THE EXISTENCE AND UNIQUENESS OF SOLUTIONS TO DIFFERENTIAL EQUATIONS

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Abstract. I expound on a proof given by Arnold on the existence and uniqueness of the solution to a first-order differential equation, clarifying and expanding the material and commenting on the motivations for the various components.

Contents

1. Introduction

In addition to its intrinsic mathematical interest, the theory of ordinary differential equations has extensive applications in the natural sciences, notably physics, as well as other fields. The existence and uniqueness of a solution to a first-order differential equation, given a set of initial conditions, is one of the most fundamental results of ODE. In his textbook on the subject [1][2], Vladimir Arnold provides a proof of this theorem using the concepts of contraction mappings and Picard mappings. I now examine this proof in detail.

We will investigate solutions to the differential equation

$$\begin{aligned} \dot{x}(t) &= v(t,x) \\ x(t_0) &= x_0 \end{aligned} \tag{1.1}$$

where $t \in \mathbb{R}$, $x \in \mathbb{R}^n$, and $v(t,x)$ is defined and differentiable (of class $\mathbf{C}^r,\ r \geq 1$) in a domain U of $\mathbb{R} \times \mathbb{R}^n$.

A solution will be a function $\phi : \mathbb{R} \to \mathbb{R}^n$, where

$$\begin{aligned} \dot{\phi}(t) &= v(t,\phi(t)) \\ \phi(t_0) &= x_0 \end{aligned} \tag{1.2}$$

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We will prove the following theorems, which guarantee the existence and uniqueness of the solution for any equation of the form (1.1).

Theorem 1.3. (The Existence Theorem) *Suppose the right-hand side v of the differential equation $\dot{x} = v(t, x)$ is continuously differentiable in a neighborhood of the point $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^n$. Then there is a neighborhood of the point t_0 such that a solution of the differential equation is defined in this neighborhood with the initial condition $\phi(t_0) = x$, where x is any point sufficiently close to x_0. Moreover, this solution depends continuously on the initial point x.*

Theorem 1.4. (The Uniqueness Theorem) *Given the above conditions, there is only one possible solution for any given initial point, in the sense that all possible solutions are equal in the neighborhood under consideration.*

2. Foundations

First, let us define precisely what we mean by the derivative of a function from one metric space into another, phrased in Arnold's physics-inspired terminology.

Definition 2.1. Let $\phi : I \to M$ be a differentiable mapping of an open interval I in $\mathbb{R}$ (the "time" axis) to a metric space M. ϕ is said to *leave the point* x for some $x \in M$ if $\phi(0) = x$. The *velocity vector* v of ϕ leaving point x is the time derivative of ϕ at the point $t = 0$.

$$v = \dot{\phi}(0) = \frac{d\phi}{dt}\Big|_{t=0} \tag{2.2}$$

The *tangent space* to a domain M at a point x is the set of all velocity vectors of all such curves leaving x, and is denoted $T_x M$.

Remark 2.3. The dimension of the tangent space to any point in a metric space is the same as that of the metric space.

$$\dim(T_x M) = \dim(M)$$

Definition 2.4. Let $f : U \to V$ be a differentiable mapping from a domain U of an m-dimensional metric space into a domain V of an n-dimensional metric space, and let $\phi : I \to U$ be a differentiable mapping from an interval of the time axis into the domain U which leaves the point $x \in U$ at time $t = 0$. The *derivative of the mapping f at the point x* is the mapping

$$f_{*x} : T_x U \to T_{f(x)} V$$

which carries the velocity vector v leaving the point x of the curve ϕ into the velocity vector $f_{*x}(v)$ leaving the point $f(x)$ of the curve $f(\phi)$, i.e.

$$f_{*x}(v) = f_{*x}\Big(\frac{d\phi}{dt}\Big|_{t=0}\Big) = \frac{df(\phi)}{dt}\Big|_{t=0} \tag{2.5}$$

Remark 2.6. The mapping f_{*x} is independent of the choice of any particular coordinate system, but if such a system is chosen f_{*x} may be written in terms of the coordinates of the vector $f_{*x}(v)$ as the Jacobian matrix of $(\partial f/\partial x)$:

$$(f_{*x}(v))_i = \sum_j \frac{\partial f_i}{\partial x_j} v_j \tag{2.7}$$

To see why the same index i can be used for both $f_{*x}(v)$ and f, as well as the same index j for both v and x, recall that $\dim(T_x U) = \dim(U)$, and also $\dim(T_{f(x)} V) = \dim(V)$.

We now introduce the Lipschitz condition, along with an important circumstance under which it holds.

Definition 2.8. Let $A : M_1 \to M_2$ be a mapping of the metric space M_1 (with metric ρ_1) into the metric space M_2 (with metric ρ_2), and L a positive real number. The mapping A *satisfies a Lipschitz condition with constant* L (written $A \in \text{Lip}L$) if

$$\rho_2(Ax, Ay) \le L\rho_1(x, y)\ \forall x, y \in M_1, \tag{2.9}$$

i.e. if it increases the distance between any two points of M_1 by a factor of at most L. A *satisfies a Lipschitz condition* (or simply *is Lipschitz*) if there is some L for which $A \in \text{Lip}L$. L is called the *Lipschitz constant* of A.

Theorem 2.10. *Let $f : U \to \mathbb{R}^n$ be a smooth mapping ($f \in C^r, r \ge 1$) from $U \subseteq \mathbb{R}^m$ to $\mathbb{R}^n$, $x \in U$. Then f satisfies a Lipschitz condition on each convex compact subset V of U, with Lipschitz constant L equal to the supremum of the derivative of f on V:*

$$L = \sup_{x \in V} |f_{*x}| \tag{2.11}$$

Proof. Take any two points $x, y \in V$ and join them together with a line segment:

$$z(t) = x + t(y - x),\ 0 \le t \le 1$$

Since V is convex, $z(t) \in V\ \forall t \in [0, 1]$. Now,

$$\int_0^1 \frac{d}{dt}(f(z(t)))dt = f(z(1)) - f(z(0)) = f(y) - f(x)$$

and

$$\int_0^1 \frac{d}{dt}(f(z(t)))dt = \int_0^1 \frac{df}{dz}\Big|_{z(t)} \frac{dz}{dt}(t)dt = \int_0^1 f_{*z(t)}(y - x)dt.$$

Examining the absolute magnitude of this integral, we find

$$\begin{aligned}\Big|\int_0^1 f_{*z(t)}(y - x)dt\Big| &\le \int_0^1 |f_{*z(t)}(y - x)|dt \le \int_0^1 |f_{*z(t)}||y - x|dt \\ &\le \Big(\int_0^1 |f_{*z(t)}|dt\Big)|y - x| \le \Big(\int_0^1 Ldt\Big)|y - x| \\ &= [L \cdot 1 - L \cdot 0]|y - x| = L|y - x|\end{aligned}$$

We have thus determined that for any two points $x, y \in V$,

$$|f(y) - f(x)| = \Big|\int_0^1 f_{*z(\tau)}(y - x)d\tau\Big| \le L|y - x|,$$

and hence that f satisfies a Lipschitz condition on V with constant L. □

Note 2.12. Since $f \in C^1$, the mapping $f_* = \frac{df}{dx}$, which takes a given x and returns the mapping f_{*x}, is continuous. Since V is compact, $|f_{*x}|$ actually attains its maximum value L.

3. Contraction and Picard Mappings

Now we introduce the two types of functions that will be at the heart of our final proof.

Definition 3.1. Let M be a metric space with metric ρ, and $A : M \to M$ a mapping. A is called a *contraction mapping*, or *contraction*, if there exists some constant λ, $0 < \lambda < 1$, such that

$$\rho(Ax, Ay) \leq \lambda \rho(x, y) \ \forall x, y \in M. \tag{3.2}$$

λ is called the *contraction constant* of A.

Remark 3.3. *contraction $\Rightarrow$ Lipschitz condition $\Rightarrow$ continuity*

Definition 3.4. $x \in M$ is called a *fixed point* of a mapping $A : M \to M$ if $Ax = x$.

Theorem 3.5. (Contraction Mapping Theorem) *Let $A : M \to M$ be a contraction mapping of a complete metric space M (with metric ρ) into itself. Then A has one and only one fixed point. For any point x in M, the sequence of images of the point x under applications of A,*

$$x, \ Ax, \ A^2x, \ \ldots,$$

converges to the fixed point.

Proof. Let $\rho(x, Ax) = d$, for some $x \in M$. By induction,

$$\rho(A^n x, A^{n+1}x) \leq \lambda^n d \ \forall n \in \mathbb{N}.$$

Since $0 < \lambda < 1$, the series $\sum_{n=1}^{\infty} \lambda^n$ converges, and therefore $\sum_{n=1}^{\infty} \lambda^n d$ converges as well. This means that $\forall \epsilon > 0$, there is an N_0 such that $\forall p, q \geq N_0$ (assume $q > p$),

$$\epsilon > \sum_{n=p}^{q-1} \lambda^n d \geq \sum_{n=p}^{q-1} \rho(A^n x, A^{n+1}x) \geq \rho(A^p x, A^q x),$$

the last arising from successive applications of the triangle inequality. The sequence $\{A^n x\}_{n \in \mathbb{N}}$ is thus Cauchy, and converges because of the completeness of M. Thus, for every $x \in M$ there is some X such that

$$X = \lim_{n \to \infty} A^n x.$$

Now,

$$AX = A \lim_{n \to \infty} A^n x = \lim_{n \to \infty} A^n x = X,$$

so X is a fixed point of A. If Y is also a fixed point of A,

$$\rho(X, Y) = \rho(AX, AY) \leq \lambda \rho(X, Y),$$

and $0 < \lambda < 1$, which can only be true if $\rho(X, Y) = 0$, making $X = Y$. Thus A has one and only one fixed point, given by $\lim_{n \to \infty} A^n x$ for any $x \in A$. $\square$

Definition 3.6. Given a point $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^n$ as well as a differential equation $\dot{x} = v(t, x)$, where $x \in \mathbb{R}^n$, v a vector field over $\mathbb{R} \times \mathbb{R}^n$, define the *Picard mapping* to be the mapping P that takes a function $\phi : t \to x$ to the function $P\phi : t \to x$, where

$$(P\phi)(t) = x_0 + \int_{t_0}^{t} v(\tau, \phi(\tau)) d\tau. \tag{3.7}$$

Remark 3.8. $(P\phi)(t_0) = x_0$ for any ϕ.

Theorem 3.9. *The mapping $\phi : I \to \mathbb{R}^n$ is a solution to $\dot{x} = v(t, x)$ with the initial condition $\phi(t_0) = x_0$ if and only if $\phi = P\phi$.*

Proof. Assuming $\phi = P\phi$,

$$\phi(t) = x_0 + \int_{t_0}^{t} v(\tau, \phi(\tau))d\tau,$$

meaning $\dot{\phi} = v(t, \phi(t)),\ \phi(t_0) = x_0$.

Assuming ϕ is a solution to $\dot{x} = v(t, x)$ with the initial condition $\phi(t_0) = x_0$,

$$\dot{\phi} = v(t, \phi(t)),\ \phi(t_0) = x_0,$$

meaning

$$\phi(t) = x_0 + \int_{t_0}^{t} v(\tau, \phi(\tau))d\tau,$$

and so $\phi = P\phi$. □

The solution to a first-order differential equation is the "fixed point" of a Picard mapping, and the Contraction Mapping Theorem gives us the conditions under which a contraction mapping has one and only one fixed point. Thus, if we can construct a mapping that incorporates both of these types of functions in just the right way, we could take advantage of the existence and uniqueness of a contraction mapping's fixed point to prove the existence and uniqueness of the solution to our differential equation.

4. A Priori Bounds for the Solution

If v is differentiable at the point $(t_0, x_0) \in U$ (which it must be), then some neighborhood $\mathcal{C}$ around that point must lie within U. Specifically, if we choose small enough numbers a and b,

$$\mathcal{C} = \left\{ t, x \;\middle|\; |t - t_0| \le a,\ |x - x_0| \le b \right\} \subset U \tag{4.1}$$

Geometrically, this region corresponds to the surface and enclosed volume of an $n + 1$-dimensional cylinder in $\mathbb{R} \times \mathbb{R}^n$, oriented along the t-axis and centered at (t_0, x_0). This cylinder is a closed and bounded subset of a Euclidean space, and is thus compact.

Since v is continuous over U and hence $\mathcal{C}$, $|v|$ attains its supremum over $\mathcal{C}$. Similarly, $v_* = \frac{dv}{dx}$ is continuous over $\mathcal{C}$ (since $v \in \mathbf{C}^{r \ge 1}$), and so $|v_*|$ attains its supremum over $\mathcal{C}$. Both of these suprema are thus known to be finite constant numbers; let

$$c = \sup_{\mathcal{C}} |v|,\ L = \sup_{\mathcal{C}} |v_*| \tag{4.2}$$

The choice of the letter L for the latter constant is intentional. We will construct a function based on v which satisfies a Lipschitz condition on each convex compact subset of U, including the cylinders $\mathcal{C}$, with Lipschitz constant L on those cylinders. For now, though, let us dissect $\mathcal{C}$ into some useful subregions. Consider one such region

$$K_0 = \left\{ t, x \;\middle|\; |t - t_0| \le a',\ |x - x_0| \le c|t - t_0| \right\}$$

corresponding to the cone with vertex (t_0, x_0), aperture c and height a' oriented along the t-axis. If a' is sufficiently small, this cone lies within the cylinder $\mathcal{C}$. Specifically, since $|x - x_0| \le c|t - t_0| \le ca'$, then $a' = \min(a, \frac{b}{c})$ will do.

For $|x' - x_0| \le b'$, we can construct

$$K_{x'} = \Big\{ t, x \;\Big|\; |t - t_0| \le a',\;\; |x - x'| \le c|t - t_0| \Big\} \tag{4.3}$$

with the same size, shape and alignment as K_0 but shifted to the vertex (t_0, x'). If both a' and b' are sufficiently small, all of these cones will be in $\mathcal{C}$. Taking advantage of the triangle inequality, we find

$$b = ca' + b' \ge |x - x'| + |x' - x_0| \ge |x - x_0|$$

to give the bound on x we need, so let $b' = \frac{b}{2}$ and $a' = \min(a, \frac{b}{2c})$. In fact, a' will need one more bound later on, namely the condition $a' < \frac{1}{L}$ (we are ignoring the trivial case $L = 0$), so let us go ahead and apply it now:

$$a' = \min(a, \frac{b}{2c}, \frac{1}{2L}). \tag{4.4}$$

We will look for the solution $\phi_{x'} : \mathbb{R} \to \mathbb{R}^n$ of (1.1) with initial condition $\phi_{x'}(t_0) = x'$, expressed in the form $\phi_{x'}(t) = x' + h(t, x')$, though we can now remove the prime on x:

$$\phi_x(t) = x + h(t, x) \tag{4.5}$$

Since our solution starts at some initial x within distance b' of some x_0, and is defined for at least those t within distance a' of t_0, let us construct a map

$$\phi : \Big\{ t, x \;\Big|\; |t - t_0| \le a',\;\; |x - x_0| \le b' \Big\} \to \mathbb{R}^n, \tag{4.6}$$

defined by

$$\phi(t, x) = \phi_x(t) \tag{4.7}$$

This ϕ is the "general" solution, which may be narrowed down to a "particular" solution ϕ_x by supplying the initial condition x.

Lemma 4.8. *For any solution ϕ_x, the point $(t, \phi_x(t))$ lies within the cone K_x for all t such that $|t - t_0| \le a'$.*

Proof. The initial point (t_0, x) is the cone's vertex. Since c was chosen to be the largest value of v anywhere in $\mathcal{C}$, the fact that v is the time derivative of the solution means that a tangent line to the curve can never have a slope whose magnitude is greater than c, i.e. for any point $(t, \phi_x(t))$,

$$\Big| \frac{\phi_x(t) - x}{t - t_0} \Big| \le c,$$

or equivalently

$$|\phi_x(t) - x| \le c|t - t_0|.$$

The appropriate constraint on $\phi_x(t)$ is thus satisfied (the constraint on t is covered in the lemma's statement), and so $(t, \phi_x(t))$ is in K_x for any t close enough to t_0. $\square$

5. The Contraction Mapping

Any contraction mapping takes some metric space into itself, so let us first define the metric space we will use. Recall that we are trying to equate the fixed point of our contraction mapping with the solution to (1.1), so this metric space should include all the mappings which could possibly be solutions. We will in fact proceed at a step removed, and consider the space of all functions $h(t,x)$ which, added to x, could give us a solution ϕ_x with initial condition $\phi_x(t_0) = x$.

Some necessary constraints follow from the boundary we set up above. If we are given some central initial condition (t_0, x_0), the map ϕ should take a point (t,x) from the region $|t - t_0| \leq a',\ |x - x_0| \leq b'$ to $\mathbb{R}^n$. The map h must then also be over this region.

$$h : \left\{ t, x \;\middle|\; |t - t_0| \leq a',\ |x - x_0| \leq b' \right\} \to \mathbb{R}^n \tag{5.1}$$

We have found that any $\phi_x(t) = x + h(x,t)$ must lie within the cone K_x, so

$$|\phi_x(t) - x| \leq c|t - t_0| \tag{5.2}$$

Equate $|\phi_x(t) - x| = |x + h(t,x) - x| = |h(t,x)|$ to get

$$|h(t,x)| \leq c|t - t_0| \tag{5.3}$$

Finally, since ϕ_x must be a differentiable function in order to be a solution, it must be continuous on the domain over which it is a solution; therefore, so must $\phi_x(t) - x = h(t,x)$. This turns out to be the last condition we need to consider, so denote by M (with the central point (t_0, x_0) understood) the set of all continuous mappings h satisfying conditions (5.1) and (5.3).

Remark 5.4. $h(t_0, x) = 0$ for any $h \in M,\ x \in \mathcal{C}$, where 0 is the zero vector in $\mathbb{R}^n$.

To make M a metric space, define a metric ρ on M by

$$\rho(h_1, h_2) = \|h_1 - h_2\| = \sup|h_1(t,x) - h_2(t,x)| \tag{5.5}$$

Since every h is a continuous function over a closed and bounded cylinder of a Euclidean space, this supremum is actually attained.

Remark 5.6. The metric space (M, ρ) is complete.

Define a mapping $A : M \to M$ by

$$(Ah)(t,x) = \int_{t_0}^{t} v(\tau, x + h(\tau, x))d\tau \tag{5.7}$$

for $|x - x_0| \leq b',\ |t - t_0| \leq a'$. Clearly $(\tau, x + h(\tau, x))$ is in the domain of v for any (τ, x) in the appropriate region, but we should be careful to check that Ah is in fact an element of M.

Lemma 5.8. *For all $h \in M$, $Ah \in M$.*

Proof. Take any $h \in M$. By construction, Ah is a function that satisfies (5.1).

$$\begin{aligned} |(Ah)(t,x)| = \Big| \int_{t_0}^{t} v(\tau, x + h(\tau, x))d\tau \Big| &\leq \int_{t_0}^{t} |v(\tau, x + h(\tau, x))|d\tau \\ &\leq \int_{t_0}^{t} c d\tau = |c \cdot t - c \cdot t_0| = c|t - t_0|, \end{aligned}$$

meaning Ah satisfies (5.3).

The function h is continuous for any (τ, x) in its domain, so the point $(\tau, x + h(\tau, x))$ varies continuously with (τ, x), and since v is also continuous on its domain, v varies continuously with (τ, x) as well. Taking the integral will then result in a continuous function of the boundary terms taken at (t, x) and (t_0, x). Thus Ah is a continuous function of (t, x) which satisfies (5.1) and (5.3), meaning $Ah \in M$. □

As you may have guessed from this section's title, A is the contraction mapping we have been looking for.

Theorem 5.9. *A is a contraction mapping.*

Proof. We need to show that, for any $h_1,\ h_2 \in M,\ \|Ah_1 - Ah_2\| \le \lambda \|h_1 - h_2\|$ for some constant $0 < \lambda < 1$. Let us then construct the mapping $Ah_1 - Ah_2$.

$$(Ah_1)(t,x) = \int_{t_0}^{t} v(\tau, x + h_1(\tau, x))d\tau \text{ (abbreviated } \int_{t_0}^{t} v_1 d\tau)$$

$$\Rightarrow (Ah_1 - Ah_2)(t,x) = \int_{t_0}^{t} v_1 d\tau - \int_{t_0}^{t} v_2 d\tau = \int_{t_0}^{t} (v_1 - v_2)d\tau$$

For a fixed (τ, x), v will act as a mapping that takes $h_i(\tau, x)$ to $v(\tau, x + h_i(\tau, x))$. As v was assumed to be continuously differentiable over its domain, we invoke 2.10 to find that v satisfies a Lipschitz condition on each convex compact subset of its domain, and therefore on each cylinder $\mathcal{C}$ of U (actually, since τ is fixed but $x + h_i(\tau, x)$ varies with $h_i(\tau, x)$, the domain over which v varies is only the x-portion of $\mathcal{C}$). Theorem 2.10 also gives us the Lipschitz constant, $L(\tau) = \sup_{|x - x_0| \le b} |v_*|$, where I have emphasized the fact that this L depends on the choice of the constant τ. Thus for all points (τ, x),

$$|v_1(\tau, x) - v_2(\tau, x)| \le L(\tau)\|h_1 - h_2\|$$

As seen earlier, the magnitude of any mapping in M attains its supremum at some point in its domain, so

$$\|Ah_1 - Ah_2\| = \sup|Ah_1(t,x) - Ah_2(t,x)| = |Ah_1(t_m, x_m) - Ah_2(t_m, x_m)|$$

for some $(t_m, x_m) \in \mathcal{C}$. Therefore

$$\|Ah_1 - Ah_2\| = \Big| \int_{t_0}^{t_m} (v_1(\tau, x_m) - v_2(\tau, x_m))d\tau \Big| \le \int_{t_0}^{t_m} |v_1(\tau, x_m) - v_2(\tau, x_m)|d\tau$$
$$\le \int_{t_0}^{t_m} L(\tau)\|h_1 - h_2\|d\tau = \int_{t_0}^{t_m} L(\tau)d\tau \|h_1 - h_2\|$$

In (4.2), L (without the parenthetical τ) was designated the supremum of $|v_*|$ over all of $\mathcal{C}$, i.e. over both the t and x domains, meaning that

$$\|Ah_1 - Ah_2\| \le \int_{t_0}^{t_m} L(\tau)d\tau\|h_1 - h_2\| \le \int_{t_0}^{t_m} L d\tau \|h_1 - h_2\|$$
$$= L|t_m - t_0|\|h_1 - h_2\| \le La'\|h_1 - h_2\|$$

Lastly, we take advantage of the extra bound we placed on a' in (4.4) to find that $La' \le L\frac{1}{2L} = \frac{1}{2} < 1$. Thus, for all $h_1, h_2 \in M$,

$$\|Ah_1 - Ah_2\| \le La'\|h_1 - h_2\|,\ 0 < La' < 1$$

making A a contraction mapping. □

6. The Picard Mapping; Proofs of Existence and Uniqueness Theorems

With A thus known to be a contraction mapping over a complete metric space, we now apply the Contraction Mapping Theorem and guarantee the existence and uniqueness of its fixed point $h_0 \in M$. Our goal now is to incorporate this in a Picard mapping of potential solutions to (1.1), using the existence and uniqueness of h_0 to confirm the existence and uniqueness of the fixed point of the Picard mapping, which will in turn prove our main theorems.

First recall that we are looking for solutions expressed in the form $\phi_x(t) = x + h(t,x)$. When h is a fixed point of A, this equals $x + (Ah)(t,x)$, and when the solution is a fixed point of our Picard mapping, $\phi_x(t)$ will equal $(P\phi_x)(t)$. Simply plugging in both of these identities into the desired form of our solution, in fact, gives us precisely the mapping we need.

$$\begin{aligned}(P\phi_x)(t) &= x + (Ah)(t,x) \\ &= x + \int_{t_0}^{t} v(\tau, x + h(\tau,x))d\tau \\ &= x + \int_{t_0}^{t} v(\tau, \phi_x(\tau))d\tau\end{aligned} \tag{6.1}$$

The mapping P is then a Picard mapping of functions $\phi_x(t) = x + h(t,x)$, and by theorem 3.9, ϕ_x is a solution to $\dot{x} = v(t,x)$ with $\phi_x(t_0) = x$ if and only if $\phi_x = P\phi_x$. We can now prove the existence of a solution of (1.1) satisfying any initial condition in the domain of v, as well as the fact that this solution depends continuously on the initial condition. After all the hard work spent proving the previous theorems, the present proof is surprisingly straightfoward, though we should take care to specify what is used to construct what.

Proof of 1.3 (The Existence Theorem). Given $v(t,x)$ as well as (t_0, x_0), demarcate a neighborhood $\mathcal{C}$ around the central point and use it to define the constants a', b'; also construct the metric space M, contraction mapping A, and Picard mapping P, as determined by v, $\mathcal{C}$, and the central point. Since M is a complete metric space, the fixed point h_0 of A must exist by the Contraction Mapping Theorem. The function $g : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ given by

$$g(t,x) = x + h_0(t,x)$$

is therefore always well-defined in a neighborhood of (t_0, x_0). Applying the Picard mapping,

$$(Pg)(t,x) = x + (Ah_0)(t,x) = x + h_0(t,x) = g(t,x)$$

which proves that, by theorem 3.9, g is a solution of the differential equation which satisfies the initial condition $g(t_0, x) = x$, as long as t is in a neighborhood of the point t_0 defined by $|t - t_0| \leq a'$ and x is any point such that $|x - x_0| \leq b'$. The function which returns the value x is continuous on $\mathbb{R} \times \mathbb{R}^n$, h_0 is continuous by construction, and the sum of any two continuous functions is continuous over the same domain, so g, a function of t and x, is continuous over its domain. Thus, the solution depends continuously on the initial point x. □

Uniqueness immediately follows.

Proof of 1.4 (The Uniqueness Theorem). Construct the neighborhood and mappings as above, but now set $b' = 0$, which restricts the initial x under our consideration to the specific point x_0. Find the solution $g(t, x_0) = x_0 + h_0(t, x)$. The uniqueness of the fixed point h_0 guarantees that this is the only solution with the initial condition x_0 that can be expressed in the form $x + h(t, x)$.

Now consider any solution ϕ_{x_0} with $\phi_{x_0}(t_0) = x_0$. By lemma 4.8, $\phi_{x_0}(t) \in K_0$ for all t in our neighborhood. By (5.2), $|\phi_{x_0}(t) - x_0| \leq c|t - t_0|$, and so $\phi_{x_0}(t) - x_0$ satisfies (5.3); label this quantity $h_\phi(t, x_0)$. This new function also clearly satisfies (5.1), and furthermore, since any solution ϕ must be continuous, h_ϕ is also continuous. The function h_ϕ therefore satisfies all the requirements of belonging to M, and $\phi_{x_0}(t) = x_0 + h_\phi(t, x_0)$, meaning all possible solutions to the differential equation with the given initial condition are expressable in the form $\phi_{x_0} = x_0 + h(t, x_0)$ for $h \in M$. As there is only one such function possible, the solution g is thus unique. □

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