INTEGRATION ON MANIFOLDS

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Abstract. The primary goal of this paper is to define the notion of integration on a k-dimensional manifold and to discuss the machinery required to do so. The majority of the rigorous definitions and proofs that follow involve manifolds embedded in an ambient space – specifically, in $\mathbf{R^n}$. However, it is also possible to talk about abstract manifolds, which are not embedded in any such ambient space. It can be shown that the theory of integration on abstract manifolds is merely a generalization of the theory of integration on manifolds embedded in Euclidean space. Therefore, some abstract generalizations of the definitions involving $\mathbf{R^n}$ are also discussed.

Contents

1. Defining a Manifold

Intuitively, a smooth manifold is a space that, when examined closely enough, looks like Euclidean space. In this regard, manifolds provide a natural setting for defining many of the usual notions of calculus, including differentiation, tangent spaces, vector fields, differential forms, and integration. To begin our discussion, we need the definitions of a diffeomorphism and a k-dimensional manifold in $\mathbf{R^n}$:

Definition 1.1. If U and V are open sets in $\mathbf{R^n}$, a **diffeomorphism** is a smooth (i.e., infinitely differentiable) function $h : U \to V$ with a smooth inverse $h^{-1} : V \to U$.

Definition 1.2. A subset M of $\mathbf{R^n}$ is called a **k-dimensional manifold** (in $\mathbf{R^n}$) if for every point $x \in M$, there is an open set U containing x, an open set $V \subset \mathbf{R^n}$, and a diffeomorphism $h : U \to V$ such that

$$h(U \cap M) = V \cap (\mathbf{R^k} \times \{0\}) = \{(y_1, \ldots, y_n) \in V : y_{k+1} = \cdots = y_n = 0\}.$$

In other words, $U \cap M$ is equivalent to $\mathbf{R^k}$, 'up to diffeomorphism.'

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We will use this definition of a manifold as we formally build up the machinery required to integrate on a manifold. However, it should also be noted that our definition need not rely on $\mathbf{R^n}$ as an ambient space. In fact, it is possible to define diffeomorphisms and manifolds in this abstract sense:

Definition 1.3. A function f is a **diffeomorphism** if it is bijective and smooth and if its inverse is also smooth.

Definition 1.4. An **abstract manifold** of dimension k is a second countable Hausdorff space M, together with an open cover (U_i) of M, and homeomorphisms $\varphi_i : U_i \to \mathbf{R^k}$ such that each $\varphi_j \circ \varphi_i^{-1}$ is a diffeomorphism from $\varphi_i(U_i \cap U_j)$ to $\varphi_j(U_i \cap U_j)$.

Now, for a subset in $\mathbf{R^n}$, the Euclidean definition of a manifold (**Definition 1.2**) and the abstract definition of a manifold (**Definition 1.5**) are equivalent. One direction of this fact is proven below:

Theorem 1.5. *Suppose M is a set in $\mathbf{R^n}$. If M satisfies the Euclidean definition of a manifold, then M satisfies the abstract definition of a manifold as well.*

Proof. We know that for each point $x \in M$, there is a diffeomorphism $h : U \to V$ between sets $U, V \in \mathbf{R^n}$ as in Definition 1.2. (Note that here we are using "diffeomorphism" as we have defined it in Definition 1.1). Take $(U_i)_{i \in J}$, which is an open cover of M. For each U_i, we define

$$\varphi_i = h|_{U_i \cap M} : U_i \cap M \to V_i \cap (\mathbf{R^k} \times \{0\}).$$

Clearly φ_i is a bijective, continuous function whose inverse is also continuous. Thus $(\varphi_i)_{i \in J}$ is a collection of homeomorphisms. For all $i, j \in J$, we have the map $\varphi_j \circ \varphi_i^{-1}$ defined on an open subset of $V_j \cap (\mathbf{R^k} \times \{\mathbf{0}\})$. We can extend this map's domain to an open subset of V_i with the function $h \circ h^{-1}$. Since this function is a diffeomorphism, one can show that its restriction $\varphi_j \circ \varphi_i^{-1}$ is also diffeomorphism. $\square$

The other direction of the proof – that **Definition 1.5** implies **Definition 1.2** – makes use of the Whitney embedding theorem, which states that any manifold can be smoothly embedded in $\mathbf{R^n}$

Although we will not rigorously define integration on abstract manifolds, the abstract analogues of definitions involving $\mathbf{R^n}$ will be discussed as appropriate.

2. Tangent Spaces

Another important characterization of a manifold in $\mathbf{R^n}$ uses the notion of coordinate systems, or charts. First we will define a coordinate system:

Definition 2.1. Let U be an open set in $\mathbf{R^n}$, $x \in U$, and W be an open set in $\mathbf{R^k}$. Then the function $f : W \to \mathbf{R^n}$ is called a **coordinate system**, or **chart**, around x if f is injective and smooth and if the following hold:

1) $f(W) = M \cap U$,
2) *For all* $y \in W$, Df_y *has rank* k *for each* $y \in W$,
3) $f^{-1} : f(W) \to W$ *is continuous.*

Based on this definition, the following claim can be made about manifolds: a subset M of $\mathbf{R^n}$ is a k-dimensional manifold if and only if for all $x \in M$, there are open sets U and W and a function $f : W \to \mathbf{R^n}$ which satisfy the definition above (a proof for this statement can be found in [1]). This characterization emphasizes that the derivative of any point in W always has full rank, which will help us greatly in understanding the important notion of a tangent space, explained below.

Consider a point $x \in M$, and a coordinate system $f : W \to \mathbf{R^n}$ around x such that $x = f(a)$. (The notation $(\mathbf{R^n})_x$ will henceforth denote the set $\{(x, v) | v \in \mathbf{R^n})\}$, and v_x will henceforth denote (x, v).) Since f is a coordinate system, the derivative of f at the point a has full rank; therefore, $D_a f$ is injective. So $D_a f(\mathbf{R^k})_a$ is a k-dimensional subspace of $(\mathbf{R^n})_x$, which we call the **tangent space** to M at x, denoted by $T_x M$. (Note that the tangent space to a point is independent of the particular chart chosen at that point: if we chose another coordinate system $g : V \to \mathbf{R^n}$ around x such that $g(b) = x$, then we would have

$$D_b g(\mathbf{R^k})_b = D_a f(D_b(f^{-1} \circ g)(\mathbf{R^k})_b) = D_a f(\mathbf{R^k})_a,$$

which shows that our notion of a tangent space is well defined.)

The abstract notion of a tangent space is constructed using a type of function called a **derivation**, a linear map that generalizes certain features of the derivative. More specifically, a derivation is a linear operator D, from the space C^∞ to $\mathbf{R}$ that satisfies the Leibniz rule; that is, $D(ab) = a(Db) + b(Da)$. Now, consider all the smooth curves that pass through a point x on a manifold M. Then the derivations of these curves at x yield directions that span a vector space, which we define to be the tangent space $T_x M$.

It is easy to see that $T_x M$, as we defined it earlier, must consist of vectors tangent to curves in M passing through x; therefore, we can understand intuitively why this way of defining a tangent space is equivalent to our previous method.

The existence of a tangent space at each point on a manifold can be conceptualized in the following way: if we looked closely enough at the neighborhoods around each point on a 2-dimensional manifold, we would find that in these regions, the manifold appeared 'flat' – specifically, it would appear reminiscent of Euclidean space. Since integration is already defined in Euclidean space, our strategy for defining integration on manifolds will be to connect this known definition with the notion of $\mathbf{R^n}$-like tangent spaces.

3. Vector Fields and Differential Forms

First we review some definitions that are important for defining integration in $\mathbf{R^n}$.

Definition 3.1. A **vector field** is a function $F : \mathbf{R^n} \to (\mathbf{R^n})_p$ that selects a vector at each point.

If $(e_1)_p, (e_2)_p, \ldots (e_n)_p$ is the standard basis for $(\mathbf{R^n})_p$, then any vector field $F(p)$ can be expressed as $F(p) = F^1(p) \cdot (e_1)_p + F^2(p) \cdot (e_2)_p + \ldots + F^n(p) \cdot (e_n)_p$. The vector field F is defined to be smooth if each F^i is smooth.

Definition 3.2. A **differential** k**-form** is a function $\omega : \mathbf{R^n} \to \Lambda^k((\mathbf{R^n})_p)$ that selects an alternating k-tensor at each point.

If $(e^1)_p, (e^2)_p, \dots (e^n)_p$ is the basis for the dual space of $(\mathbf{R^n})_p$, then any differential form $\omega(p)$ can be expressed as

$$\omega(p) = \sum_{i_1 < i_2 < \dots < i_k} \omega_{i_1, i_2, \dots i_k}(p) e^{i1}(p) \wedge e^{i2}(p) \wedge \dots \wedge e^{ik}(p).$$

The differential form ω is defined to be smooth if each function $\omega_{i_1, i_2, \dots i_k}$ is smooth.

The relationship between vector fields and differential forms can be described in the following way: at each point, the differential form selects a k-tensor, which 'eats' the particular vector selected by the vector field. Thus, applying a differential k-form to a k-tuple vector field yields a number at every point.

Now, we would like to talk about vector fields and differential forms as they relate to our discussion of manifolds. In order to do so, we need to modify our definitions appropriately. Firstly, note that each tangent space behaves exactly like a vector space. Thus, a vector field on the entire manifold will be a function from the manifold to the set of all tangent spaces to the manifold, which we call the **tangent bundle** and denote TM. Additionally, we must be careful that at each point on the manifold, the vector field selects a vector *in the tangent space at that point*. With these considerations, we have the following definition:

Definition 3.3. A **vector field** on a manifold M is a function $F : M \to TM$ that selects a tangent vector at each point on the manifold; that is, such that $F(x) \in T_x M$.

Again, we can express any vector field F as a linear combination of the bases of the tangent spaces contained in TM with some scalars $F^i(p)$. As usual, F is smooth if each F^i is smooth. In fact, TM is a manifold in its own right, and, as an alternative criterion for smoothness, it can be shown that F is a smooth vector field if and only if it is a smooth function of manifolds.

We define differential forms on manifolds in a similar way:

Definition 3.4. A **differential k-form** on a manifold M is a function $\omega : M \to \Lambda^k(TM)$ that selects a alternating k-tensor at each point of the manifold; that is, such that $\omega(x) \in \Lambda(T_x M)$ for each $x \in M$.

Let ω be a k-differential form on M. If $f : W \to \mathbf{R^n}$ is a chart on M, then the pullback $f^*\omega$ is a k-differential form on W. We define ω to be smooth if $f^*\omega$ is smooth.

4. Integration on Manifolds

we can define integration on manifolds. Again, we begin by reviewing some definitions that are important in defining integration in $\mathbf{R^n}$.

Definition 4.1. A **singular n-cube** in a subset $A \subset \mathbf{R^n}$ is a continuous function $c : [0,1]^n \to A$. An **n-chain** is a finite sum of singular n-cubes with integer coefficients of the form $\alpha_1 c_1 + \alpha_2 c_2 + \dots + \alpha_n c_n$.

In $\mathbf{R^n}$, we define integration of a differential form ω over a singular k-cube c in A in the following way:

$$\int_c \omega = \int_{[0,1]^k} c^*\omega,$$

and we define integration of ω over a chain $c = \sum_i a_i c_i$ in A in the following way:

$$\int_c \omega = \sum_i a_i \int_{c_i} \omega.$$

As expected, integration on a manifold M will be defined in almost precisely the same way, with the specification that the singular n-cube maps into the manifold; that is, $c : [0,1]^n \to M$.

Thus the formal definition of integration over a singular p-cube in a manifold is as follows: Let ω be a p-form on a k dimensional manifold M, and let c be a singular p -cube in M. Then

$$\int_c \omega = \int_{[0,1]^p} c^* \omega,$$

and integration over chains is also precisely the way we defined it before.

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References

[1] M. Spivak. Calculus on Manifolds. Addison-Wesley Publishing Company. 1995.