

HAUSDORFF DIMENSION AND ITS APPLICATIONS

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ABSTRACT. The theory of Hausdorff dimension provides a general notion of the size of a set in a metric space. We define Hausdorff measure and dimension, enumerate some techniques for computing Hausdorff dimension, and provide applications to self-similar sets and Brownian motion. Our approach follows that of Stein [4] and Peres [3].

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1. HAUSDORFF MEASURE AND DIMENSION

The theory of Hausdorff measure and dimension was invented in order to provide a notion of size not captured by existing theories, such as that of Lebesgue measure. The idea is measure the size of a set through choosing some α -dependent measure μ which selects sets of dimension α . From the perspective of μ , sets of dimension $< \alpha$ should be ‘small’, i.e. have measure zero, and sets of dimension $> \alpha$ should be ‘large’, i.e. have measure ∞ . Lebesgue measure accomplishes this, but only in \mathbb{R}^d . Moreover, Lebesgue measure can only give an integer value for dimension, and hence misses out on some structure. Hausdorff measure takes the idea of looking at the volume of coverings by rectangles and generalizes it to arbitrary metric spaces and fractional α .

Notation 1.1. We write $|E|$ to denote the diameter of a set E .

Definition 1.2. Let $E \subset X$ be a subset of a metric space. For every $\alpha, \delta \geq 0$, let $\mathcal{H}_\alpha^\delta := \inf \{ \sum_{i=1}^\infty |F_i|^\alpha : E \subset \bigcup F_i, |F_i| < \delta \}$. Then $m_\alpha^*(E) := \sup_{\delta \geq 0} \mathcal{H}_\alpha^\delta(E) = \lim_{\delta \downarrow 0} \mathcal{H}_\alpha^\delta(E)$ is the α -**Hausdorff content** of the set E .

Note that m_α^* is monotonic and countably subadditive, hence defines an outer measure on X . We would like to obtain a measure such that the Borel sets are measurable. Our approach is dictated by the following theorem, which we state without proof.

Definition 1.3. $d(A, B) := \inf \{ d(x, y) : x \in A, y \in B \}$.

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Definition 1.4. An outer measure μ_* is a **metric outer measure** if $\forall A, B \subset X$ with $d(A, B) > 0$, $\mu_*(A \cup B) = \mu_*(A) + \mu_*(B)$.

Theorem 1.5. If μ_* is a metric outer measure, then the Borel sets are μ_* -measurable.

Hence by Caratheodory's theorem μ_* restricted to the Borel sets is a measure. We now show m_α^* is a metric outer measure.

Proposition 1.6. If $d(E_1, E_2) > 0$, then $m_\alpha^*(E_1 \cup E_2) = m_\alpha^*(E_1) + m_\alpha^*(E_2)$.

Proof. We have \leq by subadditivity. To prove \geq , first fix $\epsilon > 0$ with $\epsilon < d(E_1, E_2)$. Given any cover of $E_1 \cup E_2$ by $\{F_i\}$ with $|F_i| < \delta$ for some $\delta < \epsilon$, let $F'_i = E_1 \cap F_i$ and $F''_i = E_2 \cap F_i$. Then $\{F'_i\}$ and $\{F''_i\}$ are covers for E_1 and E_2 and are disjoint. Hence $\sum |F'_i|^\alpha + \sum |F''_i|^\alpha \leq \sum |F_i|^\alpha$. Taking infimums and letting $\delta \rightarrow 0$ we conclude $m_\alpha^*(E_1) + m_\alpha^*(E_2) \leq m_\alpha^*(E_1 \cup E_2)$. \square

Exercise 1.7. For $E \subset X$, define $\mathcal{H}_\alpha^\infty(E) := \inf \{\sum_{i=1}^\infty |F_i|^\alpha : E \subset \bigcup F_i\}$, the **unlimited α -Hausdorff content** of E . Show that $\mathcal{H}_\alpha^\infty(E)$ is an outer measure on X but not necessarily a metric outer measure. This justifies the inclusion of a bound on the diameter of sets in our covering of E .

We write m_α to denote α -Hausdorff measure on X . Seeing the similarity between the construction of Hausdorff and Lebesgue measure we are led to make the following useful observation.

Proposition 1.8. Hausdorff measure is translation and rotation invariant. Moreover, it scales as follows: $\forall \lambda > 0$, $m_\alpha(\lambda E) = \lambda^\alpha m_\alpha(E)$.

Proof. The proposition follows from the corresponding statement for the diameter of a set. \square

The uniqueness of Haar measure up to a scalar then implies (after showing m_d is a Radon measure on \mathbb{R}^d) that there exists a constant c_d dependent only on the dimension such that $c_d m_d = m$. In fact $c_d = v_d/2^d$, where v_d is the measure of the unit ball in \mathbb{R}^d . We can easily prove a weaker form of this result.

Lemma 1.9. $\forall \epsilon, \delta > 0 \exists$ a covering of $E \subset \mathbb{R}^d$ by balls $\{B_i\}$ such that $|B_i| < \delta$, while $\sum m(B_i) \leq m^*(E) + \epsilon$.

Proof. This is an essential consequence of the Vitali covering lemma. \square

Theorem 1.10. If E is a Borel subset of \mathbb{R}^d , then $m_d(E) \approx m(E)$, in the sense that $c_d m_d(E) \leq m(E) \leq 2^d c_d m_d(E)$.

Proof. Find a cover of E as in the lemma. Then $\mathcal{H}_d^\delta(E) \leq \sum |B_i|^d = c_d^{-1} \sum m(B_i) \leq c_d^{-1}(m(E) + \epsilon)$. Letting $\delta, \epsilon \rightarrow 0$, we get $m_d(E) \leq c_d^{-1} m(E)$. For the reverse direction, let $\{F_i\}$ be a covering of E with $\sum |F_i|^d \leq m_d(E) + \epsilon$. Find closed balls B_i so that $B_i \supset F_i$ and $|B_i| = 2|F_i|$. Then $m(E) \leq \sum m(B_i) = c_d \sum |B_i|^d = 2^d c_d \sum |F_i|^d \leq 2^d c_d (m_d(E) + \epsilon)$. Letting $\epsilon \rightarrow 0$, we get $m(E) \leq 2^d c_d m_d(E)$. \square

Remark 1.11. The stronger statement $c_d m_d = m$ follows from the isodiametric inequality $m(E) \leq c_d |E|^d$, which expresses the intuitive fact that among sets of a given diameter the ball has the largest volume.

The advantage of Hausdorff measure over Lebesgue measure is that it isn't defined in relation to the dimension of the embedding space. This allows us to make sense out of the concept of fractional dimension.

Definition 1.12. Let E be a Borel set. Then $\alpha := \sup \{\beta : m_\beta(E) = \infty\} = \inf \{\beta : m_\beta(E) = 0\}$ is the **Hausdorff dimension** of E . If $0 < m_\alpha(E) < \infty$ we say E has **strict Hausdorff dimension** α .

Intuitively this definition expresses a set to be large in relation to sets of lower dimension and small in relation to sets of higher dimension. The following proposition shows that the α in the definition is unique, and so Hausdorff dimension is well-defined.

Proposition 1.13. 1) If $m_\alpha^*(E) < \infty$ and $\beta > \alpha$, then $m_\beta^*(E) = 0$. 2) If $m_\alpha^*(E) > 0$ and $\beta < \alpha$, then $m_\beta^*(E) = \infty$.

Proof. Let $\{F_i\}$ be a covering of E with $|F_i| < \delta$. We have $\sum |F_i|^\beta = \sum |F_i|^{\beta-\alpha} |F_i|^\alpha \leq \delta^{\beta-\alpha} \sum |F_i|^\alpha$, so $H_\delta^\beta(E) \leq \delta^{\beta-\alpha} H_\delta^\alpha(E)$. Letting $\delta \rightarrow 0$ gives the first claim. Interchanging α and β gives the second claim. \square

Exercise 1.14. Find a set without a strict Hausdorff dimension.

At the moment we can compute the Hausdorff dimension of an important class of sets in \mathbb{R}^d : those with positive finite Lebesgue measure. The power of Hausdorff dimension lies in its ability to distinguish between sets of zero Lebesgue measure, which morally should not be of the same ‘size’. For example, the \mathbb{R}^2 Lebesgue measure of planar Brownian motion is zero almost surely, yet planar Brownian motion is neighborhood recurrent. A space-filling curve is the same as a point from the viewpoint of Lebesgue measure! We now proceed to list some techniques for computing the Hausdorff dimension of more general sets.

2. COMPUTING HAUSDORFF DIMENSION

In general obtaining an upper bound for Hausdorff dimension is the easier task; for the posited dimension α , show $H_\alpha^\delta(E) < \infty$ by finding an efficient cover of E for any δ . The infimum in the definition of Hausdorff measure makes finding lower bounds more difficult. The first technique we describe is based on constructing an appropriate function with domain or range a set with known Hausdorff dimension. This allows us to compute upper or lower bounds, respectively.

Definition 2.1. A function $f : (E_1, \rho_1) \rightarrow (E_2, \rho_2)$ between metric spaces is called **γ -Hölder continuous** if $\exists C > 0$ such that $\rho_2(f(x), f(y)) < C \rho_1(x, y)^\gamma \forall x, y \in E_1$.

Proposition 2.2. If $f : (E_1, \rho_1) \rightarrow (E_2, \rho_2)$ is surjective and γ -Hölder continuous with constant $C > 0$, then $\forall \alpha \geq 0$, $m_{\alpha/\gamma}(E_2) \leq C^{\alpha/\gamma} m_\alpha(E_1)$, and hence $\dim(E_2) \leq \frac{1}{\gamma} \dim(E_1)$.

Proof. Let $\{F_i\}$ be a covering of E_1 . Then $\{f(E_1 \cap F_i)\}$ covers E_2 , and $|f(E_1 \cap F_i)| < C |F_i|^\gamma$. Hence $\sum |f(E_1 \cap F_i)|^{\alpha/\gamma} \leq C^{\alpha/\gamma} \sum |F_i|^\alpha$, and the statement follows. \square

The next technique, the *mass distribution principle*, is similar in flavor to the above. Instead of a function, we construct an appropriate measure. γ -Hölder continuity is replaced by a suitable bound on the measure of sets with respect to their diameter.

Definition 2.3. A Borel measure μ is a **mass distribution** on a metric space E if $0 < \mu(E) < \infty$.

Theorem 2.4. (Mass distribution principle) Suppose E is a metric space and $\alpha \geq 0$. If there is a mass distribution μ on E and constants $C, \delta > 0$ such that $\mu(V) \leq C|V|^\alpha$ for all closed sets V with $|V| \leq \delta$, then $\mathcal{H}_\alpha^\delta(E) \geq \mu(E)/C > 0$, and hence $\dim E \geq \alpha$.

Proof. Let $\{F_i\}$ be a covering of E . WLOG we can take the F_i to be closed since $|\overline{F_i}| = |F_i|$. We have $0 < \mu(E) \leq \mu(\bigcup F_i) \leq \sum \mu(F_i) \leq C \sum |F_i|^\alpha$, and the statement follows. \square

The mass distribution principle requires to spread a positive finite mass over a set such that local concentration is bounded from above. The next technique, the *energy method*, is essentially a computational means of measuring the local concentration of the mass.

Definition 2.5. Let μ be a mass distribution on a metric space E and $\alpha \geq 0$. The α -**potential** of a point $x \in E$ with respect to μ is defined as $\phi_\alpha(x) := \int \frac{d\mu(y)}{\rho(x,y)^\alpha}$. The α -**energy** of μ is $I_\alpha(\mu) := \int \phi_\alpha(x) d\mu(x) = \iint \frac{d\mu(x)d\mu(y)}{\rho(x,y)^\alpha}$.

The idea of the energy method is that mass distributions with $I_\alpha(\mu) < \infty$ spread the mass so that at each point the concentration is sufficiently small to overcome the singularity of the integrand.

Theorem 2.6. (Energy method) Let $(E, \rho), \mu, \alpha$ be as above. Then $\forall \delta > 0$,

$$\mathcal{H}_\alpha^\delta(E) \geq \mu(E) \left/ \iint_{\rho(x,y) < \delta} \frac{d\mu(x)d\mu(y)}{\rho(x,y)^\alpha} \right/.$$

Hence if $I_\alpha(\mu) < \infty$ then $m_\alpha(E) = \infty$, and so $\dim E \geq \alpha$.

Proof. Suppose $\{F_n\}_{n \in \mathbb{N}}$ is a pairwise disjoint covering of E with $|F_n| < \delta$. Then

$$\iint_{\rho(x,y) < \delta} \frac{d\mu(x)d\mu(y)}{\rho(x,y)^\alpha} \geq \sum_{n=1}^{\infty} \iint_{F_n \times F_n} \frac{d\mu(x)d\mu(y)}{\rho(x,y)^\alpha} \geq \sum_{n=1}^{\infty} \frac{\mu(F_n)^2}{|F_n|^\alpha}.$$

Also, $\mu(E) \leq \sum_{n=1}^{\infty} \mu(F_n) = \sum_{n=1}^{\infty} |F_n|^{\alpha/2} \frac{\mu(F_n)}{|F_n|^{\alpha/2}}$. By Cauchy-Schwarz,

$$\mu(E) \leq \sum_{n=1}^{\infty} |F_n|^\alpha \sum_{n=1}^{\infty} \frac{\mu(F_n)^2}{|F_n|^\alpha} \leq H_\alpha^\delta(E) \iint_{\rho(x,y) < \delta} \frac{d\mu(x)d\mu(y)}{\rho(x,y)^\alpha}.$$

\square

We now prove a converse to the mass distribution principle, i.e. starting from a lower bound on the Hausdorff measure we construct a mass distribution on a set with bounded local concentration. From here on we work in \mathbb{R}^d .

Theorem 2.7. (Frostman's lemma) Let $E \subset \mathbb{R}^d$ be a closed set with $m_\alpha(E) > 0$. Then \exists a Borel probability measure μ supported on E and a constant $C > 0$ such that $\mu(D) \leq C|D|^\alpha \forall$ Borel sets D .

Remark 2.8. Frostman's lemma holds more generally for Borel subsets of \mathbb{R}^d , though we do not prove this fact here.

The proof is based on the representation of compact subsets of \mathbb{R}^d by trees. We first set up some machinery.

Definition 2.9. A **tree** $T = (V, E)$ is a connected graph described by an at most countable set of **vertices** V , which includes a distinguished vertex ρ designated as the **root**, and a set of ordered **edges** $E \subset V \times V$, such that

- $\forall v \in V$ the set $\{w \in V : (w, v) \in E\}$ consists of exactly one element \bar{v} , the **parent**, except for ρ , which has no parent;
- $\forall v \in V \exists$ a unique self-avoiding path from ρ to v ;
- $\forall v \in V$ the set of **children** $\{w \in V : (v, w) \in E\}$ is finite.

Every infinite self-avoiding path starting at ρ is called a **ray**. The set of rays is denoted ∂T , the **boundary** of T .

Definition 2.10. Let capacities be assigned to the edges of a tree T in the form of a mapping $C : E \rightarrow [0, \infty)$. A **flow** of strength $c > 0$ through a tree with capacities C is a mapping $\theta : E \rightarrow [0, c]$ such that

- for the root ρ we have $\sum_{\bar{w} \in \rho} \theta(\rho, w) = c$, for every other vertex $v \neq \rho$ we have $\theta(\bar{v}, v) = \sum_{w: \bar{w}=v} \theta(v, w)$, i.e. the flow into and out of each vertex other than the root is conserved;
- $\theta(e) \leq C(e)$, i.e. the flow through the edge e is bounded by its capacity.

A set Π of edges is called a **cutset** if every ray includes an edge from Π .

We omit the proof of the following theorem, a famous result in graph theory. It will be of critical importance in the proof of Frostman's lemma.

Theorem 2.11. (Max-flow min-cut)

$$\max\{\text{strength}(\theta) : \theta \text{ flow with capacities } C\} = \inf \left\{ \sum_{e \in \Pi} C(e) : \Pi \text{ cutset} \right\}.$$

We will need the following measure extension theorem, which we state without proof.

Definition 2.12. A collection of sets S is a **semi-algebra** if $\forall A, B \in S, A \cap B \in S$ and A^c is a finite disjoint union of sets in S .

Theorem 2.13. Let S be a semi-algebra and let $\tilde{\mu}$ be a measure on S . Then $\tilde{\mu}$ extends uniquely to a measure μ on $\sigma(S)$.

Proof. (of Frostman's lemma) WLOG let $E \subset [0, 1]^d$, and define the dyadic cubes in the usual manner. We construct a tree $T = (V, E)$ as follows. Let the root be associated with the cube $[0, 1]^d$. Then grow the tree such that every vertex has 2^d out-edges, each leading to a vertex associated with one of the 2^d subcubes of half sidelength. Finally, erase the edges ending in vertices associated with subcubes that do not intersect E . Note that the rays in ∂T correspond to sequences of nested compact cubes.

There is a canonical mapping $\Phi : \partial T \rightarrow E$, which maps sequences of nested cubes to their intersection. Clearly Φ is surjective. Now given an edge $e \in E$ at level n define the capacity $C(e) = 2^{-n\alpha}$. Associate to every cutset Π a covering of E consisting of those cubes associated with the initial vertices of the edges in the cutset. We have $\inf \left\{ \sum_{e \in \Pi} C(e) : \Pi \text{ cutset} \right\} \geq \inf \left\{ \sum_j |E_j|^\alpha : E \subset \bigcup E_j \right\}$, and as $m_\alpha(E) > 0$ this is bounded from zero. Thus, by the max-flow min-cut theorem, \exists a flow $\theta : E \rightarrow [0, \infty)$ of positive strength such that $\theta(e) \leq C(e) \forall$ edges $e \in E$.

Given an edge $e \in E$ associate a set $T(e) \subset \partial T$ consisting of all rays containing e . Define $\tilde{\nu}(T(e)) := \theta(e)$. Note that $\tilde{\nu}$ is countably additive and that the collection $\mathcal{C}(\partial T)$ of subsets $T(e) \subset \partial T \forall e \in E$ is a semi-algebra on ∂T . Thus by Theorem 2.12, $\tilde{\nu}$ extends to a measure ν on $\sigma(\mathcal{C}(\partial T))$. Define a Borel measure $\mu := \nu \circ \Phi^{-1}$ on E . We have $\mu(C) = \theta(e)$, where C is the cube associated with the initial vertex of the edge e .

Suppose now that D is a Borel subset of \mathbb{R}^d and n is the integer such that $2^{-n} < |D \cap [0, 1]^d| \leq 2^{-(n-1)}$. Then $D \cap [0, 1]^d$ can be covered with 3^d dyadic cubes of sidelength 2^{-n} . Using the bound, we have $\mu(D) \leq d^{\alpha/2} 3^d 2^{-n\alpha} \leq d^{\alpha/2} 3^d |D|^\alpha$, so μ is as required after normalization. \square

Frostman's lemma now allows us to prove that the energy method is sharp for closed subsets of \mathbb{R}^d (or Borel subsets of \mathbb{R}^d , if one assumes the more general version).

Definition 2.14. The α -**capacity** of a set E is defined to be

$$\text{Cap}_\alpha(E) := \sup \{I_\alpha(\mu)^{-1} : \mu \text{ Borel probability measure supported on } E\}.$$

Note that the energy method states that a set of positive α -capacity has dimension at least α .

Theorem 2.15. For any closed set $E \subset \mathbb{R}^d$, $\dim E = \sup \{\alpha : \text{Cap}_\alpha(E) > 0\}$.

Proof. It only remains to show \leq , and for this it suffices to show that if $\dim E > \alpha$, then \exists a Borel probability measure μ on E such that $I_\alpha(\mu) < \infty$. By assumption, for some sufficiently small $\beta > \alpha$ we have $m_\beta(E) > 0$. By Frostman's lemma, \exists a Borel probability measure μ supported on E and a constant C such that $\mu(D) \leq C |D|^\beta \forall$ Borel sets D . WLOG let the support of μ have diameter < 1 . Fix $x \in E$, and for $k \geq 1$ let $S_n(x) := \{y : 2^{-n} < |x - y| \leq 2^{1-n}\}$. We have

$$\int \frac{d\mu(y)}{|x - y|^\alpha} = \sum_{n=1}^{\infty} \int_{S_n(x)} \frac{d\mu(y)}{|x - y|^\alpha} \leq \sum_{n=1}^{\infty} \mu(S_n(x)) 2^{n\alpha} \leq 2^{2\beta} C \sum_{n=1}^{\infty} 2^{n(\alpha-\beta)} < \infty,$$

proving the theorem. \square

3. SELF-SIMILAR SETS

Definition 3.1. A mapping $S : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is a **similarity with ratio** $r > 0$ if $|S(x) - S(y)| = r|x - y|$.

Definition 3.2. A set E is **self-similar** if \exists finitely many similarities $\{S_i\}_{i=1}^n$ with the same ratio r such that $E = \bigcup S_i(E)$.

Example 3.3. Let E be the middle-thirds Cantor set. The similarities $S_1(x) = x/3$ and $S_2(x) = x/3 + 2/3$ (with ratio $r = 1/3$) show that E is self-similar.

The definition requires us to produce similarities for a given set E , but it turns out we can dispense with particular sets and instead speak solely of similarities. Similarities are easy to define, so in effect the following theorem gives us a wide range of self-similar sets to investigate.

Theorem 3.4. Let $\{S_i\}_{i=1}^n$ be n similarities of common ratio $r < 1$. Then \exists a unique non-empty compact set E such that $E = \bigcup S_i(E)$.

The S_i are contraction mappings, suggesting a proof in the way of a fixed point argument. We collect some preliminary lemmas below.

Lemma 3.5. \exists a closed ball B so that $S_i(B) \subset B$ for all i .

Proof. We have $|S_i(x)| \leq |S_i(x) - S_i(0)| + |S_i(0)| \leq r|x| + |S_i(0)|$. If we require $|x| \leq R$ to imply $|S_i(x)| \leq R$, it suffices to choose R so that $R \geq |S_i(0)|/(1-r)$. Take the maximum of the R so obtained. \square

Now, for any set A let $\tilde{S}(A) = \bigcup S_i(A)$. Note that if $A \subset A'$, then $\tilde{S}(A) \subset \tilde{S}(A')$. Note further that \tilde{S} is a mapping from subsets of \mathbb{R}^d to subsets of \mathbb{R}^d . There is a natural metric on compact subsets of \mathbb{R}^d , the Hausdorff metric, that allows us to exploit the property of contraction. This notion will prove useful for showing uniqueness.

Definition 3.6. For each $\delta > 0$ and set A , let $A^\delta := \{x : d(x, A) < \delta\}$. For compact sets A, B , define $\text{dist}(A, B) := \inf \{\delta : B \subset A^\delta \text{ and } A \subset B^\delta\}$.

Lemma 3.7. If $\{S_i\}_{i=1}^n$ are n similarities with common ratio r , then

$$\text{dist}(\tilde{S}(A), \tilde{S}(B)) \leq r \text{dist}(A, B).$$

Proof. Exercise. \square

Proof. (of Theorem 3.4) Choose B as in Lemma 3.5, and let $E_k := \tilde{S}^k(B)$ (\tilde{S} composed k times). Each E_k is compact, non-empty, and $E_k \subset E_{k-1}$. Thus $E := \bigcap_{k=1}^\infty E_k$ is compact and non-empty. Clearly, $\tilde{S}(E) = E$. It remains to prove uniqueness. Suppose G is another compact set so that $\tilde{S}(G) = G$. Then by Lemma 3.7, $\text{dist}(F, G) \leq r \text{dist}(F, G)$. Since $r < 1$, $\text{dist}(F, G) = 0$, so $F = G$. \square

The scaling property of self-similar sets suggests the following preliminary line of attack for computing the Hausdorff dimension. Suppose for a self-similar set E that the $S_i(E)$ are disjoint. Then $m_\alpha(E) = \sum_{i=1}^n m_\alpha(S_i(E))$. Since each S_i scales by r , we have $m_\alpha(S_i(F)) = r^\alpha m_\alpha(F)$. Hence, $m_\alpha(F) = nr^\alpha m_\alpha(F)$. If $0 < m_\alpha(F) < \infty$, we would have $\alpha = \frac{\log n}{\log 1/r}$. Of course, for many interesting examples (such as the Sierpinski triangle) the $S_i(E)$ are not disjoint. It turns out that we can relax the disjointness condition, as follows.

Definition 3.8. Similarities $\{S_i\}$ are **separated** if \exists a bounded open set \mathcal{O} such that $\mathcal{O} \supset \bigcup S_i(\mathcal{O})$ and the $S_i(\mathcal{O})$ are disjoint. Note that we do not require \mathcal{O} to contain E .

Theorem 3.9. Let $\{S_i\}_{i=1}^n$ be n separated similarities of common ratio $r < 1$, and let E be as in Theorem 3.4. Then E has Hausdorff dimension $\frac{\log n}{\log 1/r}$.

Proof. Let $\alpha = \frac{\log n}{\log 1/r}$.

$m_\alpha(E) < \infty$: We do not require the separation assumption. In the terminology of Theorem 3.4, note that E_k is the union of n^k sets of diameter less than $|B|r^k$, each of the form $S_{m_1} \circ \dots \circ S_{m_k}(B)$, where $1 \leq m_i \leq n$. Thus, if $|B|r^k \leq \delta$, then $\mathcal{H}_\alpha^\delta(E) \leq \sum_{m_1, \dots, m_k} |S_{m_1} \circ \dots \circ S_{m_k}(B)|^\alpha \leq |B|^\alpha n^k r^{\alpha k} = |B|^\alpha$, and we get $m_\alpha(E) < \infty$.

$m_\alpha(E) > 0$: We first set up some machinery. Fix a point $x \in E$. We define

the k -**vertices** as the n^k points that lie in E and are given by $S_{m_1} \circ \dots \circ S_{m_k}(x)$, where $1 \leq m_i \leq n$. Similarly, we define the k -**open sets** to be the n^k sets given by $S_{m_1} \circ \dots \circ S_{m_k}(\mathcal{O})$, where $1 \leq m_i \leq n$ and \mathcal{O} is fixed and chosen to satisfy the separation condition. Note that the k -open sets are disjoint, and if $k \geq l$, each l -open set contains n^{k-l} k -open sets. For a k -vertex v , let $\mathcal{O}(v)$ denote the k -open set with the same label (m_1, \dots, m_k) . Since \bar{x} is at fixed distance from \mathcal{O} , and \mathcal{O} has a finite diameter, we find that: a) $d(v, \mathcal{O}(v)) \leq Cr^k$, and b) $C'r^k \leq |\mathcal{O}(v)| \leq Cr^k$ ($|\mathcal{O}(v)| \approx r^k$), for some constants C, C' .

By compactness, it suffices to prove that if $\{B_j\}_{j=1}^N$ is a collection of balls covering E with diameters less than δ , then $\sum_{j=1}^N |B_j|^\alpha > 0$. Let \mathcal{B} be such a covering, and choose k so that $r^k \leq \min_{1 \leq j \leq N} |B_j| < r^{k-1}$.

Lemma 3.10. *Suppose B is a ball in \mathcal{B} that satisfies $r^l \leq |B| < r^{l-1}$ for some $l \leq k$. Then \exists constant C such that B contains at most Cn^{k-l} k -vertices.*

Proof. Let $v \in B$ be a k -vertex. Properties a) and b) above imply \exists a fixed dilate B^* of B such that $\mathcal{O}(v) \subset B^*$ and B^* contains the l -open set that contains $\mathcal{O}(v)$. Since B^* has volume $C'r^{dl}$ for some constant C' , and each l -open set has volume $\approx r^{dl}$ by property b), B^* contains at most C l -open sets for some constant C . Hence B^* contains at most Cn^{k-l} k -open sets. We conclude B contains at most Cn^{k-l} k -vertices. \square

Now let N_l denote the number of balls in \mathcal{B} so that $r^l \leq |B_j| < r^{l-1}$. By Lemma 3.10, we see that the total number of k -vertices covered by \mathcal{B} can be no more than $C_{\max} \sum_l N_l n^{k-l}$. Since all n^k k -vertices belong to E , $C_{\max} \sum_l N_l n^{k-l} \geq n^k$, and hence $\sum_l N_l n^{-l} > 0$. The definition of α gives $r^{l\alpha} = n^{-l}$, so $\sum_{j=1}^N |B_j|^\alpha \geq \sum_l N_l n^{-l} > 0$, completing the proof. \square

4. BROWNIAN MOTION

Definition 4.1. A random variable X is **normally distributed with expectation μ and variance σ^2** if $\forall x \in \mathbb{R}, \mathbb{P}\{X > x\} = \frac{1}{\sqrt{2\pi\sigma^2}} \int_x^\infty e^{-(u-\mu)^2/2\sigma^2} du$.

Definition 4.2. A real-valued stochastic process $\{B(t) : t \geq 0\}$ is called a **linear Brownian motion** with start in $x \in \mathbb{R}$ if the following holds:

- $B(0) = x$;
- the process has **independent increments**, i.e. \forall times $0 \leq t_1 \leq \dots < t_n$, the increments $\{B(t_k) - B(t_{k-1})\}_{k=2}^n$ are independent random variables;
- $\forall t, h \geq 0$, the increments $B(t+h) - B(t)$ are normally distributed with expectation zero and variance h ;
- almost surely, the function $t \mapsto B(t)$ is continuous.

We say that $\{B(t) : t \geq 0\}$ is a **standard Brownian motion** if $x = 0$.

Definition 4.3. If B_1, \dots, B_d are independent linear Brownian motions started in x_1, \dots, x_d , then the stochastic process $\{B(t) : t \geq 0\}$ given by $B(t) = (B_1(t), \dots, B_d(t))$ is called a **d-dimensional Brownian motion** started in (x_1, \dots, x_d) .

Remark 4.4. It is a nontrivial theorem that Brownian motion exists. We will be content with assuming this fact.

There are two ways to look at this definition. The first is to view Brownian motion as a family of (uncountably many) random variables $\omega \mapsto B(t, \omega)$ defined on a single probability space $(\Omega, \mathcal{A}, \mathbb{P})$. The second is to view it as a *random function* with the sample functions defined by $t \mapsto B(t, \omega)$. This is the view we will take.

We are interested in the Hausdorff dimension of a set after applying Brownian motion. The net content of the following theorems is that Brownian motion works to double the dimension of a set. We begin with the upper bound.

Definition 4.5. A function $f : [0, \infty) \rightarrow \mathbb{R}$ is **locally γ -Hölder continuous** at $x \geq 0$ if $\exists \epsilon, C > 0$ such that $\forall y \geq 0$ with $|y - x| < \epsilon$, $|f(x) - f(y)| \leq C|x - y|^\gamma$.

Theorem 4.6. For $\gamma < 1/2$, d -dimensional Brownian motion is everywhere locally γ -Hölder continuous almost surely.

Proof. The proof is reliant upon details going into the construction of Brownian motion and is omitted. \square

Corollary 4.7. For a d -dimensional Brownian motion $\{B(t) : t \geq 0\}$ and any fixed set $A \subset [0, \infty)$, almost surely $\dim B(A) \leq \min(2 \dim(A), d)$.

Proof. Proposition 2.2 and Theorem 4.6. \square

To prove the lower bound we use the energy method. The energy method proves useful for random sets since to obtain a lower bound it suffices to show the finiteness of a single integral. In particular, given a random set E and a random measure μ on E , $\mathbb{E}[I_\alpha(\mu)] < \infty$ implies, almost surely, $I_\alpha(\mu) < \infty$ and hence $\dim E \geq \alpha$.

Theorem 4.8. For a d -dimensional Brownian motion $\{B(t) : t \geq 0\}$ and a closed set $A \subset [0, \infty)$, almost surely $\dim B(A) = \min(2 \dim(A), d)$.

Proof. It remains to prove the lower bound. Let $\alpha < \min(\dim(A), d/2)$. By Theorem 2.15, \exists a Borel probability measure μ on A such that $I_\alpha(\mu) < \infty$. Denote by μ_B the measure defined by $\mu_B(D) := \mu(\{t \geq 0 : B(t) \in D\})$ for all Borel sets $D \subset \mathbb{R}^d$. Then

$$\mathbb{E}[I_{2\alpha}(\mu_B)] = \mathbb{E}\left[\int \int \frac{d\mu_B(x)d\mu_B(y)}{|x - y|^{2\alpha}}\right] = \mathbb{E}\left[\int \int \frac{d\mu(t)d\mu(s)}{|B(t) - B(s)|^{2\alpha}}\right],$$

where the second equality can be verified by a change of variables. Note that the denominator on the right hand side has the same distribution as $|t - s|^\alpha |Z|^{2\alpha}$, where Z is a d -dimensional standard normal random variable. Since $2\alpha < d$, we have that $\mathbb{E}[|Z|^{-2\alpha}] = \frac{1}{(2\pi)^{d/2}} \int |y|^{-2\alpha} e^{-|y|^2/2} dy < \infty$. Hence, using Fubini's theorem,

$$\mathbb{E}[I_{2\alpha}(\mu_B)] = \int \int \mathbb{E}[|Z|^{-2\alpha}] \frac{d\mu(t)d\mu(s)}{|t - s|^\alpha} \leq \mathbb{E}[|Z|^{-2\alpha}] I_\alpha(\mu) < \infty,$$

and so $I_{2\alpha}(\mu_B) < \infty$ almost surely. Moreover, μ_B is supported on $B(A)$ because μ is supported on A . By the energy method, $\dim B(A) \geq 2\alpha$ almost surely. Letting $\alpha \uparrow \min(\dim(A), d/2)$ completes the proof. \square

Remark 4.9. If we assume the stronger version of Frostman's lemma, Theorem 4.8 holds for all Borel sets $A \subset [0, \infty)$.

Exercise 4.10. We remarked earlier that the Lebesgue measure of planar (two-dimensional) Brownian motion is zero almost surely. How is this not a contradiction with Theorems 1.10 and 4.8?

To conclude we mention a powerful generalization of Theorem 4.8. Note that in Theorem 4.8, the null probability set depends on A , while in Theorem 4.11, dimension doubles *simultaneously* for almost all sets. This allows us to work with arbitrary random sets.

Theorem 4.11. *Let $\{B(t) : t \geq 0\}$ be Brownian motion in dimension $d \geq 2$. Almost surely, for any $A \subset [0, \infty)$, $\dim B(A) = 2 \dim A$.*

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