UNSTABLE FIRST-ORDER THEORIES

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Abstract. In this paper, we examine the divide between stable and unstable first-order theories in model theory. We begin by defining a stable theory and proving Shelah's theorem, which reduces the question of stability to a problem of examining a single formula. Afterwards, we will provide some applications of the stability/instability divide to other model-theoretic questions, such as the question of categoricity in power or the existence of saturated models. Along the way, we will explore other model-theoretic topics pertinent to the questions at hand, and examine numerous examples to keep our bearing.

Contents

1. Introduction

We will assume the reader is familiar with the basics of first-order logic. That is, we assume the reader understands the definition of a first-order language, and of formulas and sentences over those languages. Given a set Γ of sentences and a sentence φ, we assume the reader is familiar with the statement $\Gamma \vdash \varphi$, or that φ is a consequence of Γ. Given a language $\mathcal{L}$, we will use script letters such as $\mathcal{M}$ to denote an $\mathcal{L}$-structure, and Latin upper-case such as M to denote the underlying set or domain.

Given a structure $\mathcal{M}$ and a sentence φ, we assume the reader understands the statement $\mathcal{M} \models \varphi$, or φ is true in $\mathcal{M}$. Given a set Γ of sentences, we say that $\mathcal{M} \models \Gamma$ if for all $\varphi \in \Gamma$, $\mathcal{M} \models \varphi$.

We define $Th(\mathcal{M})$, or the *theory of* $\mathcal{M}$ as the set of all sentences true in $\mathcal{M}$. We say two models are *elementary equivalent*, written $\mathcal{M} \equiv \mathcal{N}$, if $Th(\mathcal{M}) = Th(\mathcal{N})$. Finally, two models are *isomorphic* if there is a bijective map f between them that preserves all relations and functions. Two isomorphic models are elementary equivalent.

Uncertainties can be resolved by reading the first chapter of almost any model theory book. Specifically, I recommend section 1.3 of Chang and Kiesler's Model Theory.

1.1. **Basic Information.** We use this section to cover some basic information that will set the groundwork for the remainder of the topic. Unless otherwise noted, we work in the language $\mathcal{L}$.

Definition 1.1. Let Γ be a set of sentences. We say Γ is *consistent* if $\Gamma \nvdash \varphi \wedge \neg\varphi$ for all sentences φ. By a *theory* T we mean a consistent set of sentences, closed under logical implication. We call a theory T *complete* if for any sentence φ, $\varphi \in T$ or $\neg\varphi \in T$.

We call Γ a set of axioms for T if, for any sentence φ, we have $\Gamma \vdash \varphi \iff \varphi \in T$.

We begin with three theorems, which we give without proof but are of fundamental importance to first-order logic.

Theorem 1.2. (Completeness Theorem). *Let Γ be a set of sentences. Then Γ is consistent if and only if there is an $\mathcal{L}$-structure $\mathcal{M}$ with $\mathcal{M} \models \Gamma$.*

Theorem 1.3. (Compactness Theorem). *Let Γ be a set of sentences. Then there is an $\mathcal{L}$-structure with $\mathcal{M} \models \Gamma$ if and only if for all finite sets $\Gamma_0 \subset \Gamma$, there is a structure $\mathcal{M}_0 \models \Gamma_0$.*

Theorem 1.4. (Lowenheim-Skolem Theorem). *Let $\mathcal{M}$ be an* infinite *$\mathcal{L}$-structure. Then for every cardinal $\lambda \geq |\mathcal{L}|$, there is a model $\mathcal{N}$ of cardinality λ with $\mathcal{M} \equiv \mathcal{N}$.*

The completeness theorem and L-S theorem offer a sufficient condition for a theory being complete, called the Los-Vaught test:

Proposition 1.5. *Let T be a theory over a countable language $\mathcal{L}$ that has only infinite models, and assume there is an infinite cardinal λ so that T has only one model of size λ up to isomorphism. Then T is complete.*

Proof. If T was not complete, then there would be a sentence φ where both $T \cup \{\varphi\}$ and $T \cup \{\neg\varphi\}$ were consistent. Close the above sets under logical deduction to get theories T_1 and T_2. By completeness, both T_1 and T_2 have models $\mathcal{M}_1$ and $\mathcal{M}_2$, and both of these satisfy T so they must be infinite. By the L-S theorem, then both T_1 and T_2 have models in each infinite cardinal, and their models are non-isomorphic. But then T has two non-isomorphic models in the cardinality λ. □

Throughout this paper, we will work only with complete theories (Buechler notes that we could extend our analysis to incomplete theories, but that "the benefits of the added generality are negligible"). The above proposition shows that a few canonical examples of theories are complete.

Examples 1.6. *Dense Linear Order without Endpoints*: Let $\mathcal{L}$ be the language with a single binary relation $<$. Let DLO be the theory of the dense linear order without endpoints: that is, the theory saying that $<$ is a non-reflexive total order, and that

$$(\forall x_1 < x_2)(\exists y_1\, y_2\, y_3)(y_1 < x_1 < y_2 < x_2 < y_3).$$

It is clear that any model of DLO is infinite, and it can be checked using a back-and-forth argument that any countable dense linear order is isomorphic to $\mathbb{Q}$, so DLO is complete.

Infinite Random Graph: Let $\mathcal{L}$ be a theory with a single binary relation R. Let T be the theory of an infinite random graph; i.e., let T have a sentence φ_n saying there are at least n distinct elements in any model of T, a sentence saying that R is reflexive, and for each n a sentence

$$\psi_n = (\forall x_1 \dots x_n y_1 \dots y_n)\left(\left(\bigwedge_{1\le i,j\le n} x_i \neq y_j\right) \to (\exists z)\left(\bigwedge_{i=1}^{n} \left(R(x_i,z) \wedge \neg R(y_i,z)\right)\right)\right)$$

It can be shown via a back-and-forth argument that any two infinite random graphs are isomorphic, so the theory of the infinite random graph is complete.

Equivalence Relation with Infinitely Many Infinite Equivalence Classes: Let $\mathcal{L}$ be a language with a single binary relation $\sim$. Let T be the theory saying $\sim$ is an equivalence relation where each equivalence class is infinite, and there are infinitely many equivalence classes. Then any countable model of T consists of countably many countable equivalence classes, so any two such models are isomorphic. Thus, T is complete.

2. Addition of Constants, Types, and Instability

Throughout this section, $\mathcal{L}$ is a language, T is a theory over $\mathcal{L}$, and $\mathcal{M}$ is an $\mathcal{L}$-structure.

We will first discuss adding constants to a language. Let $A \subset M$ be a set of elements in $\mathcal{M}$. We set

$$\mathcal{L}_A = \mathcal{L} \cup \{c_a : a \in A\}$$

as the language with constants added for each element of A. Then $\mathcal{M}$ induces a natural $\mathcal{L}_A$-structure which we call $\mathcal{M}_A$. Let

$$T_A = Th(\mathcal{M}_A) = \bigcup_{n<\omega} \{\varphi(c_{a_1},\dots,c_{a_n}) : \mathcal{M} \models \varphi(a_1,\dots,a_n)\},$$

or the collection of all sentences with parameters in A true in $\mathcal{M}$.

We now define types:

Definition 2.1. We call p an *n-type* if it is a consistent collection of formulas in n free variables; we call p *complete* if for any formula ψ in n free variables, either $\psi \in p$ or $\neg\psi \in p$. Given a structure $\mathcal{M}$, we say $\mathcal{M}$ *realizes* p if there are $a_1,\dots,a_n \in M$ such that for all $\psi \in p$, $\mathcal{M} \models \psi(a_1,\dots,a_n)$; otherwise, we say $\mathcal{M}$ *omits* p. We say a type p is *consistent with a theory* T if there is a model of T realizing p.

It is useful to think about a type as a potential element of a model. For instance, consider the language $\mathcal{L}$ having a single binary relation $<$ and constants c_q that name each element $q \in \mathbb{Q}$, and let the model $\mathcal{M}$ be just the model $\mathbb{Q}$, where every element is named by a constant. Let p be the 1-type

$$p(x) = \{c_q < x : q < \pi\} \cup \{c_q > x : q > \pi\}.$$

This type is consistent with $Th(\mathcal{M})$ since we can extend $\mathcal{M}$ to a model $\mathcal{N}$ whose underlying set is the real numbers. $\mathcal{N}$ is a model of $Th(\mathcal{M})$ since the relations between the constants are unchanged and it has an element realizing p, namely π.

It is often useful to examine the consistency of a type using addition of constants and the compactness theorem. For instance, let $p(x)$ be a 1-type. To test consistency, we can add a constant c to our language, and define the theory

$$T' = T \cup \{\psi(c) : \psi \in p\}$$

so that any model of T' will also be a model of T and will have an element realizing p. Now, using compactness, we need only to check that each finite subset of T' is satisfiable, so it suffices to check that

$$T \cup \{\psi(c) : \psi \in p_0\} \text{ where } p_0 \subset p \text{ is finite}$$

is satisfiable. One can therefore prove a type to be consistent with a theory by checking a single model of the theory and finding an element that realizes any finite subset of the type.

This paper will deal with stability and instability, which concern themselves with the number of complete types when we expand the language to include parameters from a given model. We therefore set:

Definition 2.2. Let T be a theory with $\mathcal{M} \models T$, and let $A \subset M$. We call p a *consistent n-type with parameters in A* if p is an n-type in the language $\mathcal{L}_A$, consistent with T_A. We write $S_n^{\mathcal{M}}(A)$ for the collection of all complete consistent n-types over A, though we may suppress the mention of $\mathcal{M}$ when the model is clear.

As it turns out, it will suffice to restrict our attention to a single formula. This requires one further definition:

Definition 2.3. Let $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_m)$, and let $\varphi(x; y)$ be a formula in $n + m$ free variables. Let T be a theory, $\mathcal{M} \models T$, and $A \subset M$. By a φ-type over A, we mean a set of the form

$$\{\varphi(x, b) : b \in B\} \cup \{\neg\varphi(x, c) : c \in C\}$$

where B, C are disjoint subsets of A^m (in other words, a φ-type over A is a collection of formulas of the form φ or $\neg\varphi$ with parameters in A). We say a φ-type p is consistent with T if there is a model of $Th(\mathcal{M}_A)$ that realizes it, and we call a type complete if for every $b \in A^m$, either $\varphi(x; b) \in p$ or $\neg\varphi(x; b) \in p$. We set $S^{\varphi}(A)$ as the set of complete consistent φ-types over A.

Having discussed types, we now turn to instability:

Definition 2.4. Let T be a theory and let λ be an infinite cardinal. We say T is *λ-unstable* if there is an $\mathcal{M} \models T$ and $A \subset M$, with $|A| = \lambda$ but $|S_n(A)| > \lambda$. We say T is *unstable* if it is unstable in every infinite cardinality.

We say φ is *λ-unstable over T* if there is a model $\mathcal{M} \models T$ with parameters $A \subset M$, and with $|S^{\varphi}(A)| > |A| = \lambda$. A formula is *unstable* if it is unstable in every infinite cardinality.

Example 2.5. The theory DLO of dense linear orders without endpoints is $\aleph_0$-unstable. Consider the model $\mathbb{Q}$, with $A = \mathbb{Q}$. We claim that $|S_1(A)| \geq 2^{\aleph_0}$. Pick any $r \in \mathbb{R}$. Then every finite subset of the type

$$\{x < q : q \in \mathbb{Q},\, r < q\} \cup \{x > q : q \in \mathbb{Q},\, r > q\}$$

is realized in $\mathbb{Q}$, so the type is consistent. Each can be extended to (at least one) complete type, and there are continuum many such types.

In fact, this shows that

$$\varphi(x; y) = \text{“}x < y\text{”}$$

is $\aleph_0$-unstable over DLO.

From the above example, it should be clear that if a theory T has a formula that is unstable in a given cardinality, then the theory itself is unstable in that cardinality, since having too many types over a single formula automatically produces too many types in total. Therefore, a theory with an unstable formula is automatically unstable.

Interestingly enough, the converse also holds: an unstable theory has an unstable formula. We will build the proof over the next two sections. The following section will discuss characterizations of formulas, each of which will turn out equivalent to instability.

3. The Order Property and Tree Property

In this section, we will build equivalent characterizations of formulas, which in the next section we will show are equivalent to instability. In particular, we will investigate properties of unstable formulas that do *not* involve types. Throughout this section, we will take $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_m)$, and $\varphi(x;\, y)$ a formula in $n + m$ free-variables. Moreover, we will always be working over a theory T; that is, all parameters in this section will be taken from a model $\mathcal{M} \models T$.

Definition 3.1. (i). We say φ satisfies the *order property* if there is a model $\mathcal{M} \models T$ and sequences $\{a_i\}_{i<\omega}$, $\{b_j\}_{j<\omega} \subset M$ such that

$$\mathcal{M} \models \varphi(a_i;\, b_j) \text{ iff } i < j.$$

(ii). Given a simply ordered set $(X, <)$, we say φ satisfies the X *order property* if there is a model $\mathcal{M} \models T$ and sequences $\{a_x\}_{x\in X}$, $\{b_y\}_{y\in X}$ so that

$$\mathcal{M} \models \varphi(a_x;\, b_y) \text{ iff } x < y.$$

To my knowledge, the terminology of 3.1.ii is not standard in the literature.

Definition 3.2. Let φ, T be as above, and α be any ordinal. We say $\Gamma(\varphi,\, \alpha)$ holds if, for some structure $\mathcal{M} \models T$, we can create a binary tree of height α satisfying the following:

1. (1) Each node η is a formula of the form $\varphi(x;\, b_\eta)$ or $\neg\varphi(x;\, b_\eta)$ for some $b_\eta \in M^m$, and the two nodes directly above the same node are negations of one another.
2. (2) For each branch $B \in 2^\alpha$ through the tree, there is an $a_B \in M^n$ so that $\mathcal{M} \models \varphi(a_B, b_\eta)$ if $\varphi(x, b_\eta)$ is in the branch B, and $\mathcal{M} \models \neg\varphi(a_B,\, b_\eta)$ if $\neg\varphi(x,\, b_\eta)$ is in B.

We note that, for finite n, the property $\Gamma(\varphi,\, n)$ can be expressed as a sentence in $\mathcal{L}$. Hodges defines the *branching index for* φ as the largest n for which $\Gamma(\varphi,\, n)$ holds; thus the branching index is uniquely defined either as some $n \in \mathbb{N}$ or as ∞.

Example 3.3. Let T be the theory of dense linear orders without endpoints and $\varphi(x;\, y) =$ "$x < y$". Then φ clearly satisfies the order property over T because we can take the model $\mathbb{Q}$ with $a_i = i$ and $b_j = j + \frac{1}{2}$.

Moreover, $\Gamma(\varphi,\, \omega)$ is satisfied. Consider $\mathcal{M} = (\mathbb{R}, <)$ as a model of DLO. Let the base node have the formula $0 < x$; let the first level have nodes $1/2 < x$ and $\neg(1/2 < x)$. Construct a tree where parameters at level n are of the form $\frac{2k+1}{2^n}$ inductively as follows. Above the node of the form

$$\frac{2k+1}{2^n} < x \text{ use parameter } \frac{4k+3}{2^{n+1}}$$

and above the node of the form

$$\neg\left(\frac{2k+1}{2^n} < x\right) \text{ use parameter } \frac{4k+1}{2^{n+1}}.$$

Thus, if a and a' are parameters along a branch, with a occurring in a formula $x < a$ and a' occurring in a formula $x \geq a'$, then $a' < a$. Given a branch B, let a_B be the supremum of all parameters occurring in formulas of the form $x \geq a$, so that a_B satisfies the type along B.

The construction of the tree in the above example is rather important, as we will use a transfinite generalization to construct trees of arbitrary height in the (2) $\implies$ (3) step of theorem 3.5 below.

Example 3.4. On the other hand, let $\mathcal{L}$ be a language with a single relation $\sim$ and let T' be the theory that says $\sim$ is an equivalence relation, each equivalence class is infinite, and there are infinitely many classes. Let

$$\psi(x;\, y) = x \sim y.$$

We claim that ψ satisfies neither the order property nor $\Gamma(\psi, \omega)$. Assume there were sequences $\{a_i\}_{i<\omega}$, $\{b_j\}_{j<\omega}$ satisfying the order property. Then we must have

$$\psi(a_2;\, b_1) \text{ and } \psi(a_4;\, b_1),$$

so $a_2 \sim b_1 \sim a_4$. But we must also have

$$\neg\psi(a_2;\, b_3) \text{ and } \psi(a_4;\, b_3)$$

so $a_2 \not\sim b_3 \sim a_4$, a contradiction. Therefore, ψ cannot satisfy the order property over T'.

To see that $\Gamma(\psi,\, \omega)$ is not satisfied, pick any desired tree as in definition 3.2 with parameters in any model of T'. We will show that even the paths through the first three levels cannot form a consistent type. One of the formulas on level 2 is of the form $x \sim a_1$. Consider the two nodes immediately above this formula. One of them is $x \sim a_2$ and the other is $x \not\sim a_2$. If $a_2 \sim a_1$, then the type $\{x \sim a_1,\, x \not\sim a_2\}$ is inconsistent, and if $a_2 \not\sim a_1$, then the type $\{x \sim a_1,\, x \sim a_2\}$ is inconsistent, so one of the paths is an inconsistent type.

The above examples motivate the following theorem:

Theorem 3.5. *Let T be a theory and $\varphi(x;\, y)$ be a formula. Then the following are equivalent over T:*

(1) φ satisfies the order property.
(2) φ satisfies the X order property for any simply ordered set X.
(3) $\Gamma(\varphi,\, \alpha)$ holds for all cardinals α.
(4) $\Gamma(\varphi,\, \omega)$ holds.

Proof. We will show (1) $\implies$ (2) $\implies$ (3) $\implies$ (4) $\implies$ (1).

(1) $\implies$ (2): Let X be a simply-ordered set and let $\mathcal{L}'$ the be language

$$\mathcal{L} \cup \{c_x\}_{x\in X} \cup \{d_y\}_{y\in X}.$$

Consider the $\mathcal{L}'$ theory

$$T' = T \cup \{\varphi(c_x;\, d_y) :\, x < y\} \cup \{\neg\varphi(c_x;\, d_y) : x \geq y\}.$$

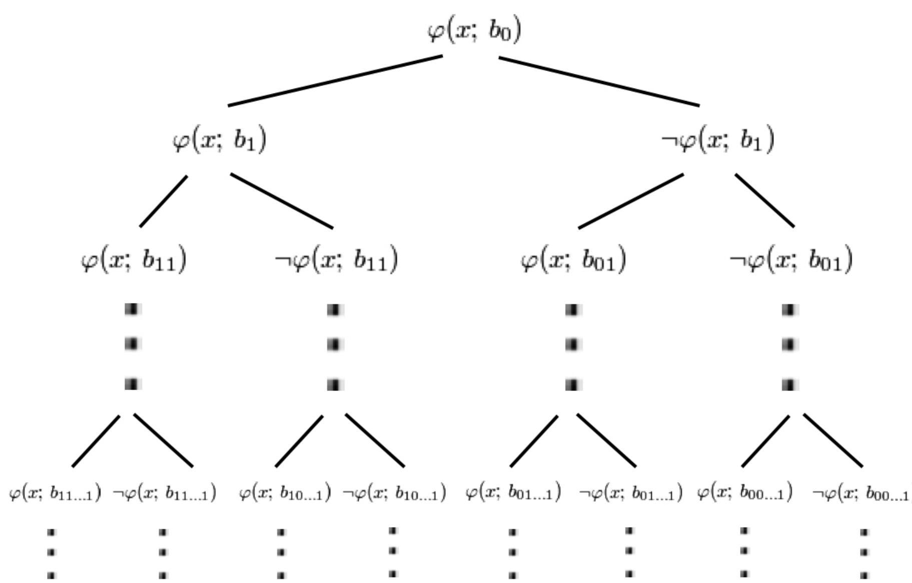
To show T' is consistent, we use compactness to show any finite subset is consistent. Any finite subset T'_0 references only finitely many of the added constants; let these constants have subscrips $x_1, \ldots, x_n$ with $x_1 < \ldots < x_n$. Then we have

$$T'_0 \subset T \cup \{\varphi(c_{x_i};\, d_{x_j}) : i < j\} \cup \{\varphi(c_{x_i};\, d_{x_j}) :\, i \geq j\}.$$

To see that T'_0 is consistent, consider the model $\mathcal{M}$ where we take the parameters $\{a_i\}$, $\{b_j\}$ that fulfill the order property, and let $\mathcal{M}'$ be the $\mathcal{L}'$ structure where we realize $c_{x_i}^{\mathcal{M}} = a_i$, $d_{x_j}^{\mathcal{M}} = b_j$. Because φ satisfies the order property with parameters a_i, b_j, then $\mathcal{M}' \models T'_0$, completing the proof.

(2) $\implies$ (3): Let α be any cardinal, let $X = 2^\alpha$ (i.e. the set of α-indexed strings of 0 and 1's), and given $x, y \in 2^\alpha$, let $x < y$ if there is an ordinal $\beta < \alpha$ where $x[\beta] < y[\beta]$ and $x\,|_{<\beta} = y\,|_{<\beta}$ (that is, if the two are the same on a substring, after which y takes the value 1 and x takes the value 0). Let $\{a_x\}_{x \in X}$, $\{b_x\}_{x \in X}$ satisfy the X order property.

Construct a tree of height α using transfinite induction as follows. At the base node, put the formula $\varphi(x;\, b_0)$, and on the first level, put the formulas $\varphi(x;\, b_1)$ and $\neg\varphi(x;\, b_1)$ (where having the expansion terminate means that it is followed by all 0's). We will construct a tree so that at all nodes, the expansion terminates with a 1 followed by a string of 0's.



At all successor ordinals, assume the previous node is of the form $\varphi(x;\, b_{\sigma*1})$ (where "$*$" denotes the operation of concatenation); then let the next two nodes be of the form $\varphi(x;\, b_{\sigma*01})$ and $\neg\varphi(x;\, b_{\sigma*01})$. If the previous node is of the form $\neg\varphi(x;\, b_{\sigma*1})$; then let the next two nodes be of the form $\varphi(x;\, b_{\sigma*11})$ and $\neg\varphi(x;\, b_{\sigma*11})$.

On the other hand, assume β is a limit ordinal. Pick a branch and, for $\gamma < \beta$ define $\sigma(\gamma)$ as the string of length γ so that the parameter along the branch at height γ is $b_{\sigma(\gamma)*1}$. Then $\sigma(\gamma) \prec \sigma(\gamma')$ if $\gamma < \gamma'$. Let $\bar{\sigma}$ be the union of all the $\sigma(\gamma)$.

Let the nodes at level β (i.e, at the end of the branch) be of the form $\varphi(x; b_{\bar{\sigma}*1})$ and $\neg\varphi(x; b_{\bar{\sigma}*1})$.

We must show that there is an a_B realizing each branch B as a type. Given a branch, note that if $x_1, x_2 \in X$ are such that b_{x_1} occurs in a positive instance of φ and b_{x_2} occurs in a negative instance of φ, then $x_1 < x_2$. Therefore, let x^* be the least upper bound of all those $x \in X$ for which b_x occurs positively. Letting a_B be a_{x^*} from the order property finishes the proof, because $x^* < x'$ for all x' occurring in negative instances of φ.

(3) $\implies$ (4) is obvious.

(4) $\implies$ (1). Call an n-ladder a collection $(a_1, \dots, a_n,\, b_1, \dots, b_n)$ satisfying the order property. By compactness, it suffices to find an n-ladder for all n. We will show that if we can find a tree of height $2^{n+1} - 2$, then we can find an n-ladder. The proof will rely heavily on the combinatorics of binary trees.

We call a map $f : 2^{<m} \to 2^{<n}$ a tree map if, given $\sigma, \tau \in 2^{<m}$, we have $\sigma \prec \tau$ iff $f(\sigma) \prec f(\tau)$. Given a set N of nodes in $2^{<n}$, we say N contains an m-tree if there is a tree map $f : 2^{<n} \to 2^{<m}$ where the images of all nodes are in N. We begin with:

Lemma 3.6. *Let H be an $n+k$ tree, and partition the nodes into sets N, P. Then either N contains an n-tree or P contains a k-tree.*

Proof. We work by induction. The case $n = k = 0$ is clear. For $n + k \geq 1$, assume WLOG the base node is in N. Let H_0 be the half-tree above the node (0) and H_1 be above the node (1). Then H_i is a tree of height $n + k - 1$, so either $N \cap H_i$ contains an $n-1$ tree or $P \cap H_i$ contains a k tree. If the latter holds for either i we are done since then P has a k-tree. Otherwise, both half-trees have an $n - 1$-tree with nodes in N; combining them and putting in the base node gives us an n-tree contained in N. □

We now complete the proof. Assume we have a tree of size $2^{n+1} - 2$. We will show, by induction on $n - r$, that the following situation S_r holds: there are

$$a'(0), b'(0), \dots a'(q-1), b'(q-1), H, a'(q), b'(q), \; \dots, a'(n-r-1),\; b'(n-r-1) \tag{3.7}$$

satisfying

1. H is a $2^{r+1} - 2$ tree for φ
2. For $i, j < n - r$, we have $\mathcal{M} \models \varphi(a'(i),\, b'(j))$ iff $i \leq j$
3. For any node b of H, we have $\mathcal{M} \models \varphi(a'(i), b)$ iff $i < q$.
4. For any branch a of H, we have $\mathcal{M} \models \varphi(a, b'(j))$ iff $j \geq q$.

(It is best to think of the induction as building the ladder using nodes remaining in the tree H at each stage).

The base case S_n simply says there is an $2^{n+1} - 2$-tree for φ. In the S_1-case, we have a 2-tree for φ, so we have a node b and a branch a with $\mathcal{M} \models \varphi(a, b)$, so we may put a, b in that order between $b'(q-1)$ and $a'(q)$ to get a ladder.

It remains to show the inductive step. For a branch a of H, let $H(a)$ be the collection of nodes $b \in H$ with $\mathcal{M} \models \varphi(a, b)$. We consider two cases:

Case 1: There is a branch a where $H(a)$ contains a $2^n - 1$-tree. Pick node b in the branch a and let $H' \subset H(a)$ be the $2^n - 2$ tree not containing b; then replacing H with a, b, H' in that order in 3.7 satisfies S_{r-1}.

Case 2: There is no branch a where $H(a)$ contains a $2^n - 1$-tree. Let b be the bottom node of H, let a be any branch of the half-tree above (0), and let N be the set of nodes in the half-tree above (0). Since $N \cap H(a)$ contains no $2^n - 1$-tree, then $N \setminus H(a)$ contains an $2^n - 2$-tree H' for φ. Replace H by H', a, b to see that S_{r-1} is satisfied.

This completes the proof. □

4. Definable Types and Stability of Formulas

Having examined numerous equivalent characterizations for formulas over a theory, we now relate them all to instability. Specifically, the goal of this section is to show that a formula is unstable if and only if one of the above 4 equivalent properties in theorem 3.5 holds.

Theorem 4.1. *If $\Gamma(\varphi, \alpha)$ holds for all infinite cardinals α over a theory T, then φ is unstable over T.*

Proof. Let λ be any cardinal, let α be minimal so that $2^\alpha > \lambda$ (such a λ exists since the cardinals are well-ordered, and we know $\alpha \leq \lambda$). Since $\Gamma(\varphi, \alpha)$ holds, we can create a tree as desired. Each branch extends to a maximal consistent φ-type, and any two distinct branches are inconsistent, so we have $2^\alpha > \lambda$ many consistent types over the parameters given.

We now need to count the parameters. There is a bijection between the parameters used and the nodes of the tree, and the number of nodes is:

$$\sum_{\beta < \alpha} 2^\beta \leq \alpha \cdot \lambda \leq \lambda^2 = \lambda.$$

Therefore, there are $2^\alpha > \lambda$ many types over the parameters in the tree, of which there are at most λ, so φ is λ-unstable. □

We now must show the other direction; that if a theory is unstable, then there is a formula satisfying the four equivalent properties of theorem 3.5. The main tool for this proof will be definability of types.

Definition 4.2. Let p be a complete n-type over some set of parameters $A \subset M$, and let $\varphi(x;\, y)$ be an $\mathcal{L}$-formula in $n+m$ free variables. We say $d_p\varphi$ is a *φ-definition* of p if it is an $\mathcal{L}_A$-formula in m free variables satisfying

$$\text{for all } b \in A^m, \text{ we have } \varphi(x,\, b) \in p \text{ iff } A \models d_p\varphi(b).$$

We say p is *definable* if there is a φ-definition of p for all φ, whence we call d_p a *definition schema*.

We note that a definition schema is a map from the collection of $\mathcal{L}$-formulas in $n + m$ free variables to the collection of $\mathcal{L}_A$-formulas in m free variables.

Example 4.3. Consider the theory DLO with model $\mathcal{M} = (\mathbb{Q}, <)$ and parameter set $A = \mathbb{Q}$. Let $p(x)$ be the type of a positive infinitessimal element; that is, for all positive q we have "$x < q$" $\in p(x)$ and for all other q we have "$x > q$" $\in p(x)$. It is clear that this type is finitely satisfiable and therefore consistent.

Let $\varphi(x;\, y) =$ "$x < y$". Then $d_p\varphi(y) =$ "$y > 0$" is a φ-definition for p since the lone parameter comes from $\mathbb{Q}$ and we have "$x < b$" $\in p(x)$ iff $b > 0$.

On the other hand, consider the type where "$x < q$" $\in p(x)$ if $q > \pi$ and "$x > q$" $\in p(x)$ if $x > \pi$. This type is not definable. For assume there were a φ-definition for p, where as above $\varphi(x; y) =$ "$x < y$". Then we would need $\mathcal{M} \models d_p\varphi(b)$ iff $b < \pi$. It can be shown that any formula is equivalent over DLO to a quantifier-free formula; then $d_p\varphi(y)$ would be equivalent to a finite combination of formulas of the form $y < q$ and $y > q$, where $q \in \mathbb{Q}$ is some parameter, in disjunctive normal form. But no such combination of such formulas can be equivalent to $y < \pi$ (or, for that matter, $y < \alpha$ for any irrational α).

We will now show that, if there is no sentence φ so that $\Gamma(\varphi, \omega)$ holds, then all types over T are definable. We begin with some definitions.

Definition 4.4. Let φ be as in section 3 and let $\theta(x)$ be a formula using parameters from M and having $x = (x_1, \ldots, x_n)$ as free variables. We say $\Gamma(\varphi, \theta, n)$ holds if there is a tree of height n for φ using parameters from M, all of whose branches satisfy θ.

We remark that $\Gamma(\varphi, \theta, n)$ can be expressed in an $\mathcal{L}$-sentence with parameters in M. Hodges calls the maximal n for which $\Gamma(\varphi, \theta, n)$ holds the *relativized branching index* for φ and θ. It is clear that $\Gamma(\varphi, \theta, n)$ implies $\Gamma(\varphi, n)$.

Lemma 4.5. *Assume $\mathcal{M} \not\models \Gamma(\varphi, \psi, n + 1)$. Let $c \in M^m$, let $\theta_c(x) = \psi(x) \wedge \varphi(x, c)$, and let $\theta'_c(x) = \psi(x) \wedge \neg\varphi(x, c)$. Then either $\mathcal{M} \models \neg\Gamma(\varphi, \theta_c, n)$ or $\mathcal{M} \models \neg\Gamma(\varphi, \theta'_c, n)$.*

Proof. Assume we could find trees of height n whose nodes are (positive or negative) instances of φ, all of whose branches satisfy $\psi(x) \wedge \varphi(x, c)$, and another tree all of whose branches satisfy $\psi(x) \wedge \neg\varphi(x, c)$. Then we can create an $n+1$-tree with c at the bottom node, and using the two above trees as half-trees; this is a tree for φ, all of whose nodes satisfy ψ. $\square$

Proposition 4.6. *If $\Gamma(\varphi, \omega)$ does not hold for any φ, then all complete types p are definable.*

Proof. By compactness, if $\Gamma(\varphi, k)$ holds for all n, then $\Gamma(\varphi, \omega)$ holds, so there is an n for which $\Gamma(\varphi, k)$ does not hold. Therefore, the relativized branching index of φ and ψ is finite for all ψ; pick $\psi \in p$ so that the relativized branching index with φ is minimal, and let it be n. For $b \in A^m$, define $\theta_b(x) = \psi(x) \wedge \varphi(x, b)$ and let $d_p\varphi(b) = \Gamma(\varphi, \theta_b, n)$.

We claim $d_p\varphi$ is a φ-definition for p. First, if $\varphi(x, b) \in p$, then $\theta_b(x) = \psi(x) \wedge \varphi(x, b) \in p$, and since we picked ψ to have minimal relative branching index with φ, we must have $\Gamma(\varphi, \theta_b, n)$ holding, whence $\mathcal{M} \models d_p\varphi(b)$. Second, if $\varphi(x, b) \notin p$, then its negation is in p, and thus so is $\theta'_b(x) = \psi(x) \wedge \neg\varphi(x, b)$. Thus, the same argument as above shows $\Gamma(\varphi, \theta'_b, n)$ holds. If $\Gamma(\varphi, \theta_b, n)$ also held, then $\Gamma(\varphi, \psi, n+1)$ would hold by lemma 4.5. This contradicts that the relativized branching index of φ and ψ is precisely n, so we must have $\mathcal{M} \not\models \Gamma(\varphi, \theta_b, n) = d_p\varphi(b)$, so that $d_p\varphi$ is a φ-definition for p. $\square$

Finally, we can show that this implies stability:

Proposition 4.7. *Assume all types over T are definable. Then T is stable in arbitrarily high cardinals.*

Proof. Let λ be any cardinal; we will show that T is $\lambda^{|\mathcal{L}|}$-stable. Let A be a parameter set of size $\lambda^{|\mathcal{L}|}$. Since all types are definable, and distinct types must have distinct definition schema, then it suffices to count the number of definition schema. Since a definition scheme is a map from the collection of $\mathcal{L}$-formulas, of which there are $|\mathcal{L}|$, to the collection of $\mathcal{L}_A$-formulas, of which there are $|A| = \lambda^{|\mathcal{L}|}$, then the total number of definition schema is

$$\left(\lambda^{|\mathcal{L}|}\right)^{|\mathcal{L}|} = \lambda^{|\mathcal{L}|\cdot|\mathcal{L}|} = \lambda^{|\mathcal{L}|} = |A|.$$

Thus, T is stable in $\lambda^{|\mathcal{L}|}$ for all λ. □

Therefore, we have shown the following theorem:

Theorem 4.8. *The following are equivalent:*

(1) T is unstable.
(2) There is an undefinable type over T.
(3) There is an unstable formula φ over T.
(4) There is a formula φ satisfying $\Gamma(\varphi, \omega)$ over T.
(5) There is a formula φ satisfying the order property over T.

Proof. (3) $\implies$ (1) was discussed in section 2. (1) $\implies$ (2) (or rather, $\neg(2) \implies \neg(1)$) is proposition 4.7. (2) $\implies$ (4) was proposition 4.6. (4) $\implies$ (3) is a combination of 4.1 and 3.5, and then (4) $\iff$ (5) by 3.5. □

There is little to no reason that this result should be true offhand; the existence of many types over parameter sets of all cardinalities is a 'global' property giving information about many different models. The order property, however, depends only on parameters from a single model; it is a 'local' property.

5. $\aleph_0$-Stable Theories

The goal of this section is to prove that an $\aleph_0$-stable theory in a countably language is stable in all cardinalities. As above, the bulk of the proof will rely on building a tree to create 'too many' types over a set of parameters.

Lemma 5.1. *Let $|A| = \kappa$. Given a formula φ in n free variables the language $\mathcal{L}_A$, define*

$$[\varphi] = \{p \in S_n(A) : \varphi \in p\}$$

and let κ be an infinite cardinal. Given φ with $|[\varphi]| > \kappa$, there is a formula ψ in n free variables with $|[\varphi \wedge \psi]|, |[\varphi \wedge \neg\psi]| > \kappa$.

Proof. Suppose not. Define $p = \{\psi : [\psi \wedge \varphi] > \kappa\}$; then for each formula ψ, either $\psi \in p$ or $\neg\psi \in p$, but not both. We will show p is finitely satisfiable; let $\psi_1, \ldots, \psi_m \in p$. We see

$$\begin{aligned} &\neg(\psi_1 \wedge \ldots \wedge \psi_m) \in p \\ \implies &|[\varphi \wedge (\neg\psi_1 \vee \ldots \vee \neg\psi_m)]| > \kappa \\ \implies &|[(\varphi \wedge \neg\psi_1) \vee \ldots \vee (\varphi \wedge \neg\psi_m)]| > \kappa \end{aligned}$$

so that at least one of the $[\varphi \wedge \neg\psi_k]$ has cardinality $> \kappa$, a contradiction. It follows that $\psi_1 \wedge \ldots \wedge \psi_m \in p$, so that $\varphi \wedge \psi_1 \wedge \ldots \wedge \psi_m$ is contained in more than κ-many types (and therefore in *some* type) and is therefore satisfiable. Thus, p is consistent.

Moreover, we have

$$[\varphi] = \{p\} \cup \bigcup_{\psi \notin p} [\varphi \wedge \psi].$$

Since there are κ many total formulas, then we have the union of at most κ-many sets, each of size at most κ, so $[\varphi] \leq \kappa$, a contradiction. □

Theorem 5.2. *Let T be an $\aleph_0$-theory in a countable language. Then T is stable in all cardinalities.*

Proof. Assume T is κ-unstable, let $\mathcal{M} \models T$, $A \subset M$ with $|S_n^{\mathcal{M}}(A)| > |A| = \kappa$.

We will construct a binary tree of formulas with parameters from A inductively as follows where each node φ satisfies $|[\varphi]| > \kappa$ as follows:

Step 0: Since there are κ-many formulas with parameters in A and $|S_n(A)| > \kappa$, then there is a φ_0 with $|[\varphi_0]| > \kappa$. Let φ_0 be the base node.

Step n: Assume we have an $(n-1)$-tree. For each terminal node φ, pick ψ so that $|[\varphi \wedge \psi]|$, $|[\varphi \wedge \neg\psi]| > \kappa$; let $\varphi \wedge \psi$ and $\varphi \wedge \neg\psi$ be the subsequent nodes.

This yields a tree of height $\aleph_0$, with formulas φ_σ at node σ, satisfying:

(1) If $\tau \prec \sigma$, then $\varphi_\sigma \vdash \varphi_\tau$
(2) $\varphi_{\sigma * i} \vdash \neg\varphi_{\sigma*(1-i)}$, and
(3) $|[\varphi_\sigma]| > \kappa$.

Let B be any branch of the tree, and let B_0 be a finite subset. Let σ^* be the longest string in B_0. Then $\varphi_{\sigma^*} \vdash \varphi_\sigma$ for all other $\sigma \in B_0$, and φ_{σ^*} is contained in infinitely many consistent types, hence it must be consistent. Thus, each branch forms a finitely consistent, and therefore consistent, type. Moreover, since any two branches split and splitting nodes contradict one another, all such types must be distinct. Therefore, we have $2^{\aleph_0}$ many consistent types over the parameters present in the tree. But since we have at most countably many formulas, each of which has finitely many parameters, then the set of parameters A_0 is countable, but $|S_n^{\mathcal{M}}(A_0)| > |A_0| = \aleph_0$, violating $\aleph_0$-stability. $\square$

Thus, over a countable language, any $\aleph_0$-stable theory is stable in all cardinalities κ. Moreover, we have shown that *any* stable theory is stable in all cardinalities κ with $\kappa^{\aleph_0} = \kappa$. Shelah showed that these were the only possibilities over a countable language; that is, either a theory is unstable, or it is stable in all cardinalities κ with $\kappa^{\aleph_0} = \kappa$, or it is stable in all cardinalities. He dubbed those theories stable in all cardinalities as *superstable.*

6. Consequences of Stability to Model Theoretic Questions

Having examined criteria equivalent to stability, we should answer the question: why is stability important? Why should model theorists care about the stability/instability divide?

In addition to providing a useful means for categorizing theories, the stability/instability divide helps answer two very important questions in model theory. First, the question of categoricity: when does a theory T have a unique model in some cardinality, up to isomorphism? Second, the question of the existence of saturated models: when does T have a model realizing 'as many types as possible'?

First, we will show that if T is unstable, it is not categorical in any uncountable cardinality. We begin by constructing a model $\mathcal{M}$ such that that realizes at most $|A|$ many types for any subset $A \subset M$.

Definition 6.1. Let X be any ordered set. We call $\{a_x : x \in X\}$ a *sequence of indiscernibles* if for any $x_1 < \ldots < x_n$ and $y_1 < \ldots < y_n$, the tuples $(a_{x_1}, \ldots, a_{x_n})$ and $(a_{y_1}, \ldots, a_{y_n})$ satisfy the same type.

Proposition 6.2. *Let T be a theory with an infinite model and X an ordered set. Then there is a model of T with an X-indexed sequence of indiscernibles.*

Proof. Let $\mathcal{L}'$ be the language with added constants $\{c_x : x \in X\}$, and let

$$T' = T \cup \bigcup_{\varphi} \{\varphi(c_{x_1}, \ldots, c_{x_n}) \leftrightarrow \varphi(c_{y_1}, \ldots, c_{y_n}) : x_1 < \ldots < x_n,\, y_1 < \ldots < y_n\}$$

(where the latter union is over all $\mathcal{L}$-formulas φ).

We must show T' is consistent. Using compactness, it suffices to show that any finite subset is satisfiable. Let T_0 be a finite subset of T'. Let $x_1, \ldots, x_n$ be the elements of x mentioned in T_0 and let $\varphi_1, \ldots, \varphi_m$ be the formulas mentioned.

Let $\mathcal{M}$ be an infinite model of T. Consider

$$D = \{a \in M^n : \mathcal{M} \models \varphi_1(a)\}.$$

Then either D or its complement is infinite; pick the infinite component, and call it D_1. Then either

$$\{a \in D_1 : \mathcal{M} \models \varphi_2(a)\}$$

or its complement is infinite; call the infinite set D_2. Repeat until we get a set D_m; then any tuple in D_m satisfies T_0, completing the proof. □

Definition 6.3. Let $\mathcal{L}$ be a language and T a theory over $\mathcal{L}$. We say T has *built in Skolem functions* if for any formula φ in n free variables, there is an n-ary function f_φ such that

$$(\forall v_1, \ldots, v_n)(\exists x \varphi(x, v_1, \ldots v_n) \rightarrow \varphi(f_\varphi(v_1, \ldots, v_n), v_1, \ldots, v_n) \in T.$$

That is, if there is a witness to $\exists x \varphi(x, a_1, \ldots, a_n)$, then $f_\varphi(a_1, \ldots, a_n)$ outputs such a witness.

If T has built-in Skolem functions, $\mathcal{M} \models T$, and $A \subset M$, then we define the Skolem Hull $H(A)$ to be the smallest set closed under Skolem functions containing A. The nature of Skolem functions ensures that the collection of elements in $H(A)$ forms an *elementary submodel* $\mathcal{N}$ of $\mathcal{M}$ (written $\mathcal{N} \succ \mathcal{M}$); that is

$$\text{for all } a = (a_1, \ldots, a_n) \in H(A), \mathcal{N} \models \varphi(a) \iff \mathcal{M} \models \varphi(a)$$

This can be shown by induction on the length of the formula.

Exercise 6.4. Let $\{\mathcal{M}_\beta\}_{\beta<\kappa}$ be a sequence of models so that $\mathcal{M}_\beta \prec \mathcal{M}_\alpha$ for $\beta < \alpha$, and let $\mathcal{M} = \bigcup_{\beta<\kappa} \mathcal{M}_\beta$. Then $\mathcal{M}_\beta \prec \mathcal{M}$ for all β.

Proposition 6.5. *Let T be a theory in a countable language $\mathcal{L}$, and κ a cardinal. Then there is a model $\mathcal{M} \models T$ with $|M| = \kappa$ such that for all $A \subset M$, $\mathcal{M}$ realizes at most* $\max\{|A|, \aleph_0\}$ *many types over A.*

Proof. Let $\mathcal{L}'$ be the language closed under the addition of Skolem functions, and let T' be the corresponding $\mathcal{L}'$-theory. Let X be a set of indiscernibles of size κ indexed by a well-ordered set I in some model $\mathcal{N}'$ for T'; let $\mathcal{M}'$ be the model formed by the Skolem hull of X; we will consider the model $\mathcal{M}$, or the structure $\mathcal{M}'$ reduced to the original language $\mathcal{L}$.

Pick any $A \subset M$. Since each $a \in A$ is in the Skolem hull of X, each a is the value of some term in finitely many elements of X. Therefore, A is contained in the Skolem hull of some $Z \subset X$, where $|Z| = \max\{|A|, \aleph_0\}$; we may assume WLOG that A actually is the Skolem hull. Moreover, every element of A is some term from Z. Therefore, two tuples elements satisfy the same type over A iff they satisfy the same type over Z.

How many types can be realized? Let $x = x_1 < \ldots < x_m$ and $y = y_1 < \ldots < y_m$ be two sequences from X. Call them equivalent over Z if for all $z \in Z$, we have

$$y_i < z \text{ iff } x_i < z \text{ and } y_i = z \text{ iff } x_i = z.$$

Because X is a sequence of indiscernibles, given two tuples of terms $(t_1(x), \ldots, t_n(x))$ and $(t_1(y), \ldots, t_n(y)) \in M^n$, then if x and y are equivalent over Z, then the two tuples realize the same type. Therefore, to complete the proof, it suffices to show that the number of equivalence classes of sequences from X is at most $\max\{|Z|, \aleph_0\}$. For each $x \in X$, define

- $x' =$ the least $z \in Z$ with $x < z$, if one exists, and
- $x' = \infty$ otherwise.

then sequences x and y are equivalent if $x'_i = y'_i$ for all i; thus there are $\max\{|Z|, \aleph_0\}$ many equivalence classes, completing the proof. □

Lemma 6.6. *Let T be a theory, $\mathcal{M} \models T$ and $A \subset M$. Let $\Gamma \subset S_n(A)$. Then there is a model $\mathcal{N} \succ \mathcal{M}$ that realizes all $p \in \Gamma$.*

Proof. By exercise 6.4, it suffices to find a model $\mathcal{N} \succ \mathcal{M}$ satisfying a single type. We will then have a chain of elementary extensions

$$\{\mathcal{M}_\beta\}_{\beta < |\Gamma|} \text{ where } \mathcal{M}_\beta \prec \mathcal{M}_{\beta+1};$$

then $\mathcal{M}^* = \bigcup \mathcal{M}_\beta$ will be an elementary extension of each $\mathcal{M}_\beta$ realizing all types in Γ.

Let $p \in \Gamma$, let $\Delta = Th(\mathcal{M}_M) \cup p$, and let $\Delta_0 \subset \Delta$ be finite. Then WLOG Δ_0 is a single formula

$$\varphi(x_1, \ldots, x_n, a_1, \ldots, a_m) \wedge \psi(a_1, \ldots, a_m, b_1, \ldots, b_k)$$

where $a_i \in A$, $b_j \in M \setminus A$, $\varphi \in p$ and $\mathcal{M} \models \psi(a, b)$.

Then $\exists w \psi(a, w) \in Th(\mathcal{M}_A)$, and since p is consistent with $Th(\mathcal{M}_A)$, the statement $(\exists w)(\varphi(v, a) \wedge \psi(a, w))$ is consistent, so we have a model $\mathcal{N}_0 \models (\exists w)(\varphi(v, a) \wedge \psi(a, w))$; interpreting the b_i as the witnesses shows that Δ_0 is consistent. Therefore, Δ is consistent.

If $\mathcal{N} \models \Delta$, them $\mathcal{N}$ is an elementary extension of $\mathcal{M}$ since $\mathcal{N} \models Th(\mathcal{M})$, and clearly $\mathcal{N}$ realizes p. □

Remark 6.7. It should be noted that, for any cardinality $\kappa \geq |M| + |\Gamma|$, we can find a model $\mathcal{N} \succ \mathcal{M}$ realizing all types in Γ with $|N| = \kappa$. Let $\mathcal{L}'$ be the language with constants $\{c_m\}_{m \in M} \cup \{d_p\}_{p \in \Gamma}$, and let T' be $Th(\mathcal{M}_M)$ along with sentences saying that d_p realizes p for each p. Then the above proposition tells us that T' is satisfiable, and therefore by Lowenheim-Skolem there is a model of each cardinality $\kappa \geq |\mathcal{L}'| = |M| + |\Gamma|$.

In fact, if $\mathcal{M} \models T$ and $A \subset M$ and $\Gamma \subset S_n(A)$, we can find a model realizing all types in Γ of each cardinality $\geq |A| + |\Gamma|$, since, by adding constants for each element of A to our language, we can find a model $\mathcal{N}$ of cardinality $|A|$ with $\mathcal{N}_A \equiv \mathcal{M}_A$, then use the above remark to get the result.

Theorem 6.8. *Let T be unstable (or, in fact, even $\aleph_0$-unstable). Then T is not categorical in any uncountable cardinality.*

Proof. Let $\mathcal{M} \models T$ and $B \subset M$ countable with $|S_n(B)| \geq \aleph_1$. Pick $\Gamma \subseteq S_n(B)$ of cardinality $\aleph_1$. By the above remark, we can find a model $\mathcal{N}$ of any cardinality $\kappa \geq \aleph_1$ with $\mathcal{N}_B \equiv \mathcal{M}_B$ that realizes all $p \in \Gamma$. On the other hand, by proposition

5.5, there is a model $\mathcal{N}'$ of cardinality κ so that for any $A \subset N'$, only $|A|$-many types are realized. These two models cannot be isomorphic. $\square$

Therefore, no unstable theory is categorical in any uncountable cardinality. This yields immediately that DLO is not categorical in any uncountable cardinality (though it is categorical in $\aleph_0$).

On the other hand, this also yields that the theory of algebraically closed fields of characteristic 0 is stable, since an algebraically closed field is uniquely determined by its transcendence degree, and for uncountable κ a field of transcendence degree κ has cardinality κ. Proving the stability of ACF_0 without theorem 5.8 requires the Nullstillensatz.

A second application of stability involves saturated models.

Definition 6.9. Let $\mathcal{M}$ be a structure. We say $\mathcal{M}$ is *κ-saturated* if for all subsets $A \subset \mathcal{M}$ with $|A| < \kappa$, then $\mathcal{M}$ realizes all types over A.

We say $\mathcal{M}$ is saturated if it is $|M|$-saturated.

We wish to show, for a class of cardinalities κ, that if T is κ-stable, then T has a saturated model of cardinality κ. We begin with a lemma.

Lemma 6.10. *Let $\mathcal{M}$ be a structure and κ an infinite cardinal. Then the following are equivalent;*

(1) $\mathcal{M}$ is κ-saturated.
(2) If $A \subset M$, $|A| < \kappa$, and $p \in S_1^{\mathcal{M}}(A)$, then $\mathcal{M}$ realizes p.

Proof. (1) $\implies$ (2) is clear.

We show (2) $\implies$ (1) by induction on n. The base case follows immediately from (2) Let $p \in S_n^{\mathcal{M}}(A)$, and let $q \in S_{n-1}^{\mathcal{M}}(A)$ be the type $\{\varphi(v_1, \dots, v_{n-1}) : \varphi \in p\}$. By induction, q is satisfied by some $a = (a_1, \dots, a_n) \in M$. Let $r \in S_1(A \cup \{a_1, \dots, a_{n-1}\})$, (also a set of cardinality $< \kappa$) with $r = \{\varphi(a_1, \dots, a_{n-1}, v_n) : \varphi \in p\}$. By (2), r is satisfied by some $b \in M$, whence the tuple $(a_1 \dots, a_{n-1}, b)$ realizes p. $\square$

Definition 6.11. Let κ be a cardinal. We say κ is regular if every subset $X \subset \kappa$ with $|X| < \kappa$ is bounded. Otherwise, we call κ singular.

Theorem 6.12. *Let κ be a regular cardinal and let T be κ-stable. Then T has a saturated model of size κ. In particular, an $\aleph_0$-stable theory has a saturated model in all cardinalities.*

Proof. Let $\mathcal{M}_0 \models T$ be a model of cardinality $\leq \kappa$. Use transfinite induction to define an elementary chain of models $\{\mathcal{M}_\beta\}_{\beta<\kappa}$ so that:

(1) $\mathcal{M}_{\beta+1} \succ \mathcal{M}_\beta$ and every type in $S_1(M_\beta)$ is satisfied in $\mathcal{M}_{\beta+1}$.
(2) For a limit ordinal δ, $\mathcal{M}_\delta = \bigcup_{\beta<\delta} \mathcal{M}_\delta$.
(3) $|M_\beta| \leq \kappa$

By the inductive hypothesis, we can satisfy (1) and (3) simultaneously by remark 6.7 since $|M_\beta|$ is no greater than κ and therefore has no more than κ many types. Moreover, for δ a limit ordinal, M_δ is a limit of at most κ sets of cardinality at most κ, so (2) and (3) can be simultaneously satisfied, completing the inductive step.

Let

$$\mathcal{M} = \bigcup_{\beta<\kappa} \mathcal{M}_\beta.$$

Then $\mathcal{M}$ is an elementary extension of each $\mathcal{M}_\beta$ and its cardinality is κ.

We claim $\mathcal{M}$ is saturated. Take any set of parameters A with $|A| < \kappa$. Then there is an β_a with $a \in \mathcal{M}_{\beta_a}$ for all a, and because κ is regular the sequence $\{\beta_a : a \in A\}$ is bounded by some β^*. Then $A \subset M_{\beta^*}$, so that any 1-type over A is realized in $\mathcal{M}_{\beta^*+1} \prec \mathcal{M}$. $\square$

7. Concluding Remarks and Acknowledgements

We have shown that a theory is unstable if and only if it has an unstable formula, or, equivalently, a formula with the order property or an undefinable type. This result of Shelah reduces the question of stability, a fundamental question examining models in all infinite powers, to the examination of potentially a single formula and countably many parameters in some model. The existence of some formula with the order property allows us to deduce the existence of some tree of arbitrary height and the existence of 'too many' types over a parameter set. On the other hand, the lack of such a formula tells us that all types over T are definable and, therefore, that all parameter sets A have at most $|A|$ consistent types over A.

The proof of theorem 3.5 was suggested by Professor M. Malliaris, though I made a couple of my own additions. The use of the X order property to construct a tree of arbitrary height was my own idea, although it is probably not original. Showing that $\Gamma(\varphi, \omega) \implies$ "φ has the order property" was taken from Hodges, as was the rest of the 'loop' of theorem 4.10. Many of the examples were offered as exercises by Professor Malliaris.

The question of stability and instability has repercussions in other model-theoretic questions. For instance, an unstable theory cannot be categorical in any uncountable power, whereas a κ-stable theory has a saturated model of size κ. Recalling the proof of proposition 4.7, a stable theory is in fact stable in arbitrarily high power, so that a stable theory has arbitrarily large saturated models. These proofs were from Buechler, although I also used Marker to fill some nontrivial gaps in my understanding of Buechler's reasoning (for instance, Buechler did not explain why any set of consistent types can be simultaneously realized in the same mode).

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References

Steven Buechler. Essential Stability Theory. Springer. 1996.
Chen Chung Chang and H. Jerome Keisler. Model Theory. Elsevier. 1990.
Wilfrid Hodges. Model Theory. Cambridge University Press. 1993.
David Marker. Model Theory: An Introduction. Springer. 2000.
Saharon Shelah. Classification Theory. Elsevier. 1990.