# SOLUTIONS TO THE HEAT AND WAVE EQUATIONS AND THE CONNECTION TO THE FOURIER SERIES 

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#### Abstract

We discuss two partial differential equations, the wave and heat equations, with applications to the study of physics. First we derive the equations from basic physical laws, then we show different methods of solutions. Finally, we show how these solutions lead to the theory of Fourier series.


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## 1. Introduction

The study of waves in an elastic medium and heat propagation in a body are important in physics. We can represent the motion of waves and diffusion of heat as partial differential equations, PDEs, if we use basic physical laws. While there is no general method for solving PDEs, it can be shown that unique solutions to the wave and heat equations exist. In this paper we will solve the wave equation using traveling waves and superposition of standing waves. The latter method will lead us to the theory of the Fourier series. We will then turn our focus to the heat equation. First we will discuss the maximum principle, then we will solve the heat equation by finding a particular solution and then constructing the general

[^0]solution. A second method of solution to the heat equation for a bounded interval will be presented using separation of variables and eigenfunction expansion.

In this expository paper I present some basic results from the theory of PDEs. For the theorems and method of proof I rely on Partial Differential Equations by Walter A. Strauss, Fourier Analysis by Elias M. Stein and Rami Shakarchi and Partial Differential Equations by Fritz John. I present the material from a different perspective and attempt to highlight the connection to the study of physics and Fourier Analysis.

## 2. Derivation of the Wave Equation

We consider an ideal elastic string of length $l$, density $\rho$, and tension $T$. In order to model the situation, we subdivide the string into $N$ masses each of mass $\rho h$. Each mass is placed a distance of $h=l / N$ apart so that the $n^{t h}$ mass is located at $x_{n}=n h$. We assume the masses can only move in the transverse $y$ direction. We denote the vertical displacement of the $n^{t h}$ particle from equilibrium by $y_{n}(t)$.


Figure 1. A vibrating string modeled by a discrete system of masses.
We make the simplifying assumption that the force on each mass is only from the two neighboring masses. This is reasonable if we assume the transverse motion is small. We define $\theta$ as the angle the segment of string from $x_{n}$ to $x_{n+1}$ makes with the horizontal. The transverse component of the force on mass $n$ due to the tension in the string from mass $n+1$ is

$$
\begin{equation*}
T \sin \theta \approx T \theta \approx T \tan \theta=T \frac{y_{n+1}-y_{n}}{h} \tag{2.1}
\end{equation*}
$$

where we have used the assumption that the transverse motion is small to approximate sine and tangent by their first-order Taylor expansions.

Since the tension in the string on either side of the mass is a restoring force, we can use Newton's second law of motion to write the acceleration of the mass as a function of its position

$$
\begin{equation*}
\rho h \frac{\partial^{2}}{\partial t^{2}} y_{n}(t)=\frac{T}{h}\left(\left(y_{n+1}-y_{n}\right)-\left(y_{n}-y_{n-1}\right)\right) \tag{2.2}
\end{equation*}
$$

If we assume the transverse motion is smooth enough we can take the continuum limit as the spacing between the particles goes to zero and the situation reduces to the original problem of the ideal string that we wanted to model. In the limit

$$
\begin{equation*}
\frac{\left(\left(y_{n+1}-y_{n}\right)-\left(y_{n}-y_{n-1}\right)\right)}{h^{2}} \longrightarrow \frac{\partial^{2}}{\partial x^{2}} y(x, t) \text { as } h \longrightarrow 0 . \tag{2.3}
\end{equation*}
$$

We conclude the motion of the string satisfies

$$
\begin{equation*}
\frac{1}{c^{2}} \frac{\partial^{2} y}{\partial t^{2}}=\frac{\partial^{2} y}{\partial x^{2}} \quad \text { where } \quad c=\sqrt{\frac{T}{\rho}} \tag{2.4}
\end{equation*}
$$

## 3. Derivation of The Heat Equation

In a bounded region $D \subset \mathbb{R}^{3}$ let $u(x, y, z, t)$ be the temperature at a point $(x, y, z) \in D$ and time $t$, and $H(t)$ be the amount of heat in the region at time $t$. Then we can write the heat function as a volume integral

$$
\begin{equation*}
H(t)=\iiint_{D} c \rho u(x, y, z, t) d x d y d z \tag{3.1}
\end{equation*}
$$

where $c>0$ is the "specific heat" of the material and $\rho$ is the density (mass per unit volume). Since this is a physical system we will assume that $u$ is sufficiently smooth so that both $u$ and $\frac{\partial u}{\partial t}$ are continuous, and that we can differentiate with respect to time under the integral. The rate of heat flow into $D$ is

$$
\begin{equation*}
\frac{\partial H}{\partial t}=\iiint_{D} c \rho \frac{\partial u}{\partial t} d x d y d z \tag{3.2}
\end{equation*}
$$

We can use Fourier's law and the law of conservation of energy to calculate the heat flux across the boundary of $D$. Fourier's law states that heat flows from hot to cold regions proportionally to the gradient of the temperature. We know by the law of conservation of energy that $D$ can only lose heat through its boundary. Therefore the rate of change of heat energy in $D$ is equal to the heat flux across the boundary

$$
\begin{equation*}
\frac{\partial H}{\partial t}=\iint_{\partial D} \kappa(\nabla u \cdot n) d S \tag{3.3}
\end{equation*}
$$

where $\kappa$, a positive constant, is the heat conductivity. If we apply the divergence theorem we find

$$
\begin{equation*}
\iiint_{D} c \rho \frac{\partial u}{\partial t} d x d y d z=\iiint_{D} \nabla \cdot(\kappa \nabla u) d x d y d z \tag{3.4}
\end{equation*}
$$

which is true for any arbitrary region $D$. If we take the region to be arbitrarily small we find the two integrands must be identically equal, yielding the PDE

$$
\begin{equation*}
c \rho \frac{\partial u}{\partial t}=\nabla \cdot(\kappa \nabla u) \tag{3.5}
\end{equation*}
$$

which is the heat equation where $\nabla \cdot \nabla$ denotes the Laplacian of $u$.

## 4. Linearity

Definition 4.1. If $\mathcal{L}$ is a function from real numbers to real numbers then we say $\mathcal{L}$ is a linear operator if for any real-valued functions $u_{i}$ and constants $\alpha_{i}$

$$
\begin{equation*}
\mathcal{L}\left(\sum_{i=1}^{n} \alpha_{i} u_{i}\right)=\sum_{i=1}^{n} \alpha_{i} \mathcal{L}\left(u_{i}\right) . \tag{4.2}
\end{equation*}
$$

For example, the operator that takes $u$ to $\frac{\partial u}{\partial x}$ is a linear operator. If a PDE can be written in the form

$$
\begin{equation*}
\mathcal{L}\left(u_{i}\right)=a \tag{4.3}
\end{equation*}
$$

where $a$ is a constant then we call it an homogeneous linear equation. Since the solutions to these types of equations form a linear subspace, we can sum over all of the particular solutions to find the general solution. The wave equation $\frac{\partial^{2} u}{\partial x^{2}}-\frac{1}{c^{2}} \frac{\partial u^{2}}{\partial t^{2}}=0$ and the heat equation $\frac{\partial u}{\partial t}-k \frac{\partial^{2} u}{\partial x^{2}}=0$ are homogeneous linear equations, and we will use this method to find solutions to both of these equations. Furthermore, we can solve a problem with arbitrary boundary conditions if we sum
the general homogeneous solution and the solution to the problem with arbitrary boundary conditions.

## 5. Solution to Wave Equation by Traveling Waves

We will use the method employed by [1] to solve the wave equation

$$
\begin{equation*}
\frac{\partial^{2} u}{\partial x^{2}}-\frac{1}{c^{2}} \frac{\partial^{2} u}{\partial t^{2}}=0 \tag{5.1}
\end{equation*}
$$

on the real line given initial position $u(x, 0)=\phi(x)$ and velocity $\frac{\partial}{\partial t} u(x, 0)=\psi(x)$. The differential equation can be factored

$$
\begin{equation*}
\left(\frac{\partial}{\partial t}+c \frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t}-c \frac{\partial}{\partial x}\right) u(x, t)=0 . \tag{5.2}
\end{equation*}
$$

Using the method of characteristic coordinates we reduce the PDE to a simpler ordinary differential equation that we are able to solve. We define the characteristic coordinates $\xi=x+c t$ and $\eta=x-c t$ and a new function $v(\xi, \eta)=u(x, t)$. As a consequence of the chain rule,

$$
\begin{equation*}
\frac{\partial^{2} v}{\partial \eta \partial \xi}=0 \tag{5.3}
\end{equation*}
$$

This is a simple problem that can be solved by integrating twice to obtain

$$
\begin{equation*}
v(\xi, \eta)=\iint \frac{\partial^{2} v}{\partial \eta \partial \xi} d \xi d \eta=F(\xi)+G(\eta) \tag{5.4}
\end{equation*}
$$

and finally changing back to the original variables we find

$$
\begin{equation*}
u(x, t)=F(x+c t)+G(x-c t) \tag{5.5}
\end{equation*}
$$

The solution is the sum of two traveling waves, $F$ and $G$, moving in opposite directions. Using the initial conditions we can write the sum of the two traveling waves as a function of the initial position function and the initial velocity function.

$$
\begin{equation*}
F(x)+G(x)=\phi(x) \text { and } c \frac{\partial}{\partial t} F(x)-c \frac{\partial}{\partial t} G(x)=\psi(x) \tag{5.6}
\end{equation*}
$$

We can solve for $F(x)$ and $G(x)$

$$
\begin{equation*}
F(x)=\frac{1}{2}\left(\frac{1}{c} \int_{0}^{x} \psi(y) d y+\phi(x)\right)+d \text { and } G(x)=\frac{1}{2}\left(\phi(x)-\frac{1}{c} \int_{0}^{x} \psi(y) d y\right)+e \tag{5.7}
\end{equation*}
$$

where $d, e$ are constants of integration. If we use the initial condition $\phi(x)=u(x, 0)$ we can solve for the constants and find $d+e=0$. We conclude that the solution is

$$
\begin{equation*}
u(x, t)=\frac{1}{2}(\phi(x+c t)+\phi(x-c t))+\frac{1}{2 c} \int_{x-c t}^{x+c t} \psi(y) d y \tag{5.8}
\end{equation*}
$$

6. Solution To Wave Equation by Superposition of Standing Waves (Using Separation of Variables and Eigenfunction Expansion)

The wave equation on a finite interval can also be solved by the superposition of standing waves as shown in [2]. We consider standing waves on a string fixed at both ends $u(0, t)=u(l, t)=0$, with initial velocity $u_{t}(x, 0)=\psi(x)$. For simplicity we normalize velocity appropriately so that $c=1$. In order to solve the equation we write $u$ as the product of two functions, each of one variable only, to make the problem simpler to solve. Then we can use the linearity of the solution to sum over
all of the solutions in order to find the general solution. Assuming $u$ can be written as the product of one function of time only, $f(t)$ and another of position only, $g(x)$, then we can write $u(x, t)=f(t) g(x)$. We plug this guess into the differential wave equation

$$
\begin{equation*}
\frac{g_{x x}(x)}{g(x)}=\frac{f_{t t}(t)}{f(t)} \tag{6.1}
\end{equation*}
$$

In order for this equation to hold, both fractions must equal some constant, $\lambda$, that neither depends on $t$ nor on $x$. We can then set the equations equal to zero and try to find solutions.

$$
\begin{align*}
g_{x x}(x)-\lambda g(x) & =0  \tag{6.2}\\
f_{t t}(t)-\lambda f(t) & =0 \tag{6.3}
\end{align*}
$$

We will only consider values of $\lambda<0$ because these are the only values that cause $u$ to exhibit wave behavior. If $\lambda=0$ then $u(x, t)$ will be linear and we will not have oscillation. If $\lambda>0$ then $u$ will increase exponentially to infinity which from a physical standpoint does not make sense in our problem. This reduces the problem to two ordinary differential equations that can be solved by linear combinations of trigonometric functions. If we define $\lambda=-m^{2}$ then we know solutions exist of the form

$$
\begin{align*}
g(x) & =C \cos m x+D \sin m x  \tag{6.4}\\
f(t) & =A \cos m t+B \sin m t \tag{6.5}
\end{align*}
$$

The boundary condition $g(0)=g(l)=0$ implies $C=0$ and $m=\frac{n \pi}{l}$ where $n \in \mathbb{N}$. A linear combination of solutions is also a solution so the most general solution must be

$$
\begin{equation*}
u(x, t)=\sum_{n=1}^{\infty}\left(A_{n} \cos \frac{n \pi t}{l}+B_{n} \sin \frac{n \pi t}{l}\right) \sin \frac{n \pi x}{l} \tag{6.6}
\end{equation*}
$$

If we are given the initial position of the string $u(x, 0)=\phi(x)$ then

$$
\begin{equation*}
u(x, 0)=\phi(x)=\sum_{n=1}^{\infty} A_{n} \sin \frac{n \pi x}{l} \tag{6.7}
\end{equation*}
$$

and we can solve for $A_{m}$ if we multiply both sides by $\sin \frac{m \pi x}{l}$ and integrating from 0 to $l$

$$
\begin{equation*}
\int_{0}^{l} \phi(x) \sin \frac{m \pi x}{l} d x=\int_{0}^{l} \sum_{n=1}^{\infty} A_{n} \sin \frac{n \pi x}{l} \sin \frac{m \pi x}{l} d x=A_{m} \frac{l}{2} \tag{6.8}
\end{equation*}
$$

We have used the fact

$$
\int_{0}^{l} \sin \frac{n \pi x}{l} \sin \frac{m \pi x}{l}= \begin{cases}0 & \text { for } n \neq m  \tag{6.9}\\ \frac{l}{2} & \text { for } n=m\end{cases}
$$

Since each term of the sum is zero when $n \neq m$ and $\frac{\pi}{2}$ when $n=m$ we find

$$
\begin{equation*}
A_{m}=\frac{2}{l} \int_{0}^{l} \phi(x) \sin \frac{m \pi x}{l} d x \tag{6.10}
\end{equation*}
$$

We can use a similar method to solve for $B_{m}$. We differentiate $u$ with respect to time and use the same integration technique that was used to solve for $A_{m}$.

$$
\begin{align*}
u_{t}(x, 0) & =\psi(x)=\sum_{n=1}^{\infty} \frac{n \pi}{l} B_{n} \sin \frac{n \pi x}{l}  \tag{6.11}\\
\int_{0}^{l} \psi(x) \sin \frac{m \pi x}{l} d x & =\int_{0}^{l} \sum_{n=1}^{\infty} \frac{n \pi}{l} B_{n} \sin \frac{n \pi x}{l} \sin \frac{m \pi x}{l} d x=B_{m} \frac{n \pi}{2}  \tag{6.12}\\
B_{m} & =\frac{2}{n \pi} \int_{0}^{l} \psi(x) \sin \frac{m \pi x}{l} d x \tag{6.13}
\end{align*}
$$

We have written a function defined on the interval $[0, l]$ as an infinite trigonometric series. It is not clear a priori if this series converges, or what limitations we must place on the function to ensure convergence of the series. The function represents the initial shape of a string and from a physical perspective this places some very strong restrictions on the function, $\phi(x)$.

We ignore the question of convergence of the infinite trigonometric series in equation (6.7) for the moment and let $l=\pi$ as a simplification. If $\phi(x)$ is an odd function, then we can extend the function to the interval $[-\pi, \pi]$ by instead summing from $n=-\infty$ to $n=\infty$. If $f(x)$ were an even function we would hope to be able to use the exact same technique and replace the sine terms with cosine terms. Since we can write any function as the sum of an odd and even function, by applying Euler's identity we hope to be able to express any function in this manner

$$
\begin{equation*}
f(x)=\sum_{n=-\infty}^{\infty} a_{n} e^{i n x} \quad \forall x \in[-\pi, \pi] \tag{6.14}
\end{equation*}
$$

We can solve for the coefficients by multiplying both sides by $e^{-i m x}$ and integrating to get

$$
\begin{equation*}
\int_{-\pi}^{\pi} f(x) e^{-i m x} d x=\sum_{-\infty}^{\infty} \int_{-\pi}^{\pi} a_{n} e^{i n x} e^{-i m x} d x=2 \pi a_{m} \tag{6.15}
\end{equation*}
$$

Formally we expect to write each coefficient $a_{m}$ as

$$
\begin{equation*}
a_{m}=\frac{1}{2 \pi} \int_{-\pi}^{\pi} f(x) e^{-i m x} d x \tag{6.16}
\end{equation*}
$$

The series in equation (6.14) only converges if we require $\phi(x)$ to be an element of a particular space of functions. However, it is beyond the scope of this paper to delve further into this subject.

## 7. Maximum Principle and the Uniqueness of the Solution to the Heat Equation

We now turn our attention to the heat equation. Since the solution is more difficult than the wave equation we will first show the existence of a unique solution to the problem. We rely on an important property of the heat equation, the maximum principle, throughout this section. This property does not hold for the wave equation.

## Weak Maximum Principle.

Theorem 7.1. If $u(x, t)$ satisfies the heat equation $\frac{\partial u}{\partial t}=k \frac{\partial^{2} u}{\partial x^{2}}$ on the rectangle $(x, t) \in[0, l] \times[0, T]$ then the maximum value of $u(x, t)$ is equal to either the maximum value at time $t=0$ or the maximum value on either of the lateral sides $x=0$ or $x=l$.

From an intuitive perspective, the maximum principle states that if we have an insulated rod of length $l$ that can only be heated at the ends of the rod when $x=0$ or $x=l$ then the hottest temperature is equal to either the hottest temperature initially or at the ends of the rod. If one of the ends is heated, then that end is the hottest and the rest of the rod is cooler. If the hottest temperature occurs initially, not necessarily on one of the ends, then the rod will cool as time progresses.

Proof. The method of proof is due to [2]. To prove the principle we will look at a new function $v(x, t)$ that is a slight perturbation of the original function $u(x, t)$. We define $v(x, t)=u(x, t)+\epsilon x^{2}$ for $\epsilon>0$. Since $u(x, t)$ is a continuous function it must obtain its maximum on the closed rectangle $[0, l] \times[0, T]$. We denote the maximum value in the rectangle by $M$. It follows

$$
\begin{equation*}
v(x, t) \leq M+\epsilon x^{2} \text { when } t=0 \text { and } x=0 \text { or } x=l . \tag{7.2}
\end{equation*}
$$

Next we plug $v$ into the differential heat equation and from the definitions of $u$ and $v$

$$
\begin{equation*}
\frac{\partial v}{\partial t}-k \frac{\partial^{2} v}{\partial x^{2}}=\frac{\partial u}{\partial t}-k \frac{\partial^{2}}{\partial x^{2}}\left(u+\epsilon x^{2}\right)=\frac{\partial u}{\partial t}-k \frac{\partial^{2} u}{\partial x^{2}}-2 \epsilon k=-2 \epsilon k<0 \tag{7.3}
\end{equation*}
$$

The inequality $\frac{\partial v}{\partial t}-k \frac{\partial^{2} v}{\partial x^{2}}=-2 \epsilon k<0$ is essential to our proof and will be referred to as the heat inequality. Next we use calculus combined with this inequality to prove that the maximum of $v$ can neither occur on the interior of the rectangle nor on the top edge of the rectangle when $t=T$.
Suppose the maximum of $v$ occurs on the interior of the rectangle at $(a, b)$. Then

$$
\begin{equation*}
\frac{\partial v}{\partial x}(a, b)=0 \quad \frac{\partial v}{\partial t}(a, b)=0 \quad \frac{\partial^{2} v}{\partial x^{2}}(a, b) \leq 0 \tag{7.4}
\end{equation*}
$$

but this contradicts the heat inequality so the maximum cannot occur at an interior point. Next we consider the top of the rectangle when $t=T$. In this case by the same reasoning as before $\frac{\partial v}{\partial x}(a, T)=0$ and $\frac{\partial^{2} v}{\partial x^{2}}(a, T) \leq 0$. In order for $v$ to have a maximum on the boundary of the rectangle, at the point $(a, T)$ we must have

$$
\begin{equation*}
v(a, T)>v(a, T-h) \quad \forall h>0 \tag{7.5}
\end{equation*}
$$

Then by the definition of the derivative

$$
\begin{equation*}
\frac{\partial v}{\partial t}=\lim _{h \rightarrow 0^{+}} \frac{v(a, T)-v(a, T-h)}{h} \geq 0 \tag{7.6}
\end{equation*}
$$

but this implies $\frac{\partial v}{\partial t}-k \frac{\partial^{2} v}{\partial x^{2}} \geq 0$ which contradicts the heat inequality. Since $v$ is a continuous function on a compact rectangle it can only attain its maximum on the edges when $t=0$ or when $x=0$ or $x=l$. By the definition of $v$

$$
\begin{gather*}
v(x, t)=u(x, t)+\epsilon x^{2} \leq M+\epsilon l^{2}  \tag{7.7}\\
u(x, t) \leq M+\epsilon\left(l^{2}-x^{2}\right) \quad \forall \epsilon>0 \tag{7.8}
\end{gather*}
$$

Then by taking epsilon arbitrarily small we prove $u(x, t) \leq M$. The maximum of $u(x, t)$ is obtained on either the bottom side of the rectangle when $T=0$ or on the edges of the rectangle when $x=0$ and $x=l$.

We can prove a corollary by applying the maximum principle to $[-u(x, t)]$.
Theorem 7.9. If $u(x, t)$ satisfies the heat equation $\frac{\partial u}{\partial t}=k \frac{\partial^{2} u}{\partial x^{2}}$ on the rectangle $(x, t) \in[0, l] \times[0, T]$ then the minimum value of $u(x, t)$ is equal to either the minimum value at time $t=0$ or the minimum value on either of the lateral sides $x=0$ or $x=l$.

Uniqueness. We will use the maximum principle to prove the uniqueness of the solution to the heat equation.

Proposition 7.10. There is at most one solution to the heat equation

$$
\begin{equation*}
\frac{\partial u}{\partial t}-k \frac{\partial^{2} u}{\partial x^{2}}=f(x, t) \quad \text { for } x \in(0, l) \text { and } t>0 \tag{7.11}
\end{equation*}
$$

with the boundary conditions

$$
\begin{equation*}
u(0, t)=g(t) \quad u(l, t)=h(t) \tag{7.12}
\end{equation*}
$$

and initial condition

$$
\begin{equation*}
u(x, 0)=\phi(x) . \tag{7.13}
\end{equation*}
$$

Proof. Assume that $y(x, t)$ and $v(x, t)$ are two solutions to the problem and define $w=y-v$. By linearity we know $w$ must also satisfy the homogeneous heat equation, in particular it must satisfy the maximum principle. Plugging $w$ into the differential heat equation

$$
\begin{equation*}
\frac{\partial w}{\partial t}-k \frac{\partial w^{2}}{\partial x^{2}}=\frac{\partial y}{\partial t}-k \frac{\partial y^{2}}{\partial x^{2}}-\frac{\partial v}{\partial t}+k \frac{\partial v^{2}}{\partial x^{2}} \tag{7.14}
\end{equation*}
$$

Furthermore we know

$$
\begin{equation*}
w(x, 0)=g(t)-g(t)=0 \quad w(0, t)=0 \quad w(l, t)=0 . \tag{7.15}
\end{equation*}
$$

By the maximum principle, the maximum of $w$ must be equal to the maximum of $w$ on the bottom of the rectangle or on the edges which implies $w(x, t) \leq 0$. By the same reasoning the minimum principle implies that $w(x, t) \geq 0$ so $w$ must be identically zero. Therefore $y(x, t)=v(x, t)$ if $t \geq 0$.

Stability. Physically, the ability to detect differences between measurements of quantities is determined by the sensitivity of the measuring device to small changes. If we are given initial conditions sufficiently close such that our measuring device cannot distinguish them as different, then we would like the two separate solutions we formulate to the problem to remain close to each other so that our measuring device cannot distinguish the solutions for all time afterwards. We will use the maximum principle to prove the stability of the heat equation.
Definition 7.16. The solution to a PDE is uniformly stable if $u_{1}$ and $u_{2}$ are two solutions to the PDE in the interval $[0, l]$ satisfying the initial conditions $u_{1}(x, 0)=$ $\phi_{1}(x)$ and $u_{2}(x, 0)=\phi_{2}(x)$ then

$$
\begin{equation*}
\max _{x \in[0, l]}\left|u_{1}(x, t)-u_{2}(x, t)\right| \leq \max _{x \in[0, l]}\left|\phi_{1}(x)-\phi_{2}(x)\right| \quad \forall t>0 \tag{7.17}
\end{equation*}
$$

Proposition 7.18. The solution to the heat equation is uniformly stable for all time $t>0$.

Proof. As in the proof of the maximum principle we assume $y, v$ are two solutions to the problem with boundary conditions $u(0, t)=u(l, t)=0$ but $y(x, 0)=\phi_{1}(x)$ and $v(x, 0)=\phi_{2}(x)$. On the rectangle we define $w=y-v$ with $w=0$ on the lateral sides. On the bottom of the rectangle when $t=0, w=\phi_{1}-\phi_{2}$ so by the maximum principle we have

$$
\begin{equation*}
y(x, t)-v(x, t) \leq \max \left|\phi_{1}-\phi_{2}\right| \tag{7.19}
\end{equation*}
$$

and by the minimum principle we have

$$
\begin{equation*}
y(x, t)-v(x, t) \geq-\max \left|\phi_{1}-\phi_{2}\right| \tag{7.20}
\end{equation*}
$$

which means

$$
\begin{equation*}
\max _{x \in[0, l]}|y(x, t)-v(x, t)| \leq \max _{x \in[0, l]}\left|\phi_{1}(x)-\phi_{2}(x)\right| \quad \forall t>0 . \tag{7.21}
\end{equation*}
$$

## 8. Solution to the Heat Equation on the Real Line

We will solve the heat equation $\frac{\partial u}{\partial t}=k \frac{\partial^{2} u}{\partial x^{2}}$ for the whole real line, $x \in(-\infty, \infty)$ and $t>0$ given the initial value function $u(x, 0)=\phi(x)$. Five invariance properties that result from the linearity of the heat equation, whose proofs are left as an exercise, will enable us to find the solution. The method of solution is due to [2].

## Lemma 8.1.

(1) A linear combination of solutions is a solution.
(2) Any translation of a solution by a fixed number is a solution.
(3) If $u(x, t)$ is a solution then the dilated function $u(x \sqrt{a}, a t)$ is a solution for any positive $a$.
(4) A derivative of a solution is a solution.
(5) The integral $v(x, t)=\int_{-\infty}^{\infty} S(x-y, t) g(y) d y$, where $S(x, t)$ is a solution, $S(x-y, t)$ is a translation by $y$, and $g(y)$ is any function, is also a solution if the integral converges. This follows from the fact that the integral is a limit of the sum of solutions.

To solve the heat equation we will first find a particular solution $Q(x, t)$ then integrate to find the general solution. Suppose that

$$
Q(x, 0)= \begin{cases}0 & \text { if } x<0  \tag{8.2}\\ 1 & \text { if } x>0\end{cases}
$$

We define

$$
\begin{equation*}
Q(x, t)=g(p) \text { where } p=\frac{x}{\sqrt{4 k t}} \tag{8.3}
\end{equation*}
$$

Plugging $g(p)$ back into the heat equation, we get

$$
\begin{equation*}
\frac{\partial^{2} Q}{\partial x^{2}}=\frac{g^{\prime \prime}(p)}{4 k t} \text { and } \frac{\partial Q}{\partial t}=\frac{-x g^{\prime}(p)}{2 t \sqrt{4 k t}} \tag{8.4}
\end{equation*}
$$

Then we can convert the heat equation into the following ordinary differential equation

$$
\begin{equation*}
g^{\prime \prime}(p)+2 p g^{\prime}(p)=0 . \tag{8.5}
\end{equation*}
$$

This can be solved using the integrating factor $e^{\int 2 p d p}=e^{p^{2}}$. We notice that

$$
\begin{equation*}
\frac{\partial}{\partial p}\left(e^{p^{2}} g^{\prime}(p)\right)=e^{p^{2}}\left(g^{\prime \prime}(p)+2 p g^{\prime}(p)\right)=0 \tag{8.6}
\end{equation*}
$$

The function multiplied by the integrating factor in this form is much easier to solve than the original function. When we integrate both sides twice we find

$$
\begin{align*}
& \int \frac{\partial}{\partial p}\left(e^{p^{2}} g^{\prime}(p)\right) d p=\int 0 d p  \tag{8.7}\\
& g^{\prime}(p)=c e^{-p^{2}}  \tag{8.8}\\
& g(p)=\int_{o}^{p} c_{1} e^{-p^{2}} d p+c_{2} \tag{8.9}
\end{align*}
$$

where $c_{1}$ and $c_{2}$ are constants.
Next we use our assumptions about $Q(x, 0)$ to solve for the constants. Since $Q$ is not defined when $t \leq 0$ we take the limit of $Q$ as $t$ approaches zero from positive values. We have two cases:

$$
\begin{align*}
& \text { If } x>0, \quad g(p)=1 \text { then } \lim _{t \rightarrow 0^{+}} \int_{0}^{\infty} c_{1} e^{-p^{2}} d p+c_{2}=c_{1} \frac{\sqrt{\pi}}{2}+c_{2}=1  \tag{8.10}\\
& \text { If } x<0, \quad g(p)=0 \text { then } \lim _{t \rightarrow 0^{+}} \int_{0}^{-\infty} c_{1} e^{-p^{2}} d p+c_{2}=-c_{1} \frac{\sqrt{\pi}}{2}+c_{2}=0 \tag{8.11}
\end{align*}
$$

We now have two equations and two unknowns so we can solve for the constants, $c_{1}=\frac{1}{\sqrt{\pi}}$ and $c_{2}=\frac{1}{2}$. We have now solved for $Q(x, t)$ and we find

$$
\begin{equation*}
Q(x, t)=\frac{1}{\sqrt{\pi}} \int_{0}^{p} e^{-p^{2}} d p+\frac{1}{2} \text { for } t>0 \tag{8.12}
\end{equation*}
$$

By lemma 8.1 if we let $S=\frac{\partial Q}{\partial x}$ then $S=\frac{1}{\sqrt{4 \pi k t}} e^{-x^{2} / 4 k t}$ is also a solution. Furthermore the translation of $S$ by $y$ is also a solution. We get the solution

$$
\begin{equation*}
u(x, t)=\int_{-\infty}^{\infty} \frac{\partial Q}{\partial x}(x-y, t) \phi(y) d y \quad \text { for } t>0 \tag{8.13}
\end{equation*}
$$

We must check if our solution satisfies the initial condition. Since $Q$ is a solution to a physical situation we can assume that it is sufficiently smooth so that we can change the order of differentiation

$$
\begin{equation*}
u(x, 0)=\phi(x)=-\int_{-\infty}^{\infty} \frac{\partial}{\partial x}(Q(x-y, 0) \phi(y)) d y \tag{8.14}
\end{equation*}
$$

and integrate by parts to find

$$
\begin{equation*}
\phi(x)=-\left.\phi(y) Q(x-y, 0)\right|_{-\infty} ^{\infty}+\int_{-\infty}^{\infty} \phi^{\prime}(y) Q(x-y, 0) d y . \tag{8.15}
\end{equation*}
$$

If we assume the initial value function $\phi(y)$ goes to zero for all $|y|>c$ where $c$ is some large positive constant then

$$
\begin{align*}
\phi(x)= & \int_{-\infty}^{\infty} \phi^{\prime}(y) Q(x-y, 0) d y  \tag{8.16}\\
& =\int_{-\infty}^{x} \phi^{\prime}(y) d y=\left.\phi(y)\right|_{-\infty} ^{x}=\phi(x) \tag{8.17}
\end{align*}
$$

and our solution satisfies the initial conditions. The assumption that $\phi(y)$ goes to zero when $|y|$ is large is reasonable if we measure temperature on the Kelvin scale. If the heat source is located at some fixed point on the real line then at a distance
far enough away from the heat source the temperature approaches absolute zero. Our solution to the problem is

$$
\begin{equation*}
u(x, t)=\frac{1}{\sqrt{4 \pi k t}} \int_{-\infty}^{\infty} e^{-(x-y)^{2} / 4 k t} \phi(y) d y \tag{8.18}
\end{equation*}
$$

This is the solution to the heat equation on an unbounded domain, so the maximum principle can no longer be used to prove uniqueness. In fact, we find that there are infinitely many solutions to the problem. If we assume an a priori bound on the solution, then we can use the maximum principle to show that this is the unique solution. For a proof of this theorem, see [3].

## 9. Solution To Heat Equation by Separation of Variables and Eigenfunction and Expansion

We credit [2] for a second solution to the heat equation in a bounded domain $x \in(0, l)$ for all time $t>0$. If we are given initial conditions $u(0, t)=u(l, t)=0$ and $u(x, 0)=\phi(x)$ then we can separate variables and write $u$ as the product of one function of time only and one function of position only

$$
\begin{equation*}
u(x, t)=T(t) X(x) \tag{9.1}
\end{equation*}
$$

Using the same method we used to solve the wave equation, we plug in this new function to the heat equation

$$
\begin{equation*}
\frac{T^{\prime}(t)}{T(t)}=k \frac{X^{\prime \prime}(x)}{X(x)} \tag{9.2}
\end{equation*}
$$

and we find that both sides must equal a constant

$$
\begin{equation*}
T^{\prime}(t)=\lambda k T(t) \quad \text { and } \quad X^{\prime \prime}(x)=\lambda X(x) \tag{9.3}
\end{equation*}
$$

These are ordinary differential equations whose solutions are given by

$$
\begin{equation*}
T(t)=A e^{\lambda k t} \quad X(x)=C \cos m x+D \sin m x \quad \text { with } \lambda=-m^{2} \tag{9.4}
\end{equation*}
$$

The boundary conditions imply $D=0$ and $m=\frac{n \pi}{l}$. We can find a general equation for $u(x, t)$ by taking the infinite sum of all of the solutions

$$
\begin{equation*}
u(x, t)=\sum_{n=1}^{\infty} A_{n} e^{-k t\left(\frac{n \pi}{l}\right)^{2}} \sin \frac{n \pi x}{l} \tag{9.5}
\end{equation*}
$$

As in the case of the wave equation we can solve for the coefficients by using our initial value function then using the same integration technique used in the solution to the wave equation.

$$
\begin{gather*}
u(x, 0)=\phi(x)=\sum_{n=1}^{\infty} A_{n} \sin \frac{n \pi x}{l}  \tag{9.6}\\
\int_{0}^{l} \sin \frac{m \pi x}{l} \phi(x) d x=\sum_{n=1}^{\infty} A_{n} \sin \frac{m \pi x}{l} \sin \frac{n \pi x}{l} d x  \tag{9.7}\\
A_{m}=\frac{2}{l} \int_{0}^{l} \phi(x) \sin \frac{m \pi x}{l} d x \tag{9.8}
\end{gather*}
$$

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## References

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[2] Walter A. Strauss. Partial Differential Equations. John Wiley and Sons Inc. 1992.
[3] Fritz John. Partial Differential Equations. Springer-Verlag. 1982


[^0]:    Date: AUGUST 31, 2010.

