel of $dg_y$ is equal to $T_y g^{-1}(0) = T_yZ$. Thus, again, we can also use these functions in a neighborhood of $x$ to describe $f^{-1}(Z)$ as the zero set of the functions $g_1 \circ f, \ldots, g_l \circ f$. We now will consider the map $g \circ f : W \to \mathbb{R}^l$. Now, reapply Lemma 3.8; clearly transversality implies that 0 is a regular value for $g \circ f$, and thus the kernel of $d(g \circ f)_x$ is equal to $T_x(g \circ f)^{-1}(0) = T_xW$. By the chain rule, the derivative of $g \circ f$ is given by:

$$d(g \circ f)_x = dg_y \circ df_x$$

Thus the kernel of $d(g \circ f)_x$ is also the kernel for $dg_y \circ df_x$. Now, recall that the kernel for $dg_y$ is $T_yZ$, and therefore, $df_x(\ker d(g \circ f)_x) = df_x(T_xW) \subseteq T_yZ$. Furthermore, we must have $df_x(T_xX \backslash T_xW) \cap T_yZ = \emptyset$, else we would contradict that $T_xW$ is the kernel of $d(g \circ f)_x$. Thus, $df_x^{-1}(T_yZ) = T_xW$. (Note that this is not the same as saying that $df_x(T_xW) = T_yZ$; however, it does at least tell us that $df_x(T_xW) \subseteq T_yW$.) $\square$

*Remark* 4.6. If we apply this result to the case of transversal submanifolds $X$ and $Z$ of $Y$, we immediately obtain that if $x \in X \cap Z$, then

$$T_x(X \cap Z) = T_xX \cap T_xZ$$

In other words, the tangent space to the intersection of two transversal submanifolds is the intersection of the tangent spaces of two transversal submanifolds.

**Lemma 4.7.** *Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be a sequence of smooth maps of manifolds, and assume that $g$ is transversal to the submanifold $W$ of $Z$. Then $f \pitchfork g^{-1}(W)$ if and only if $g \circ f \pitchfork W$.*

*Proof.* To begin with, let $x \in f^{-1}(g^{-1}(W))$, $f(x) = y \in g^{-1}(W)$, and $(g \circ f)(x) = g(y) = z \in W$. By assumption, because $g \pitchfork W$, we have:

$$T_zZ = dg_y(T_yY) + T_zW.$$

First, suppose that $f \pitchfork g^{-1}(W)$, so that:

$$T_yY = df_x(T_xX) + T_yg^{-1}(W).$$

Thus, we may substitute for $T_yY$:

$$T_zZ = dg_y(df_x(T_xX) + T_yg^{-1}(W)) + T_zW$$
$$T_zZ = dg_y(df_x(T_xX)) + dg_y(T_yg^{-1}(W)) + T_zW.$$

By Lemma 4.6, since $g \pitchfork W$, we know that $dg_y(T_yg^{-1}(W)) \subseteq T_zW$. Thus, $dg_y(T_yg^{-1}(W)) + T_zW = T_zW$, and so we have:

$$T_zZ = dg_y(df_x(T_xX)) + T_zW$$
$$T_zZ = d(g \circ f)_x(T_xX) + T_zW.$$

Which proves that $g \circ f \pitchfork W$.

Now, instead suppose that $g \circ f \pitchfork W$, so that:

$$T_zZ = d(g \circ f)_x(T_xX) + T_zW$$
$$T_zZ = dg_y(df_x(T_xX)) + T_zW.$$

We can then substitute in for $T_yY$ and apply $(dg_y)^{-1}$ to both sides:

$$dg_y(T_yY) + T_zW = dg_y(df_x(T_xX)) + T_zW$$
$$(dg_y)^{-1}(dg_y(T_yY) + T_zW) = (dg_y)^{-1}(dg_y(df_x(T_xX)) + T_zW)$$
$$(dg_y)^{-1}(dg_y(T_yY)) + (dg_y)^{-1}(T_zW) = (dg_y)^{-1}(dg_y(df_x(T_xX))) + (dg_y)^{-1}(T_zW)$$
$$T_yY + (dg_y)^{-1}(T_zW) = df_x(T_xX) + (dg_y)^{-1}(T_zW).$$

By Lemma 4.6, we know that $(dg_y)^{-1}(T_zW) = T_yg^{-1}(W)$. And, since $(dg_y)^{-1}(T_zW) \subseteq T_yY$, we have $T_yY + (dg_y)^{-1}(T_zW) = T_yY$. Thus, we have:

$$T_yY = df_x(T_xX) + T_yg^{-1}(W).$$

Which proves that $f \pitchfork g^{-1}(W)$. □

## 5. The Transversality Theorem and The Extension Theorem

**Definition 5.1.** Let $I$ be the unit interval, $[0, 1] \subset \mathbb{R}$, and let $f_0 : X \to Y$ and $f_1 : X \to Y$. If there exists a smooth map $F : X \times I \to Y$ such that $F(x, 0) = f_0(x)$ and $F(x, 1) = f_1(x)$ then we say that $F$ is a **homotopy** and that $f_0$ and $f_1$ are **homotopic**, abbreviated $f_0$ $f_1$. We also define $f_t : X \to Y$ by $f_t(x) = F(x, t)$.

**Proposition 5.2.** *Homotopy is an equivalence relation.*

**Definition 5.3.** Suppose a map $f_0 : X \to Y$ possesses a specified property, and $f_t : X \to Y$ is a homotopy of $f_0$. If there exists an $\epsilon > 0$ such that if $t < \epsilon$ then $f_t$ also possesses the specified property, we say that the specified property is **stable**.

**Proposition 5.4.** *(The Stability Theorem) The following properties of smooth maps from a compact manifold $X$ into a manifold $Y$ are stable:*

*(1) local diffeomorphisms*
*(2) immersions*
*(3) submersions*
*(4) maps transversal to a specific submanifold $Z \subset Y$*
*(5) embeddings*
*(6) diffeomorphisms*

**Theorem 5.5.** *(Transversality Theorem) Let $F : X \times S \to Y$ be a smooth map between manifolds, let $Z$ be a submanifold of $Y$, and suppose that out of all manifolds and submanifolds mentioned, only $X$ has a boundary. If both $F \pitchfork Z$ and $\partial F \pitchfork Z$, then for almost every $s \in S$, both $f_s \pitchfork Z$ and $\partial f_s \pitchfork Z$.*

*Proof.* The preimage $F^{-1}(Z) = W$ is a submanifold of $X \times S$ with boundary $\partial W = W \cap \partial(X \times S)$. Let $\pi : X \times S \to S$ be the regular projection map.

*(1) Whenever $s \in S$ is a regular value for $\pi|_W$, then $f_s \pitchfork Z$.*
Let $f_s(x) = z \in Z$, and thus $F(x, s)$ is also equal to $z$. Now, since $F \pitchfork Z$:

$$T_zY = dF_{(x,s)}(T_{(x,s)}(X \times S) + T_zZ$$

Thus, for any vector $v \in T_zY$ there is a vector $u \in T_{(x,s)}(X \times S)$ such that:

$$dF_{(x,s)}(u) - v \in T_zZ$$

Now, $T_{(x,s)}(X \times S) = T_xX \times T_sS$, so that $u = (a, b)$ for vectors $a \in T_xX$ and $b \in T_sS$. The derivative of the projection map is:

$$d\pi_{(x,s)} : T_xX \times T_sS \to T_sS.$$

Cleary, as $d\pi_{(x,s)}$ is also a projection map it must map $T_{(x,s)}W$ onto $T_sS$. Therefore there must exist a vector $(\alpha, b) \in T_{(x,s)}W$. Since $F : W \to Z$, we have $dF_{(s,x)}(\alpha, b) \in T_zZ$. Let $\nu = a - \alpha \in T_xX$. Thus:

$$df_s(\nu) - v = dF_{(x,s)}[(a, b) - (\alpha, b)] - v = [dF_{(x,s)}(u) - v] - dF_s(\alpha, b).$$

Both $dF_{(x,s)}(u) - v$ and $dF_s(\alpha, b) \in T_zZ$. This implies that:

$$T_zY = df_s(T_xX) + T_zZ.$$

And thus $f_s \pitchfork Z$.

*(2) Whenever $s \in S$ is a regular value for $\partial\pi|_{\partial W}$, then $\partial f_s \pitchfork Z$.*
The logic for this is similar to that for the previous part.

*(3) By Sard's Theorem, almost every $s \in S$ is a regular value for both maps, which proves the theorem.*

$\square$

**Proposition 5.6.** *Let $U$ be an open neighborhood of the closed set $C \subset X$. There exists a smooth function $\gamma : X \to [0, 1]$ which equals 1 outside of $U$ and 0 on a neighborhood of $C$.*

**Theorem 5.7.** *(Extension Theorem) Let $C$ be a closed subset of $X$ and let $Z$ be a closed submanifold of $Y$, where both $Z$ and $Y$ are without boundary. Suppose that $f : X \to Y$ be a smooth map where $f|_C \pitchfork Z$ and $\partial f|_{C\cap\partial X} \pitchfork Z$. Then there exists a smooth map $g : X \to Y$ homotopic to $f$, where $g = f$ on a neighborhood of $C$, and with $g \pitchfork Z$ and $\partial g \pitchfork Z$.*

*Proof.* To begin with, let $\gamma$ be the function from the proposition, and define $\tau = \gamma^2$. Then, $d\tau_x = 2\gamma(x)d\gamma_x$, and $d\tau_x = 0$ whenever $\tau = 0$. Furthermore, we define $G : X \times S \to Y$ by $G(x, s) = F(x, \tau(x)s)$.

*(1) $f \pitchfork Z$ on a neighborhood of $C$.*
First, if $x \in C$ but $x \notin f^{-1}(Z)$, clearly this is true, as $Z$ is closed so that $X - f^{-1}(Z)$ is a neighborhood of $x$ for which $f \pitchfork Z$. If $x \in f^{-1}(Z)$ as well, then take a neighborhood of $f(x)$, call it $W$, and consider the submersion $\phi : W \to \mathbb{R}^k$ where $\phi \circ f$ is regular at a point $w \in f^{-1}(Z \cap W)$ when $f \pitchfork Z$ at $w$. Thus, $\phi \circ f$ is regular on a neighborhood of $x$ and so $f \pitchfork Z$ on a neighborhood of every point $x \in C$ and thus also for a neighborhood of $C$.

*(2) $G \pitchfork Z$.*
Let $(x, s \in G^{-1}(Z)$ and suppose $\tau(x) \neq 0$. Consider the composition of the diffeomorphism $\alpha : S \to S$ defined by $\alpha(r) = \tau(x)r$, with the submersion $\beta : S \to Y$ defined by $\beta(r) = F(x, r)$ - that is, consider $\gamma = \beta \circ \alpha : S \to Y$ for which

$\gamma(r) = F(x, \tau(x)r) = G(x, r)$. As a result of this construction, it is clear that $G$ is regular at $(x, s)$ so that $G \pitchfork Z$ at $(x, s)$.

On the other hand, let us consider $\tau(x) = 0$. Let $H : X \times S \to X \times S$ be defined as $H(x, s) = (x, \tau(x)s)$. We will now caculate $dG_{(x,s)} = d(F \circ H)_{(x,s)}$ at any element $(u, v) \in T_xX \times T_sS = T_xX \times \mathbb{R}^m$:

$$dG_{(x,s)}(u, v) = dF_{(x,\tau(x)s)} \circ dH_{(x,s)}(u, v) = dF_{(x,\tau(x)s)}(u, \tau(x)v + d\tau_x(u)s).$$

And since $\tau(x) = d\tau_x(u) = 0$, we have:

$$dG_{(x,s)}(u, v) = dF_{(x,0)}(u, 0)$$

Thus, $F$ reduces to $f$, as they are equal on $X \times \{0\}$, so that:

$$dG_{(x,s)}(u, v) = df_x(u)$$

Since $\tau(x) = 0$, we must have $x \in U$ and so $f \pitchfork Z$ at $x$, and since $dG_{(x,s)}(u, v) = df_x(u)$, we have $G \pitchfork Z$ at $(x, s)$.

*(3) $\partial G \pitchfork Z$.*

The logic for this is similar to that for the previous part.

*(4) There exists a smooth map $g : X \to Y$ homotopic to $f$, where $g = f$ on a neighborhood of $C$, and with $g \pitchfork Z$ and $\partial g \pitchfork Z$.*

By the Transversality Theorem, there exists an $s \in S$ such that $g(x) = G(x, s)$ and $g \pitchfork Z$ and $\partial g \pitchfork Z$. Clearly, $g$ is homotopic to $f$. And lastly, if $x$ is a point in a neighborhood of $C$ for which $\tau = 0$, we have $g(x) = G(x, s) = F(x, 0) = f(x)$.

□

## 6. The Degree Modulo 2 of a Mapping

**Definition 6.1.** Let $X$ and $Y$ be two submanifolds inside $Z$. Then, if $\dim X + \dim Y = \dim Z$ we say that they have **complementary dimension**. Note that if $X \pitchfork Y$, this implies that $X \cap Y$ is a manifold of zero dimension, and further, if $X$ and $Y$ are closed and if at least one of them is compact, then $X \cap Y$ is a finite collection of points. We define $\#(\mathbf{X} \cap \mathbf{Y})$ to be the number of points in $X \cap Y$.

**Definition 6.2.** Let $X$ be a compact manifold, and let $f : X \to Y$ be transversal to a closed submanifold $Z \subset Y$. Suppose also that $\dim X + \dim Z = \dim Y$. This implies that $f^{-1}(Z)$ is a closed zero-dimensional submanifold of $X$, and thus, is a finite collection of points. We define the **Intersection Number**, abbreviated $I_2(f, Z)$, to be the number of points in $f^{-1}(Z)$ modulo 2. In the case of an arbitrary smooth map $g : X \to Y$, not necessarily transversal to $Z$, simply define $I_2(g, Z) = I_2(f, Z)$, where $f$ is homotopic to $g$ and transversal to $Z$.

**Theorem 6.3.** *If $f_0, f_1 : X \to Y$ are homotopic and both are transversal to a submanifold $Z$ of $Y$, then $I_2(f_0, Z) = I_2(f_1, Z)$.*

*Proof.* By the Extension Theorem, if we take let $F : X \times I \to Y$ be a homotopy of $f_0$ and $f_1$ we may assume that $F \pitchfork Z$. Since $\partial(X \times I) = X \times \{0\} \cup X \times \{1\}$, the restriction $\partial F : \partial(X \times I) \to Y$ reduces to $f_0$ on $X \times \{0\}$ and $f_1$ on $X \times \{1\}$; thus, $\partial F \pitchfork Z$. Furthermore, since $\dim X + \dim Z = \dim Y$, the codimesion of $Z$ is $\dim X$, so that the codimension of $F^{-1}(Z)$ is also $\dim X$. Now, $\dim(X \times I) = \dim X + 1$, so the dimesion of $F^{-1}(Z)$ is simply 1. If we examine the boundary of $F^{-1}(Z)$, we find:

$$\partial F^{-1}(Z) = F^{-1}(Z) \cap \partial(X \times I) = f_0^{-1}(Z) \times \{0\} \cup f_1^{-1}(Z) \times \{1\}.$$

Since $F^{-1}(Z)$ is a one dimensional manifold, it must have an even number of boundary points; therefore, $f_0^{-1}(Z) \times \{0\} \cup f_1^{-1}(Z) \times \{1\}$ must be even, so $\# f_0^{-1}(Z) = \# f_1^{-1}(Z) \bmod 2$. □

**Corollary 6.4.** *If $g_0, g_1 : X \to Y$ are arbitrary homotopic maps, then we have $I_2(g_0, Z) = I_2(g_1, Z)$.*

*Proof.* By Definiton 6.2, $I_2(g_0, Z) = I_2(f_0, Z)$ where $f_0$ is homotopic to $g_0$ and transversal to $Z$, and likewise $I_2(g_1, Z) = I_2(f_1, Z)$ where $f_1$ is homotopic to $g_1$ and transversal to $Z$. Then, because homotopy is an equivalence relation, $f_0$ is homotopic to $f_1$, and by Theorem 6.3, $I_2(f_0, Z) = I_2(f_1, Z)$. This proves the corollary. □

**Theorem 6.5.** *Let $f : X \to Y$ is a smooth map of a compact manifold $X$ into a connected manifold $Y$. If $\dim X = \dim Y$, then $I_2(f, \{y\})$ is the same for all regular values $y \in Y$.*

*Proof.* By the Stack of Records Theorem, there is a neighborhood $V$ of $y$ such that $f^{-1}$ is a disjoint union $U_1 \cup \ldots \cup U_k$, with each $U_i$ mapped diffeomorphically onto $V$. Then, for all points $z \in V$, we have $I_2(f, \{z\}) = k \bmod 2$. Therefore, the function defined as $y \mapsto I_2(f, \{y\})$ is locally constant, and since $Y$ is connected, must be globally constant. □

**Definition 6.6.** Suppose that $f : X \to Y$ is a smooth map of a compact manifold $X$ into a connected manifold $Y$, where $\dim X = \dim Y$. By the previous Theorem, $I_2(f, \{y\})$ is the same for all regular values $y \in Y$, and thus, by Sard's Theorem, for nearly all points in $Y$. We define this number to be the **mod 2 degree of f**, abbreviated $\deg_2(f)$.

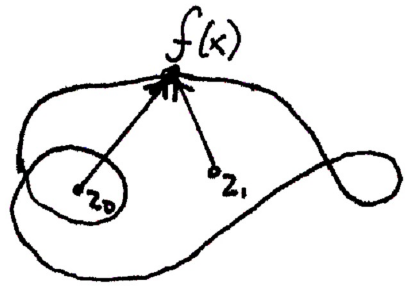
*Remark* 6.7. Note that since the intersection number is the same for homotopic maps, and since $\deg_2$ is defined as an intersection number, homotopic maps must have the same mod 2 degree.

**Definition 6.8.** Let $X$ be a compact, connected manifold of dimension $n-1$, and let $f : X \to \mathbb{R}^n$ (in this way, $f$ may very well be the inclusion map of a hypersurface into $\mathbb{R}^n$). Then, for any point $z \in \mathbb{R}^n \backslash f(X)$, we define $u : X \to S^{n-1}$:

$$u(x) = \frac{f(x) - z}{|f(x) - z|}.$$

Thus, from Definition 7.6, we know that $u$ hits nearly every point in $S^{n-1}$ the same number of times modulo 2. Therefore, we define the **mod 2 winding number** of **f** around **z** to be $W_2(f, z) = \deg_2(u)$. (See *Figure 5.*)

*Figure 5*




**Definition 6.9.** We can apply this definition to the inclusion map of a manifold into some ambient space $\mathbb{R}^n$. In this case, for any point $z \in \mathbb{R}^n \backslash X$, the function $u : X \to S^{n-1}$ becomes:

$$u(x) = \frac{x - z}{|x - z|}.$$

We therefore define the **mod 2 winding number** of **X** around **z** to be $W_2(X, z) = \deg_2(u)$.

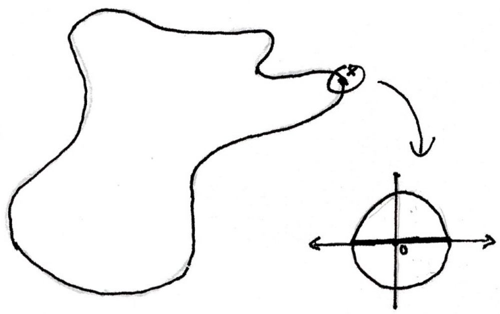
## 7. The Jordan-Brouwer Separation Theorem

**Theorem 7.1.** *(The Jordan-Brouwer Separation Theorem) Any compact, connected hypersurface $X$ in $\mathbb{R}^n$ will divide $\mathbb{R}^n$ into two connected regions; the "outside" $D_0$ and the "inside" $D_1$. Furthermore, $\bar{D}_1$ is itself a compact manifold with boundary $\partial \bar{D}_1 = X$.*

*Proof.* To begin with, we must ponder how we may locally identify $X$ with a hyperplane in $\mathbb{R}^n$.

Consider the inclusion map $i : X \to \mathbb{R}^n$. Now, $\dim X < n$, so $i$ is by definition an immersion at any point $x \in X$, and so by the Local Immersion Theorem, there exist local coordinates $\{x_1, \ldots, x_{n-1}\}$ around $x$ such that $i(x_1, \ldots, x_{n-1}) = (x_1, \ldots, x_{n-1}, 0)$. For convenience, we may translate these local coordinates so that $x = (0, \ldots, 0)$. Thus, in a neighborhood of $x$, our manifold $X$ is identified with the hyperplane $H = \{\alpha_1, \ldots, \alpha_{n-1}, 0\}$, and thus must divide our neighborhood of $x$ into two open regions: $H^+ = \{(\alpha_1, \ldots, \alpha_n) | \alpha_n > 0\}$ and $H^- = \{(\alpha_1, \ldots, \alpha_n) | \alpha_n < 0\}$. (See *Figure 6.*)

*Figure 6*



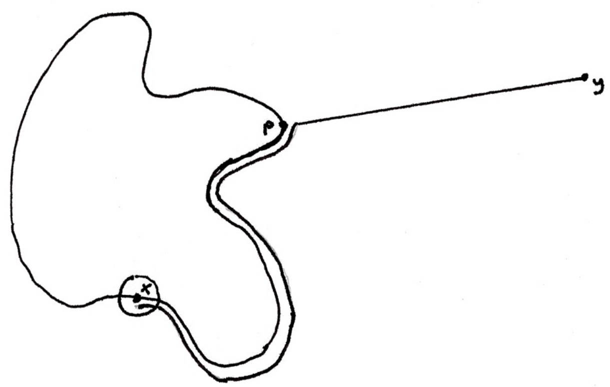
*(1) Pick a point $y \in \mathbb{R}^n \backslash X$. Then for any point $x \in X$ there is a point in the neighborhood of $x$ in $\mathbb{R}^n$ which may be connected to $y$ by a curve that does not intersect $X$. (See Figure 7.)*

To begin with, fix $y$, and rather than picking any arbitrary $x \in X$, consider the point closest to $y$ in $X$, call it $\rho$ (there must be at least one, as both $X$ and $y$ are compact; if there is more than one, just pick one). Clearly the straight line segment joining $y$ with $\rho$ must have a nonempty intersection with a neighborhood of $y$ (else $y$ would be a boundary point for its own neighborhood).

We now use this to establish a nonintersecting curve to any point $x \in X$. Since the manifold $X$ is connected, and all manifolds are locally path connected, we know that $X$ must be path connected; therefore, we can find a curve connecting our arbitrary point $x$ to $\rho$. Now all we need to do is push the curve off of $X$. To do this, we begin at $\rho$. First note the vector from $\rho$ to $y$, and call it $\vec{w}$. At $\rho$ we may locally identify $X$ with a hyperplane. which must have two normal vectors to $\rho$ pointing in opposite directions. Pick whichever normal vector points towards the same half-space which $\vec{w}$ points towards. Use this normal vector to displace the curve an arbitrary distance $\epsilon$ along the normal vector. In a neighborhood of $\rho$, since $X$ looks like a hyperplane, the curve looks like a line segment, and if we continue this procedure along the curve, locally always directing each displacement towards the same half-space, it will be the same as simply translating the line segment upwards. We may locally continue this procedure along the entirety of the path connecting $\rho$ with $x$

But can we be assured that this is actually a curve? While it is clear that it will not intersect $X$, it may still intersect itself. Provided that $\dim \mathbb{R}^n > 2$, even if the curve does intersect itself, it can not intersect itself transversally. Hence, by the Stability Theorem, we can simply displace the curve a sufficiently small amount to rid itself of any self-intersection. In the case that $\dim \mathbb{R} = 2$, a nontransversal intersection may be dealt with the same way. On the other hand, any sort of transversal intersection would imply deeper issues, such as intersection of $X$ with itself, contradicting its status as a manifold. The case that $\dim \mathbb{R}^n = 1$ is trivial. Thus, we may connect a point in a neighborhood of $x$ back to a neighborhood of $\rho$, and finally back to $y$, and since this is $\mathbb{R}^n$, there must exist a curve which connects a point in a neighborhood of $x$ back to $y$.

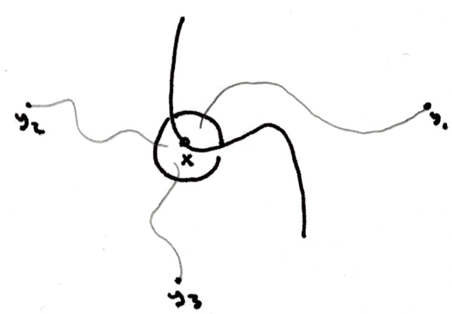
*Figure 7*




*(2) $\mathbb{R}^n \backslash X$ has, at most, two components. (See Figure 8.)*

Simply fix a point $x \in X$ and take any three points $y_1, y_2, y_3 \in \mathbb{R}^n \backslash X$. Then by the previous part, these three points may be connected to a point in a neighborhood of $x$. But as noted before, $X$ must divide a neighborhood of $x$ into two components. Therefore, two of the points $y_1, y_2, y_3$ must be connected to the same neighborhood component of $X$, and thus, these two points must belong to the same component of $\mathbb{R}^n \backslash X$.

*Figure 8*




*(3) If two points, $y_0$ and $y_1$ belong to the same component of $\mathbb{R}^n \backslash X$, then the winding number of $X$ about both $y_0$ and $y_1$ must be equal.*

Since $y_0$ and $y_1$ are part of the same connected component of $\mathbb{R}^n \backslash X$, they may be joined together by a smooth curve $\gamma : [0, 1] \to \mathbb{R}^n \backslash X$ with $\gamma(0) = y_0$ and $\gamma(1) = y_1$, and which does not intersect $X$. We claim that there exists a homotopy $U : X \times I \to S^{n-1}$ between the associated direction maps $u_0$ and $u_1$, with $U$ defined as:

$$U(x,t) = \frac{x - \gamma(t)}{|x - \gamma(t)}$$

Now, $U(x, 0) = u_0(x)$ and $U(x, 1) = u_1(x)$, so all that remains is to check that $U$ is smooth, easily verifiable through differentiation. Thus, $u_o$ and $u_1$ are homotopic, and by Corollary 6.4, they must therefore have the same intersection number. But $\deg_2$ is defined as an intersection number, so $u_0$ and $u_1$ have the same degree modulo 2, and thus, $y_0$ and $y_1$ have the same winding number.

*(4) Given a point $y \in \mathbb{R}^n \backslash X$ and a direction vector $\vec{v} \in S^{n-1}$, consider the ray emanating from $z$ in the direction of $\vec{v}$:*

$$r = [y + \vec{v}t | t \geq 0].$$

*This ray $r$ is transversal to $X$ if and only if $\vec{v}$ is a regular value for the direction map $u : X \rightarrow S^{n-1}$.*

Define a function $g : \mathbb{R} \backslash \{y\} \rightarrow S^{n-1}$ by:

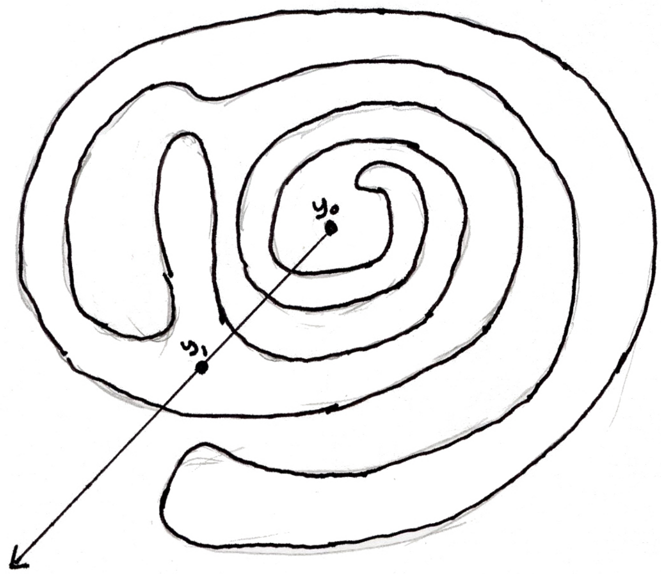
$$g(x) = \frac{x - y}{|x - y|}.$$

This definition makes it clear that, in fact, $u$ is simply $g$ composed with the inclusion map; $u = g \circ i$. Thus we have a sequence of smooth maps of manifolds, $X \xrightarrow{i} \mathbb{R}^n \backslash \{y\} \xrightarrow{g} S^{n-1}$ and with a composition $g \circ i = u$. Differentiation shows that $\vec{v}$ is clearly a regular value for $g$, so that we may write $g \pitchfork \vec{v}$. And now, we may invoke Lemma 4.8. By the Lemma, since $g \pitchfork \vec{v}$, we have $i \pitchfork g^{-1}(\vec{v})$ if and only if $g \circ i \pitchfork \vec{v}$. Rewritten, since $g^{-1}(\vec{v}) = r$, and by how we have defined transversal inclusion maps, $X \pitchfork r$ if and only if $u \pitchfork \vec{v}$, that is, $\vec{v}$ is a regular value for $u$. What is more, by Sard's Theorem, since nearly every $\vec{v}$ will be a regular value for $u$, nearly every ray from $z$ will intersect $X$ transversally.

*(5) Let $r$ be a ray emanating from a point $y_0 \in \mathbb{R}^n \backslash X$ which intersects $X$ transversally in a nonempty (necessarily finite) set. Suppose that $y_1$ is another point on $r$ (but not on $X$), and let $l$ be the number of times $r$ intersects $X$ between $y_0$ and $y_1$. Then $W_2(X, y_0) = W_2(X, y_1) + l$ mod 2. (See Figure 9.)*

We have just shown that a ray is transversal to $X$ if and only if the normalized vector of that ray is a regular value for $u$. It follows then that $\hat{r}$ is a regular value for both direction maps $u_0$ and $u_1$ associated with $y_0$ and $y_1$. Now, $\#u_0^{-1}(\hat{r}) = \#u_1^{-1}(\hat{r}) + l$, and since $\deg_2(u) = \#u^{-1}(\vec{v})$ mod 2, where $\vec{v}$ is a regular value, it follows that $\deg_2(u_0) = \deg_2(u_1) + l$ mod 2. Thus, $W_2(X, y_0) = W_2(X, y_1) + l$ mod 2.

*Figure 9*




$W_2(X, y_0) = W_2(X, y_1) + 3 \ mod \ 2$

*(6) We may now establish that* $\mathbb{R}^n \backslash X$ *has precisely two components:*

$$D_0 = \{y | W_2(X, y) = 0\} \text{ and } D_1 = \{y | W_2(X, y) = 1\}.$$

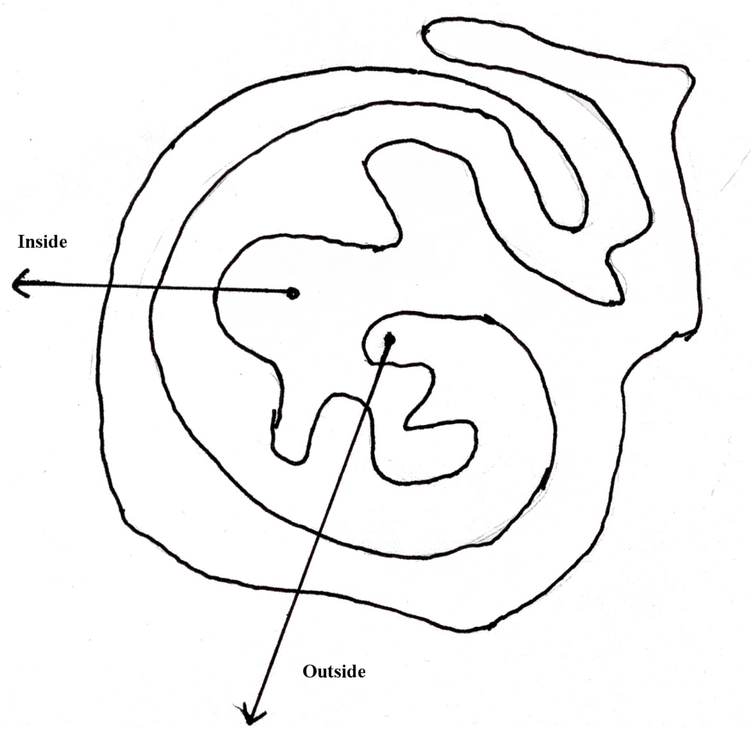
Since we have already shown that if two points are part of the same component, then there winding numbers must be equal, all we need to do now is show that $D_0$ and $D_1$ are nonempty. To do this, pick a point $x \in X$ and take a neighborhood $U \subset \mathbb{R}^n$ of $x$. As noted before, we may take local coordinates for this neighborhood so that $X$ is identified with the hyperplane $H = \{\alpha_1, \dots \alpha_{n-1}, 0\}$ and so that $x = (0, \dots, 0)$. Now, let $y_+$ be a point in $U$ whose $n^{th}$ local coordinate is greater than 0, and now, find a point $y_- \in U$ whose $n^{th}$ local coordinate is less than 0 such that the ray $r$ which emanates from $y_+$ in the direction of $y_-$ is transversal to $X$. That we can find such a point $y_-$ such that $r$ is transversal to $X$ is a result of Sard's Theorem, for if $r \pitchfork X$ then by Part 4, $\hat{r}$ is a regular value for $u$, and as we know, nearly every point in $S^{n-1}$ is a regular value for $u$, so that nearly every ray is transversal to $X$.

Thus, we have two points $y_+$ and $y_-$, and a transversal ray which passes through them both. Clearly, this ray must intersect $X$ only once between $y_+$ and $y_-$ (since between $y_+$ and $y_-$, we may identify $X$ with a hyperplane). And therefore, by Part 5, $W_2(X, y_+) = W_2(X, y_-) + 1$ mod 2, and so $D_0$ and $D_1$ are nonempty.

*(7) If the magnitude of a point* $z \in \mathbb{R}^n \backslash X$ *is very large, then* $W_2(X, z) = 0$.

Since $X$ is compact, by making $z$ very large, $u(x) = \frac{x-z}{|x-z} \approx \frac{-z}{|z|}$. Thus, $u(X)$ must lie in a small neighborhood $U$ of $\frac{-z}{|z|}$. Thus, $S^{n-1} \backslash U$ is hit by $u$ *zero* times, and since this is not a set of measure zero and since $\deg_2$ is invariant, we must have $\deg_2(u) = 0$ and likewise $W_2(X, z) = 0$. Furthermore, this gives an intuitive grasp that $D_0$ should be considered the "outside" of $X$. It follows from this that given a point $y \in \mathbb{R}^n \backslash X$, and a ray $r$ emanating from $y$ and transversal to $X$, then $y$ is "outside" $X$ if $r$ intersects $X$ in an even number of points, and $y$ is "inside" $X$ if $r$ intersects $X$ in an odd number of points. (See *Figure 10.*)

*Figure 10*




*(8) $\bar{D}_1$ is a compact manifold with boundary $\partial \bar{D}_1 = X$.*

For any point in the interior of $\bar{D}_1$, a neighborhood of that point is an open set in $\mathbb{R}^n$ and is (very obviously) diffeomorphic to an open set in $\mathbb{R}^n$. If a point $x \in \bar{D}_1$ also belongs to $X$, than as described in the beginning, by the Local Immersion Theorem, $X$ is identified with a hyperplane $H$ which divides the neighborhood of $x$ into two regions. As described in Part 9, it is clear that each region will have a winding number of either 1 or 0, and each region is an open set in either $H^+$ or $H^-$. The region with a winding number of 1 is thus diffeomorphic to a corresponding half-space, be it $H^+$ or $H^-$, which shows that the point $x \in H$ is the boundary for $\bar{D}_1$ and furthermore that $\bar{D}_1$ is a manifold with boundary.

□

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