2-DIMENSIONAL TOPOLOGICAL QUANTUM FIELD THEORIES AND FROBENIUS ALGEBRAS

CAROLINE TERRY

Abstract. Category theory provides a more abstract and thus more general setting for considering the structure of mathematical objects. 2-dimensional quantum field theories arise in physics as objects that assign vector spaces to 1-manifolds and linear maps to 2-cobordisms. From a categorical perspective, we find that they are the same as commutative Frobenius algebras. Our main goal is to explain this equivalence between the category of 2-dimensional toplological quantum field theories and the category of commutative Frobenius algebras.

Contents

1. The main theorem

Our goal in this paper is to understand the equivalence of the category of commutative Frobenius algebras and the category of 2-dimensional Topological Quantum Field Theories. We state this equivalence first as a theorem to provide motivation for the rest of our discussion and for ease of reference.

Theorem 1.1. *The category of 2-dimensional Topological Quantum Field Theories (2-TQFT) is equivalent to the category of commutative Frobenius algebras.*

To understand this theorem we need to define both of these categories, and that will be the focus of the next two sections.

2. The category of commutative Frobenius algebras

In this section we define the category of commutative Frobenius algebras over a fixed field k. In order to do so we require a few preliminary definitions:

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Definition 2.1. An *algebra* is an vector space A over k, together with multiplication $\mu\colon A \otimes_k A \to A$ and unit map $\eta\colon k \to A$ such that multiplication is associative and unital, i.e. such that the following diagrams commute:

$$\begin{tikzcd}
A \otimes_k A \otimes_k A \arrow[r, "id_A \otimes \mu"] \arrow[d, "\mu \otimes id_A"'] & A \otimes_k A \arrow[d, "\mu"] \\
A \otimes_k A \arrow[r, "\mu"'] & A
\end{tikzcd}$$

$$\begin{tikzcd}
A \otimes_k k \arrow[r, "id_A \otimes \eta"] \arrow[dr] & A \otimes_k A \arrow[d, "\mu"] \\
 & A
\end{tikzcd}
\qquad
\begin{tikzcd}
A \otimes_k A \arrow[d, "\mu"'] & k \otimes_k A \arrow[l, "\eta \otimes id_A"'] \arrow[dl] \\
A &
\end{tikzcd}$$

Example 2.2. Let $G = \{id_G = g_0, g_1, \ldots, g_n\}$ be a finite group and k a field. Let $k[G]$ be the vector space over k with basis elements the elements of G. We give $k[G]$ an algebra structure by defining a unit map $\eta\colon k \to k[G]$ by $1 \mapsto g_0$ and multiplication $\mu\colon k[G] \otimes k[G] \to k[G]$ by $\sum_i k_i g_i \otimes \sum_j k_j g_j \mapsto \sum_i \sum_j k_i k_j (g_i g_j)$. We check that this satisfies the above requirements for an algebra by calculating that $\mu \circ (id_{k[G]} \otimes \mu) = \mu \circ (\mu \otimes id_{k[G]})$.

Definition 2.3. A *coalgebra* is a vector space A over a field k, together with comultiplication $\delta\colon A \to A \otimes_k A$ and counit map $\epsilon\colon A \to k$ such that comultiplication is associative and unital, i.e. such that the following diagrams commute:

$$\begin{tikzcd}
A \otimes_k A \otimes_k A & A \otimes_k A \arrow[l, "\delta \otimes_k id_A"'] \\
A \otimes_k A \arrow[u, "id_A \otimes_k \delta"] & A \arrow[l, "\delta"] \arrow[u, "\delta"']
\end{tikzcd}$$

$$\begin{tikzcd}
A \otimes_k k & A \otimes_k A \arrow[l, "id_A \otimes_k \epsilon"'] \\
 & A \arrow[u, "\delta"'] \arrow[ul]
\end{tikzcd}
\qquad
\begin{tikzcd}
A \otimes_k A \arrow[r, "\epsilon \otimes_k id_A"] & k \otimes_k A \\
A \arrow[u, "\delta"] \arrow[ur] &
\end{tikzcd}$$

Example 2.4. Consider the vector space $k[G]$ from 2.2. We can make this into a coaglebra by defining a comultiplication $\delta\colon k[G] \to k[G] \otimes k[G]$ that sends each basis element g_i to $g_i \otimes g_i$ and a counit $\epsilon\colon k[G] \to k$ that sends each basis element g_i to 1.

Definition 2.5. Let A be a k-algebra with multiplication $\mu\colon A \otimes_k A \to A$ and unit map $\eta\colon k \to A$. A *right A-module* is a vector space M over k together with a map $\alpha\colon M \otimes_k A \to M$, such that α respects multiplication in A, i.e., such that following diagrams commute:

$$\begin{tikzcd}
M \otimes_k A \otimes_k A \arrow[r, "id_M \otimes_k \mu"] \arrow[d, "\alpha \otimes id_A"'] & M \otimes_k A \arrow[d, "\alpha"] \\
A \otimes_k A \arrow[r, "\alpha"'] & A
\end{tikzcd}$$

$$\begin{tikzcd} M \otimes_k A \arrow[r, "\alpha"] & M \\ M \otimes_k k \arrow[u, "id_M \otimes_k \eta"] \arrow[ur] & \end{tikzcd}$$

A left A-module is defined similarly with a map $\alpha\colon A \otimes_k M \to M$ that commutes with the multiplicaiton and unit maps. We can check easily that A is both a left and right A-module by replacing M above with A and α with μ.

Definition 2.6. If M and N are right A-modules with action maps $\alpha\colon M \otimes A \to M$ and $\alpha'\colon N \otimes A \to N$, then a linear map $\phi\colon M \to N$ is a *right A-module homomorphism* if it commutes with α and α', i.e., if the following diagram commutes:

$$\begin{tikzcd} A \otimes M \arrow[r, "id_A \otimes \phi"] \arrow[d, "\alpha"'] & A \otimes N \arrow[d, "\alpha'"] \\ M \arrow[r, "\phi"'] & N \end{tikzcd}$$

We define left A-module homomorphisms similarly. Now we can define a Frobenius algebra.

Definition 2.7. A *Frobenius algebra* A is an algebra and a coalgebra such that the coproduct δ is a left and right A-module homomorphism, that is, A is equipped with unit map $\eta\colon k \to A$ and multiplication $\mu\colon A \otimes A \to A$ as in Definition 2.1 and counit map $\epsilon\colon A \to k$ and comultiplication $\delta\colon A \to A \otimes A$ as in Definition 2.3 such that the following two diagrams commute:

$$\begin{tikzcd} A \otimes A \arrow[r, "id_A \otimes \delta"] \arrow[d, "\mu"'] & A \otimes A \otimes A \arrow[d, "\mu \otimes id_A"] \\ A \arrow[r, "\delta"'] & A \otimes A \end{tikzcd}$$

$$\begin{tikzcd} A \otimes A \arrow[r, "\delta \otimes id_A"] \arrow[d, "\mu"'] & A \otimes A \otimes A \arrow[d, "id_A \otimes \mu"] \\ A \arrow[r, "\delta"'] & A \otimes A \end{tikzcd}$$

We call the identities expressed in these diagrams the *Frobenius relations.*

A Frobenius algebra is commutative when the product and coproduct are commutative. This means, if we define a map $\sigma\colon A \otimes A \to A \otimes A$ by $a \otimes a' \mapsto a' \otimes a$, called the twist map, then A is a commutative Frobenius algebra if $\delta = \sigma \circ \delta$ and $\mu \circ \sigma = \mu$. In other words, we require these two diagrams to commute:

$$\begin{tikzcd} A \otimes A \arrow[r, "\sigma"] \arrow[dr, "\mu"'] & A \otimes A \arrow[d, "\mu"] \\ & A \end{tikzcd} \qquad \begin{tikzcd} A \otimes A \arrow[r, "\sigma"] & A \otimes A \\ A \arrow[u, "\delta"] \arrow[ur, "\delta"'] & \end{tikzcd}$$

We have now defined the objects of one of the categories in Theorem1.1, that of commutative Frobenius algebras. Now we define the morphisms in this category.

Given two Frobenius algebras A and A', a *Frobenius algebra homomorphism* is an algebra homomorphism $\phi\colon A \to A'$ that is also a coalgebra homomorphism, i.e. ϕ respects multiplication and unit maps along with comultiplication and counit maps.

Example 2.8. Consider again the vector space $k[G]$. From Examples 2.2 and 2.4 we know that $k[G]$ admits both algebra and coalgebra structures, so it makes sense to check whether these two structures make $k[G]$ into a Frobenius algebra:

In order for the counit map δ from Example 2.4 to be a map of right $k[G]$-modules, we need $(id_{k[G]} \otimes \mu) \circ (\delta \otimes id_{k[G]}) = \delta \circ \mu$. Applying the left side to a basis element $g_i \otimes g_j$ of $k[G] \otimes k[G]$ gives following:

$$(id_{k[G]} \otimes \mu) \circ (\delta \otimes id_{k[G]})(g_i \otimes g_j) = (\mu \otimes id_{k[G]})(g_i \otimes gi \otimes g_j) = g_i \otimes g_i g_j$$

Applying the right side to the same basis element gives the following:

$$(\delta \circ \mu)(g_i \otimes g_j) = \delta(g_i g_j) = g_i g_j \otimes g_i g_j$$

However, $g_i \otimes g_i g_j \neq g_i g_j \otimes g_i g_j$. Thus $k[G]$ is not a Frobenius algebra under these two structures.

If we instead define comultiplication by $\delta\colon g \mapsto \sum_i g g_i \otimes g_i^{-1}$ and counit by $\epsilon\colon g_0 \mapsto 1$ and $g_i \mapsto 0$ for $i \neq 0$, then $k[G]$ is indeed a Frobenius algebra (verification of this is left to the reader). When G is abelian, this definition makes $k[G]$ a commutative Frobenius algebra.

3. The category **2-TQFT**

In this section we define the objects and morphisms in the second category of Theorem 1.1, the category **2-TQFT** of 2-dimensional topological quantum field theories. First we state the definition of the objects in **2-TQFT**, then explain the components of the definition.

Definition 3.1. A *2-dimensional topological quantum field theory* (2-TQFT) is a symmetric monoidal functor from the category **2-Cob** of 2-dimensional cobordisms to the category $\mathbf{Vect}_k$ of vector spaces over a fixed field k.

Recall that a *monoidal category* is a category $\mathcal{C}$ together with a functor $\Box\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ called the monoidal product, a unit object $\mathbf{1}$, an associativity isomorphism $\alpha\colon (X \Box Y)\Box Z \to X \Box (Y \Box Z)$ for each X, Y, $Z \in \operatorname{ob}\mathcal{C}$, and isomorphisms $\lambda\colon \mathbf{1}\Box X \to X$ and $\rho\colon X \Box \mathbf{1} \to X$ for each $X \in \operatorname{ob}\mathcal{C}$. Furthermore, these structures are required to be coherent, meaning that λ and ρ are suitably related and α satisfies MacLane's pentagon axiom. A monoidal category is said to be symmetric if there exists for each X, $Y \in \operatorname{ob}\mathcal{C}$ a commutativity isomorphism $\gamma\colon X \Box Y \to Y \Box X$.

We assume the reader is familiar with the definition of a symmetric monoidal functor, and merely state that it is a functor between symmetric monoidal categories that respects the monoidal product and above isomorphisms. We now define $\mathbf{Vect}_k$ and **2-Cob** and provide their monoidal structures:

Definition 3.2. Let $\mathbf{Vect}_k$ be the category whose objects are finite dimensional k-vector spaces and whose morphisms are k-linear maps between them. $\mathbf{Vect}_k$ is a symmetric monoidal category with monoidal product $\otimes_k$, unit object k, and twist map $\sigma\colon v \otimes v' \mapsto v' \otimes v$. The associativity and unit isomorphisms α, λ and ρ are provided by the universal property of the tensor product.

The definition of **2-Cob** requires a few preliminary definitions.

Definition 3.3. Let Σ_1 and Σ_2 be two closed, oriented 1-manifolds. An *oriented cobordism* from Σ_1 to Σ_2 is a compact oriented 2-manifold M with two boundaries, one designated the in-boundary, which we always draw on the top, and the other designated the out-boundary, which we always draw on the bottom, together with a smooth map $f_1 \colon \Sigma_1 \to M$ that maps Σ_1 diffeomorphically onto the in-boundary of M and a smooth map $f_2 \colon \Sigma_2 \to M$ that maps Σ_2 onto the out-boundary of M. In order for f_1 and f_2 to respect orientations, we require f_1 to reverse the orientation of Σ_1 and f_2 to preserve the orientation of Σ_2.

We assume the result that every closed oriented 1-manifold is either empty or the disjoint union of circles. Thus both boundaries are disjoint unions of circles, including possibly the disjoint union of zero circles, or the empty 1-manifold. These disjoint unions of circles will be the objects in **2-Cob**. We still need one more definition for the morphisms:

Definition 3.4. Let M and N be two oriented cobordisms from Σ_1 to Σ_2, and let $f_1 \colon \Sigma_1 \to M$, $f_1' \colon \Sigma_1 \to M'$, $f_2 \colon \Sigma_2 \to M$, $f_2' \colon \Sigma_2 \to M'$ be diffeomorphisms mapping Σ_1 onto the in-boundaries of M and M' and Σ_2 onto their out-boundaries. Then M and M' are equivalent (in **2-Cob**), that is, in the same diffeomorphism class, if there exists an orientation preserving diffeomorphism $g \colon M \to M'$ such that the following diagram commutes:

$$\begin{tikzcd}
& M' & \\
\Sigma_1 \arrow[ur, "f_1'"] \arrow[r, "f_1"'] & M \arrow[u, "g"] & \Sigma_2 \arrow[ul, "f_2'"'] \arrow[l, "f_2"]
\end{tikzcd}$$

Now we define **2-Cob**:

Definition 3.5. Let **2-Cob** be the category whose objects are closed, oriented 1-manifolds, and whose morphisms are diffeomorphism classes of oriented cobordisms between them.

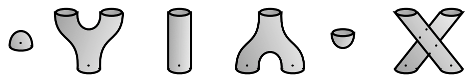
Thus, the objects in **2-Cob** are disjoint unions of circles, with the empty manifold being the disjoint union of 0 circles. We denote an object in **2-Cob** as $\mathbf{n}$, where n is the number of disjoint circles. A morphism between two such objects, say between $\mathbf{m}$ and $\mathbf{n}$, is a diffeomorphism class of an oriented 2-manifold whose in-boundary is the disjoint union of m circles and whose out-boundary is the disjoint union of n circles. We define composition of two cobordism classes by pasting representatives along their out and in boundaries, then passing to equivalence classes.

Next we give **2-Cob** the structure of a symmetric monoidal category. We take the disjoint union of cobordisms to get a functor $\amalg \colon$ **2-Cob** $\times$ **2-Cob** to **2-Cob** and let this be our monoidal product. We define for each $\Sigma_1, \Sigma_2 \in \mathrm{ob}\,$**2-Cob**, the twist map $s \colon \Sigma_1 \amalg \Sigma_2 \to \Sigma_2 \amalg \Sigma_1$ to be the class of the twist cobordism:

Here, the trapezoids stand for copies of $\Sigma_1 \times I$ and $\Sigma_2 \times I$ twisting pass each other. Finally we make the empty 1-manifold our unit object, and this makes

$(\mathbf{2\text{-}Cob}, \varnothing, \amalg, T)$ into a symmetric monoidal category. A *generating set* for a monoidal category $\mathcal{C}$ is a set of maps S such that every map in $\mathcal{C}$ is the composition and disjoint union of some elements of S. It is a result following from the classification of surfaces that there exists a generating set for the category **2-Cob**. We state the result as a theorem, then discuss why it is true.

Theorem 3.6. *The category* **2-Cob** *is generated under composition and disjoint union by the following morphisms: (we label the generators in order from left to right: $a : \varnothing \to \mathbf{1}$, $m : \mathbf{2} \to \mathbf{1}$, $i : \mathbf{1} \to \mathbf{1}$, $d : \mathbf{1} \to \mathbf{2}$, $e : \mathbf{1} \to \varnothing$, and $s : \mathbf{2} \to \mathbf{2}$)*



Our goal is to demonstrate that every cobordism is diffeomorphic to the composition and disjoint union of this set of generating morphisms. We will do this by defining a normal form for a connected cobordism determined only by the numbers of in- and out- boundaries and the number of holes, and explaining why every connected cobordism is diffeomorphic to some normal form. Then we explain why every cobordism can be expressed as the disjoint union of connected components.

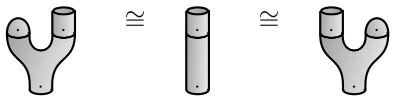
First we recall the classification of surfaces, that is, that two connected, compact, oriented surfaces without boundary are diffeomorphic if and only if they have the same genus (i.e. number of holes). For a surface with boundary, we define the genus to be the number of holes of the surface after sewing in discs onto its boundary components. Thus, to classify connected cobordisms, we need to also specify the numbers of in-boundaries and the number out-boundaries components. This means two connected 2-cobordims are diffeomorphic if and only if they have the same genus, the same number of in-boundaries, and the same number of out-boundaries.

We define the *normal form* in the following way. Given a 2-cobordism with n in boundaries, m out boundaries, and g holes, we define the normal form in three parts. The first part we define as $(\amalg_{n-2} i \amalg m) \circ (\amalg_{n-3} i \amalg m) \circ ... \circ (i \amalg m) \circ m : \mathbf{n} \to \mathbf{1}$. This gives us n in boundaries. The middle part we define as $(m \circ d) \circ (m \circ d) \circ ... \circ (m \circ d)$ where we have g copies of $(m \circ d)$. This part gives us g holes. The third part we define as $d \circ (i \amalg d) \circ (i \amalg i \amalg d) \circ ... \circ (\amalg_{m-2} i \amalg d) : \mathbf{1} \to \mathbf{m}$. This part gives us m out-boundaries. The composition of the three parts gives us a connected cobordism with n in-boundaries, m out-boundaries, and g holes. Thus we have a standard way of expressing a diffeomorphism class of a connected 2-cobordism in terms of the generating morphisms i, m, and d.

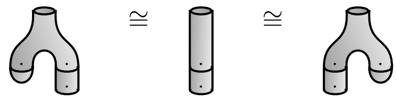
Now we consider cobordisms that are the disjoint union of connected cobordisms. We need to show that permuting the order of the boundary components does not change the diffeomorphism class of a surface. Let M and M' be two cobordisms with the same numbers of in and out boundaries and the same number of holes, such that the out-boundary components of M' are some permutation of the out-boundaries of M. Since every permutation can be written as the product of transpositions, we can compose the out-boundary of M' with the twist cobordism s in disjoint union with the cylinder i to rearrange the boundary components of M' so that their order matches those of M. Similarly we can rearrange in-boundary components.

We refer the reader to [1] for a more complete proof of the above result and of the following. We also have what are called relations, or equivalent ways of expressing a morphisms in terms of generators. The following is a complete list of relations in **2-Cob**.

These relations result from sewing a disc to one of the legs on a pair of pants:

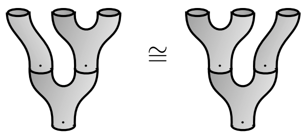


$$m \circ (a \amalg i) = i = m \circ (i \amalg a)$$

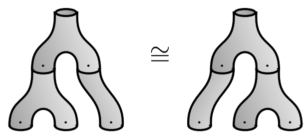


$$(e \amalg i) \circ d = i = (i \amalg e) \circ d.$$

We also have associativity and coassociativity relations:

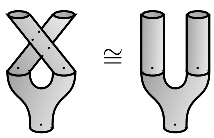


$$m \circ (i \amalg m) = m \circ (m \amalg i)$$

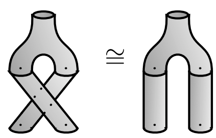


$$(d \amalg i) \circ d = (i \amalg d) \circ d.$$

We also have commutativity and cocommutativity relations:

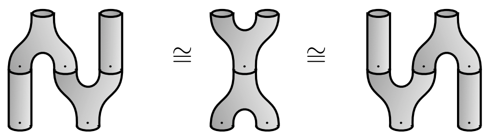


$$m \circ s = m$$



$$s \circ d = d.$$

Finally we have what is called the Frobenius relation:



$$(i \amalg m) \circ (d \amalg i) = d \circ m = (m \amalg i) \circ (i \amalg d).$$

Generators make specifying a 2-TQFT easy, since we need only specify where the TQFT sends the generating elements of **2-Cob**. For example, consider the morphism $M = (a \amalg i) \circ e \circ m$ and a TQFT $\mathcal{A}$. Since $\mathcal{A}$ is a functor, the following is true: $\mathcal{A}((a \amalg i) \circ e \circ m) = \mathcal{A}(a \amalg i) \circ \mathcal{A}(e) \circ \mathcal{A}(m)$, and since $\mathcal{A}$ is a monoidal functor, we have $\mathcal{A}(a \amalg i) \circ \mathcal{A}(e) \circ \mathcal{A}(m) = \mathcal{A}(a) \otimes \mathcal{A}(i) \circ \mathcal{A}(e) \circ \mathcal{A}(m)$. Since every morphism in **2-Cob** is the composition and disjoint union of the generating morphisms, if we know where $\mathcal{A}$ sends the generating morphisms, we know where it sends all the morphisms.

We also know how $\mathcal{A}$ acts on objects. $\mathcal{A}$ sends the circle, $\mathbf{1}$, to a vector space A. Since it is a monoidal functor, $\mathcal{A}$ sends the disjoint union of n circles, $\mathbf{n} = \mathbf{1} \amalg \cdots \amalg \mathbf{1}$, to A tensored with itself n times, $A \otimes \cdots \otimes A$. Again, since $\mathcal{A}$ is monoidal, it sends the empty union of circles $\varnothing$ to the field k. So a TQFT $\mathcal{A}$ is determined by where it sends $\mathbf{1}$ and where it sends the generating morphisms.

We have defined the objects of **2-TQFT**; now we define the morphisms.

Definition 3.7. A morphism α between two 2-TQFTs $\mathcal{A} \to \mathcal{A}'$ is a natural transformation of functors. By definition, this is a collection of morphisms $\alpha_{\mathbf{m}} : \mathcal{A}(\mathbf{m}) \to \mathcal{A}'(\mathbf{m})$, for $\mathbf{m} \in \operatorname{ob} \mathbf{2\text{-}Cob}$ such that for all 2-cobordisms $M : \mathbf{m} \to \mathbf{n}$, the following diagram commutes:

$$\begin{tikzcd} \mathcal{A}(\mathbf{m}) \arrow[r, "\alpha_{\mathbf{m}}"] \arrow[d, "\mathcal{A}(M)"'] & \mathcal{A}'(\mathbf{m}) \arrow[d, "\mathcal{A}'(M)"] \\ \mathcal{A}(\mathbf{n}) \arrow[r, "\alpha_{\mathbf{n}}"'] & \mathcal{A}'(\mathbf{n}) \end{tikzcd}$$

We now have the category of 2-TQFTs:

Definition 3.8. **2-TQFT** is the category whose objects are 2-dimensional topological quantum field theories and whose morphisms are natural transformations between them.

4. Proof of the main theorem

In this section we discuss the equivalence between the category of TQFT's and the category of commutative Frobenius algebras. We first recall what it means for two categories to be equivalent.

Definition 4.1. A functor F from a category $\mathcal{C}$ to another category $\mathcal{D}$ is *essentially surjective* if for every $Y \in \operatorname{ob}\mathcal{D}$, there exists a $X \in \operatorname{ob}\mathcal{C}$ and an isomorphism in $\mathcal{D}$ from FX to Y.

Definition 4.2. A functor F from a category $\mathcal{C}$ to another category $\mathcal{D}$ is *full* if for every pair X, $Y \in \operatorname{ob}\mathcal{C}$, $F\colon \mathcal{C}(X,Y) \to \mathcal{D}(FX,FY)$ is surjective.

Definition 4.3. A functor F from a category $\mathcal{C}$ to another category $\mathcal{D}$ is *faithful* if for every pair X, $Y \in \operatorname{ob}\mathcal{C}$, $F\colon \mathcal{C}(X,Y) \to \mathcal{D}(FX,FY)$ is injective.

Definition 4.4. Two categories $\mathcal{C}$ and $\mathcal{D}$ are *equivalent* if there exists a functor $F\colon \mathcal{C} \to \mathcal{D}$ that is full, faithful, and essentially surjective.

We define a functor F from **2-TQFT** to the category $\mathbf{Frob}_k$ of commutative Frobenius algebras in the following way. Given a TQFT, $\mathcal{A}$, define $F(\mathcal{A})$ to be the image under $\mathcal{A}$ of the circle, that is, let $F(\mathcal{A}) = \mathcal{A}(\mathbf{1})$. This gives us a vector space $A = \mathcal{A}(\mathbf{1})$, and we need to prove:

Proposition 4.5. *A is a commutative Frobenius algebra.*

Proof. Notice how the functor $\mathcal{A}$ acts on the generating morphisms of **2-Cob**: $a\colon \mathbf{0} \to \mathbf{1}$ is sent to a map $\eta\colon k \to A$, $m\colon \mathbf{2} \to \mathbf{1}$ is sent to a map $\mu\colon A \otimes A \to A$, the identity $i\colon \mathbf{1} \to \mathbf{1}$ is sent to $id_A\colon A \to A$, $d\colon \mathbf{1} \to \mathbf{2}$ is sent to a map $\delta\colon A \to A \otimes A$, $e\colon \mathbf{1} \to \mathbf{0}$ is sent to a map $\epsilon\colon A \to k$, and finally s is sent to a map $\sigma\colon A \otimes A \to A \otimes A$. These maps will be the structure maps making A a commutative Frobenius algebra.

Since $\mathcal{A}$ is a symmetric monoidal functor, it preserves the relations among these maps. For example, the (topological) Frobenius relation

$$(i \amalg m) \circ (d \amalg i) = d \circ m = (m \amalg i) \circ (i \amalg d)$$

implies the (algebraic) Frobenius relation

$$(id_A \otimes \mu) \circ (\delta \otimes id_A) = \delta \circ \mu = (\mu \otimes id_A) \circ (id_A \otimes \delta).$$

and this is exactly the requirement that δ be a map of left and right A-modules as in the definition of a Frobenius algebra. The relations

$$m \circ (a \amalg i) = i = m \circ (i \amalg a) \quad \text{and} \quad (i \amalg e) \circ d = i = (e \amalg i) \circ d$$

imply the unit and counit conditions:

$$\mu \circ (\eta \otimes id_A) = id_A = \mu \circ (id_A \otimes \eta) \quad \text{and} \quad (id_A \otimes \epsilon) \circ \delta = id_A = (\epsilon \otimes id_A) \circ \delta.$$

Similarly, (topological) associativity and coassociativity relations imply the algebraic associativity and coassociativity conditions. These are exactly what we require for A to be an algebra and a coalgebra, and thus A is a Frobenius algebra. Finally the commutativity and cocommutativity relations imply that $\mu \circ \sigma = \mu$ and $\sigma \circ \delta = \delta$, i.e. that A is commutative Frobenius algebra. $\square$

To define F on natural transformations between TQFTs, let $\alpha\colon \mathcal{A} \to \mathcal{A}'$ be a natural transformation between two TQFTs. Then by the definition of α as a natural transformation, we get a map $\alpha_{\mathbf{1}}\colon \mathcal{A}(\mathbf{1}) \to \mathcal{A}'(\mathbf{1})$, which we can check is a map of Frobenius algebras. Since $\mathcal{A}$ a symmetric monoidal functor, this map $\alpha_{\mathbf{1}}$ determines all the other component maps, since each component $\alpha_{\mathbf{n}}$ will just be $\alpha_{\mathbf{1}}$ tensored with itself n times. Let F assign each natural transformation α to its component function $\alpha_{\mathbf{1}}$.

To check that $\alpha_{\mathbf{1}}$ is a map of Frobenius algebras, consider what it means for α to be a natural transformation. It means that given a 2-cobordism M from $\mathbf{m}$ to $\mathbf{n}$, we have $\mathcal{A}'(M) \circ \alpha_{\mathbf{m}} = \alpha_{\mathbf{n}} \circ \mathcal{A}(M)$. In particular, this is true for M equal to the generators a, m, e, and d. This tells us the following diagrams commute:

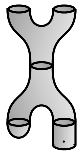
$$\begin{tikzcd} k \arrow[r, "\alpha_{\mathbf{0}}"] \arrow[d, "\mathcal{A}(a)=\eta"'] & k \arrow[d, "\mathcal{A}'(a)=\eta'"] \\ A \arrow[r, "\alpha_{\mathbf{1}}"'] & A' \end{tikzcd} \qquad \begin{tikzcd} A \otimes A \arrow[r, "\alpha_{\mathbf{2}}"] \arrow[d, "\mathcal{A}(m)=\mu"] & A' \otimes A' \arrow[d, "\mathcal{A}'(m)=\mu'"] \\ A \arrow[r, "\alpha_{\mathbf{1}}"'] & A' \end{tikzcd}$$

This shows that $\alpha_{\mathbf{1}}$ is an algebra homomorphism. Similarly, we can draw diagrams with the generators e and d that show $\alpha_{\mathbf{1}}$ is a coalgebra homomorphism. Thus, F assigns each object $\mathcal{A} \in$ **2-TQFT** to an object $F\mathcal{A} = \mathcal{A}(\mathbf{1}) \in \mathbf{Frob}_k$ and each natural transformation $\alpha\colon \mathcal{A} \to \mathcal{A}'$ to a map of Frobenius algebras $\alpha_{\mathbf{1}}\colon \mathcal{A}(\mathbf{1}) \to \mathcal{A}'(\mathbf{1})$.

To complete the proof of Theorem 1.1, we will show that the functor F is an equivalence of categories. First we check that F is essentially surjective. Given a commutative Frobenius algebra A, we want to find a TQFT $\mathcal{A}$ such that $F(\mathcal{A}) \cong A$. Define $\mathcal{A}$ on objects by sending $\varnothing$ to k, $\mathbf{1}$ to A, and $\mathbf{n}$ to $A^{\otimes n}$. Define $\mathcal{A}$ on morphisms by assigning the generating morphisms of **2-Cob** to the corresponding maps in A, that is, send a to η, m to μ, and so on. The proof of Proposition 4.5 now shows that $\mathcal{A}$ respects the relations among the generating morphisms, so this defines $\mathcal{A}$ on the whole category **2-Cob**.

Now we check that F is full. Let $\mathcal{A}$ and $\mathcal{A}'$ be two TQFTs. Let $F\mathcal{A} = A$ and $F\mathcal{A}' = A'$ be the images in $\mathbf{Frob}_k$ of $\mathcal{A}$ and $\mathcal{A}'$, i.e. let $A = \mathcal{A}(\mathbf{1})$ and $A' = \mathcal{A}'(\mathbf{1})$. We need to show that for every Frobenius algebra homomorphism $g\colon A \to A'$, there exists a natural transformation $\alpha\colon \mathcal{A} \to \mathcal{A}'$, such that $g = F\alpha$. We can construct such a natural transformation by merely defining $\alpha_{\mathbf{1}}$ to be the function $g\colon A \to A'$ and for any object $\mathbf{m}$ in **2-Cob**, defining $\alpha_{\mathbf{m}}$ to be the map g tensored with itself m times. To check naturality we need to check that for any 2-cobordism $M\colon \mathbf{m} \to \mathbf{n}$ we have $\alpha_{\mathbf{n}} \circ \mathcal{A}(M) = \mathcal{A}'(M) \circ \alpha_{\mathbf{m}}$. This follows directly from g being a Frobenius algebra homomorphism. For example let us consider the 2-cobordism

$M = (e \amalg i) \circ d \circ m$:



We want $\alpha_{\mathbf{2}} \circ \mathcal{A}(M) = \mathcal{A}'(M) \circ \alpha_{\mathbf{2}}$. This is equivalent to requiring the following diagram to commute:

$$\begin{tikzcd} A \otimes A \arrow[r, "\mu"] \arrow[d, "g\otimes g"'] & A \arrow[r, "\delta"] \arrow[d, "g"'] & A \otimes A \arrow[r, "\epsilon\otimes id_A"] \arrow[d, "g\otimes g"] & k \otimes A \arrow[d, "id\otimes g"] \\ A' \otimes A' \arrow[r, "\mu'"'] & A' \arrow[r, "\delta'"'] & A' \otimes A' \arrow[r, "\epsilon'\otimes id_{A'}"'] & k \otimes A' \end{tikzcd}$$

The above squares commute because g is a Frobenius algebra homomorphism, and so respects multiplication, comultiplication and unit maps. Since every cobordism in **2-Cob** can be expressed as the composition of generating morphisms, and these are sent to the Frobenius algebra structure maps, such diagrams will all commute.

To check that F is faithful we need to check that for every Frobenius algebra homomorphism $g\colon A \to A'$, if two natural transformations $\alpha, \alpha'\colon \mathcal{A} \to \mathcal{A}'$ map to g, then $\alpha = \alpha'$. If both α and α' map to g then $\alpha_{\mathbf{1}} = \alpha'_{\mathbf{1}} = g$. But since the rest of the component functions of α and α' are determined $\alpha_{\mathbf{1}}$ and $\alpha'_{\mathbf{1}}$ respectively, this implies $\alpha_{\mathbf{n}} = \alpha'_{\mathbf{n}}$ for each object $\mathbf{n} \in$ **2-Cob**.

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