SHORT ABSTRACTS: 2010 SUMMER VIGRE REU

All concepts mentioned in the abstracts will be carefully defined

APPRENTICE PROGRAM

Apprentices attend courses offered in the full program during week 1. A morning program of introductory material will be given in weeks 2–5. Apprentices are also welcome to attend all other courses in weeks 2–5, AND LATER. Several of the courses in the full program are designed to be accessible to apprentice participants.

Weeks 2–5: Laslo Babai

Linear algebra and combinatorics

The course will develop the usual topics of linear algebra and illustrate them on highly unusual and often striking applications to combinatorics. Emphasis will be on creative problem solving and discovery. The basic topics include permutations, determinants, linear transformations, the characteristic polynomial, Euclidean spaces, orthogonalization, the Spectral Theorem. Application areas to be highlighted include extremal set theory and the spectral theory of graphs (expansion, independence number, Shannon capacity, and more).

FULL PROGRAM

Whatever faculty have posted in past abstracts, the material actually presented has generally diverged from that originally planned so far in advance. Therefore, only quite abbreviated indications are being posted. Tentative week by week schedules have now been posted on the web page.

- 1a. Howard Masur: Weeks 1-3: hyperbolic geometry
- 1b. Benson Farb: Weeks 4-7: mapping class groups
- 2a. Dennis Hirschfeldt: Week 1: logic
- 2b. Antonio Montalban: Weeks 2 and 3: logic
- 2c. Maryanthe Malliaris: Week 4: logic
- 3a. Eva Strawbridge: Weeks 1-2: nonlinear dynamics
- 3b. Takis Souganidis: Week 3: a topic in analysis
- 3c. Gregory Lawler: Weeks 4-5: probability
- 4. Peter May: Weeks 1-8: topology
- 5. Rina Anno and Peter May: Weeks 1-8: algebra, Russian style
- 6a. Laci Babai: Weeks 3–5: the abelian sandpile model
- 6b. Laci Babai: Week 6: graphs and groups

1. Topics in geometry

1a. Weeks 1–3: Howard Masur, hyperbolic geometry

This course will be an introduction to hyperbolic geometry. In Euclidean geometry one has the famous Euclid's fifth postulate, which states that given a line and a point off the line, there is a unique line through the point and parallel to the given line. For many years mathematicians tried unsuccessfully to prove this postulate until in the 19th century geometries were developed where the postulate failed. We will study the most famous of these models, Poincare's model for hyperbolic geometry. In this model there are infinitely many lines through the point parallel to the given line. The sum of angles in a triangle is always smaller than pi and the area of a triangle is always at most pi. In contrast to Euclidean geometry where the curvature is 0, hyperbolic geometry has negative curvature. We will discuss what this means, and if time permits construct surfaces with negative curvature.

1.b. Weeks 4–7: Benson Farb, mapping class groups

This course will be an introduction to mapping class groups. This area lies at the intersection of 3-manifold theory, topology (algebraic and geometric and combinatorial), group theory, symplectic geometry, algebraic geometry, complex analysis, dynamics, etc. We will, however, focus mostly on the topological and group-theoretic aspects.

Knowledge of some point set topology and group theory would be helpful but is not necessary.

2. Topics in logic

2a. Week 1: Dennis Hirschfeldt
2b. Weeks 2 and 3: Antonio Montalban
An Introduction to Reverse Mathematics

Mathematical logic provides tools for analyzing the relative strength of mathematical theorems. This analysis often reveals surprising relationships between results in different areas, such as the tight connection between nonstandard models of arithmetic, the compactness of Cantor space, and results as seemingly diverse as the existence of prime ideals of countable commutative rings, Brouwer's fixed point theorem, the separable Hahn-Banach Theorem, and Goedel's completeness theorem, among many others. It also allows us to give mathematically precise versions of statements such as "Adding hypothesis A makes Theorem B strictly weaker", or "Technique X is essential to proving Theorem Y".

In this course we will introduce the closely related points of view of computable mathematics and reverse mathematics. Computable mathematics is concerned with questions of effectiveness. For instance, if we have a result of the form "For every A there is a B such that property P holds of A and B", then how difficult is it to compute such a B given a particular A? In reverse mathematics, we work over a weak base system (roughly corresponding to Hilbert's idea of "finitistic mathematics"), and attempt to gauge the strength of mathematical results relative to each other.

No previous knowledge of computability theory or mathematical logic will be assumed.

2c. Week 4: Maryanthe Malliaris

"Compactness" is arguably the fundamental tool in model-theoretic analysis of structures. For instance, it is a theorem proven independently by Ax and Grothendieck that every injective polynomial function from a complex vector space to itself is surjective. Proof: By compactness, one can reduce to the case of finite fields, where the assertion is obvious by counting! I'll explain what is meant by compactness, where the name comes from, and give some examples of how to use it. No prior knowledge of model theory will be assumed.

3. Topics in applied math, analysis, and probability

1a. Weeks 1–2: Eva Strawbridge, nonlinear dynamics

The study of nonlinear dynamics is intrinsically interdisciplinary. Physical and social systems such as the blinking of fireflies, chemical kinetics, neuron firing in the brain, synchronization of heart cells, economics, and even love affairs can all be modeled by nonlinear dynamical systems. Nonlinear dynamics also leads naturally to the study of fractals and chaos, a fascinating situation where a deterministic system (e.g. a set of differential equations) exhibits aperiodic behavior which may depend sensitively on initial conditions to such a degree that long-term predictions become impossible. We will study these dynamical systems geometrically and computationally to gain insight into the behavior of solutions and what this means for the systems they describe.

1b. Week 3: Takis Souganidis, a topic in analysis

3c. Weeks 4 and 5: Gregory Lawler, probability

The flow of heat can be understood very well by imagining a very large number of "heat particles" moving independently. These heat particles do random walks (or Brownian motions). One can also study heat flow without probability using a partial differential equation called (surprisingly!) the heat equation. I will discuss the relationship between probabilistic and nonprobabilistic approaches to the study of heat and discuss some of the classical ways to solve the heat equation and the Dirichlet problem (equilibrium heat distribution). Important in both approaches are the Laplacian and harmonic functions (and in probability, the related notion of a martingale). We will consider both discrete time and space (which involves diagonalization of symmetric matrices) and continuous time and space (which leads to Fourier series).

Previous exposure to probability is not required.

4. Finite topological spaces and related topics

Weeks 1–8: Peter May

There is a fascinating theory of finite topological spaces that is very little known even among research mathematicians. We will present some of the basic theory and how it relates to partially ordered sets, simplicial complexes, categories, and finite groups. We will go slowly enough that everyone can follow (I promise!!!), There are lots of unsolved problems, and maybe even some doable ones! In fact, I'll be asking for your help. Here is a part of an email (March 15, 2010), from the Publisher of the American Mathematical Society, Sergei Gelfand:
"Dear Peter, I am writing to ask if there is any progress with your text on finite topological spaces. I want to repeat that I think this will be a good and useful book, and the AMS is very interested to consider it for publication in our Student Mathematical Library series. Best regards, Sergei Gelfand"
Such a book should have lots of examples, right? Here is one: there is a space that has 6 points and the same homotopy groups as the 2-sphere (infinitely many of which are non-zero!). In fact, there is a space with $2n+2$ points that "looks like" S^n. Maybe you can find some other striking examples!!

5. Algebra, Russian style

Weeks 1–8: Rina Anno and Peter May

There are books, unknown in the west, that can be viewed as IBL texts at a high level. One is on Abel's theorem, the result that quintic equations do not generally admit solutions by radicals. It gives a geometrically focused proof, using Riemann surfaces, of something that is nowadays proven by Galois theory, using finite groups. We have in mind an experimental course with much done by the participants in which we work through these contrasting approaches to the same result.

6. Weeks 3–6: Laslo Babai, discrete mathematics

The course covers topics in discrete structures and is divided into two largely independent modules. If you missed the first module, you are still welcome to join the second.

The first module (weeks 3–5) will focus on the Abelian Sandpile Model; the second (week 6) on connections between groups and graphs. Research problems abound in both areas.

An unrelated set of puzzle problems will pepper the course.

6a. Weeks 3–5: Semigroups, graphs, matrices, avalanches, and more — the abelian sandpile model

Originating in statistical physics and nearly simultaneously and independently introduced in algebraic graph theory and in theoretical computer science in the 1990s, the Abelian Sandpile Model associates a variety of structures with a diffusion process on finite graphs. The model gives rise to a remarkably rich theory which connects the fields of graph theory, stochastic processes, commutative semigroups and groups, matrices and determinants, lattices in n-dimensional space, algorithms, number theory, discrete dynamical systems, and more. In the context of the Abelian Sandpile Model we shall learn about cyclotomic polynomials, the Jordan-Holder Theorem, the Laplacian and the matrix-tree theorem, random walks, and a lot more.

We shall learn most prerequisites along the way.

PQ: Basic linear algebra, or consent of instructor.

6b. Week 6: Groups and graphs — Symmetry in finite structures

The course will explore connections between finite groups and the combinatorial properties of finite graphs. Remarkable structural consequences of symmetry will be discovered using a minimum amount of group theory. Interactions between the finite and the infinite will be found via a limit argument; the elements of hyperbolic plane geometry will find an application to the structure of finite symmfetrical graphs. Generalizations of Rubik's cube will be studied. Combinatorial and probabilistic arguments will be applied to solve problems on finite groups.

Concepts of group theory, graph theory, probability, number theory will be developed along the way.

PQ: a first encounter with groups or consent of instructor.