# EUCLIDEAN ISOMETRIES AND SURFACES

XIN CAO

Abstract. In this paper, we attempt a classification of the euclidean isometries and surfaces. Using isometry groups, we prove the Killing-Hopf theorem, which states that all complete, connected euclidean spaces are either a cylinder, twisted cylinder, torus, or klein bottle.

Contents



## 1. Euclidean Isometries

We begin with several important definitions used throughout the paper.

**Definition 1.1.** The *euclidean plane* is the set

$$\mathbb{R}^2 = \{(x, y) \mid x, y \in \mathbb{R}\}$$

with *euclidean distance* between points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ defined as

$$d(P_1, P_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

**Definition 1.2.** A *isometry* of $\mathbb{R}^2$ is a function $f : \mathbb{R}^2 \to \mathbb{R}^2$ which preserves euclidean distance, such that

$$d(f(P_1), f(P_2)) = d(P_1, P_2) \text{ for all } P_1, P_2 \in \mathbb{R}^2.$$

We see immediately from this definition that the identity function 1, which maps every element of $\mathbb{R}^2$ to itself, is a isometry

**Definition 1.3.** If $f$ and $g$ are euclidean isometries, then their *product* $f \cdot g$ (abreviated $fg$) is also a isometry $h$, defined for all $P \in \mathbb{R}^2$ by $h(P) = f(g(P))$.

**Definition 1.4.** If $\Gamma$ is a set of isometries, then $\Gamma$ is a *isometry group* if it satisfies the following properties:

1. *Closure* If $f, g \in \Gamma$, then $f \cdot g \in \Gamma$.
2. *Associativity* If $f, g, h \in \Gamma$, then $(f \cdot g) \cdot h = f \cdot (g \cdot h)$.
3. *Identity* The identity $1 \in \Gamma$.
4. *Inverses* If $f \in \Gamma$, then there exists $g \in \Gamma$ such that $f \cdot g = g \cdot f = 1$.

Before we attempt to classify all the euclidean isometries, we begin with several familiar examples of isometries: translations, reflections, and rotations. Eventually, we show that there are actually only three types of isometries on the euclidean plane. Namely, these are translations, rotations, and glide reflections.

**Example 1.5.** We define a translation $t_{(\alpha,\beta)}$ by $t_{(x,y)} = (x+\alpha, y+\beta)$. In other words, the translation $t_{(\alpha,\beta)}$ moves the origin, $O$, to $(\alpha,\beta)$.

We verify that a translation is an isometry by showing that it preserves the square of the distance between any two points on the euclidean plane. Since,

$$\begin{aligned}(x_2' - x_1')^2 + (y_2' - y_1')^2 &= (x_2 + \alpha - (x_1+\alpha))^2 + (y_2 + \beta - (y_1+\beta))^2 \\ &= (x_2 - x_1)^2 + (y_2 - y_1)^2,\end{aligned}$$

we conclude that $d(f(P_1), f(P_2)) = d(P_1, P_2)$ for all $P_1, P_2 \in \mathbb{R}^2$.

And because,

$$t_{(\alpha,\beta)} \cdot t_{(-\alpha,-\beta)} = t_{(-\alpha,-\beta)} \cdot t_{(\alpha,\beta)} = 1,$$

the inverse of $t_{(\alpha,\beta)}$ is $t_{(-\alpha,-\beta)}$.

**Example 1.6.** We define a reflection in the $x$-axis is by $\overline{r}_{x-axis}(x,y) = (x,-y)$.

A quick calculation shows that the reflections are isometries, what's more, is that any reflection is its own inverse (i.e. $\overline{r} = \overline{r}^{-1}$).

**Example 1.7.** We define a rotation $r_\theta$ by $r_\theta(x,y) = (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta)$.

To show that euclidean distance is preserved, we square the distance, reduce the identities, and simplify to get,

$$\begin{aligned}d(f(P_1), f(P_2)) &= [(x_2\cos\theta - y_2\sin\theta) - (x_1\cos\theta - y_1\sin\theta)]^2 \\ &= [(x_2 - x_1)\cos\theta - (y_2 - y_1)\sin\theta]^2 + [(x_2 - x_1)\sin\theta + (y_2 - y_1)\cos\theta]^2 \\ &= (x_2 - x_1)^2{\cos\theta}^2 - 2(x_2 - x_1)(y_2 - y_1) + (y_2 - y_1)^2{\sin\theta}^2 \\ &\quad + (x_2 - x_1)^2{\sin\theta}^2 + 2(x_2 - x_1)(y_2 - y_1) + (y_2 - y_1)^2{\cos\theta}^2 \\ &= (x_2 - x_1)^2 + (y_2 - y_1)^2.\end{aligned}$$

Now, we show that the rotations have inverses.

**Exercise 1.8.** Show that $r_\theta r_\phi = r_{\theta+\phi}$ and hence that $r_\theta^{-1} = r_{-\theta}$

By example 1.7, after rotation by angle $\phi$ about O, we have

$$\begin{aligned}x' &= x\cos\phi - y\sin\phi, \\ y' &= x\sin\phi + y\cos\phi.\end{aligned}$$

And after rotation by angle $\theta$ about O, we have

$$\begin{aligned}x'' = x'\cos\theta - y'\sin\theta &= (x\cos\phi - y\sin\phi)\cos\theta - (x\sin\phi + y\cos\phi)\sin\theta \\ &= x(\cos\phi\cos\theta - \sin\phi\sin\theta) - y(\sin\phi\cos\theta + \sin\theta\cos\phi) \\ &= x\cos(\phi+\theta) - y\sin(\phi+\theta),\end{aligned}$$

Likewise, $y'' = x\sin(\phi+\theta) + y\cos(\phi+\theta)$, which shows that $r_\theta r_\phi = r_{\theta+\phi}$.
Further substituting $-\theta$ for $\phi$, we conclude that $r_{-\theta} = r_\theta^{-1}$.

Having found the inverses for our examples thus far, we can now express rotations and reflections at points and lines other than the origin or $x$-axis. This is possible by taking the *conjugate* of a isometry. Conjugation is the method by which we perform the same operation but in a new coordinate system. For instance, the reflection $t_{(\alpha,\beta)}\overline{r}_\theta t^{-1}_{(\alpha,\beta)}$ is the conjugate of $\overline{r}$ by $t_{(\alpha,\beta)}$. The conjugate by $t_{(\alpha,\beta)}$ treats the point $(\alpha,\beta)$ as if it were the origin of the plane.

We now want to show that reflections are the fundamental isometries, in the sense that all isometries are the products of them. But first, we begin by proving that all translations and rotations are the products of two reflections.

**Theorem 1.9.** *Any translation or rotation is the product of two reflections. Conversely, the product of two reflections is a rotation or translation.*

*Proof.* With any translation, we can define our y-axis so that it is parallel to the direction of the translation. Without loss of generality, our translation is $t_{(0,\delta)}$. We claim that $t_{(0,\delta)}$ is the product of the reflection $\overline{r}$ in the $x$-axis, and the reflection $t_{(0,\delta/2)}\overline{r}t^{-1}_{(0,\delta/2)}$ in the line $y = \delta/2$.

$$\begin{aligned} (x,y) &\mapsto (x,-y) \quad \textit{by } \overline{r} \\ &\mapsto (x,-y-\delta/2) \quad \textit{by } t^{-1}_{(0,\delta/2)} \\ &\mapsto (x,y+\delta/2) \quad \textit{by } \overline{r} \\ &\mapsto (x,y+\delta) \quad \textit{by } t_{(0,\delta/2)}. \end{aligned}$$

This verifies our claim for translations. Likewise, we can also define our origin so that the rotation is $r_\theta$. We claim that $r_\theta$ is the product of reflections $\overline{r}$ in the $x$-axis and $r_{\theta/2}\overline{r}r^{-1}_{\theta/2}$ in the line obtained by rotating the $x$-axis through angle $\theta/2$. This can also be confirmed by calculation, which completes the first half of our theorem.

Conversely, given two reflections $\overline{r}_L$ and $\overline{r}_M$, we examine two cases. If lines L and M intersect, then the situation is identical to a rotation about their intersection point by twice the angle of intersection. We begin by defining the intersection as our origin (and thus our fixed point) and one of our lines as the $x$-axis. The other line, we choose to be the line of reflection obtained by rotating the $x$-axis by angle $\delta/2$. If the two lines do not intersect, then the situation is analogous to the translation defined above, in which we take one line to be the $x$-axis and the other to be a line $\delta/2$ away. □

The following corollary results from the fact that translations and rotations are not uniquely composed: the product of different pairs of reflections can be the same isometry. This property of the isometries becomes a useful tool in later proofs.

**Corollary 1.10.** *If $\overline{r}_M\overline{r}_L$ is a rotation (expressed as the product of reflections $\overline{r}_M$ and $\overline{r}_L$), then $\overline{r}'_M\overline{r}'_L = \overline{r}_M\overline{r}_L$ for any lines $L'$, $M'$ with the same intersection as L, M and the same (signed) angle from L to M. If $\overline{r}_M\overline{r}_L$ is a translation, then $\overline{r}'_M\overline{r}'_L = \overline{r}_M\overline{r}_L$ for any lines $L'$, $M'$ parallel to L, M and the same (signed) distance apart.*

*Proof.* First take $L, M$ to be the $x$-axis and the line through $O$ at angle $\theta/2$ respectively, so

$$\overline{r}_M\overline{r}_L = r_{\theta/2}\overline{r}r^{-1}_{\theta/2} \text{ by theorem.}$$

Take $L', M'$ to be the lines through $O$ at angles $\phi, \theta/2+\phi$, respectively, so $\overline{r}_{L'} = r_\phi\overline{r}r^{-1}_\phi, \overline{r}_{M'} = r_{\theta/2+\phi}\overline{r}r^{-1}_{\theta/2+\phi}$. Then

$$\begin{aligned} \overline{r}_{M'}\overline{r}_{L'} &= r_{\theta/2+\phi}\overline{r}r^{-1}_{\theta/2+\phi} \cdot r_\phi\overline{r}r^{-1}_\phi \\ &= r_\phi \cdot r_{\theta/2}\overline{r}r^{-1}_{\theta/2}\overline{r} \cdot r^{-1}_\phi \\ &= r_\phi r_\theta r^{-1}_\phi = r_\theta = \overline{r}_M\overline{r}_L. \end{aligned}$$

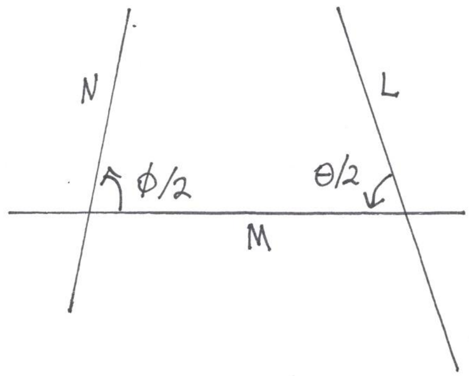


FIGURE 1

Second, take $L, M$ to be the $x$-axis and the line $y = \delta/2$. Then if we take $L', M'$ to be the lines $y = \omega, y = \omega + \delta/2$, we find that $\overline{r}_{M'}\overline{r}_{L'} = r_\theta = \overline{r}_M\overline{r}_L$. □

**Corollary 1.11.** *The set of translations and rotations is closed under product.*

*Proof.* It's clear that the product of two translations will always be a translation. To find the product of two rotations about different fixed points, we can describe these rotations as reflections about lines. Using the above corollary, given rotations $r_{P,\theta}$ and $r_{Q,\phi}$ about points P and Q respectively, we can choose to represent the product of these rotations as products of reflections in lines $L, M, N$ shown in Figure 1. Then

$$r_{P,\theta} \cdot r_{Q,\phi} = \overline{r}_N\overline{r}_M \cdot \overline{r}_M\overline{r}_L = \overline{r}_N\overline{r}_L,$$

which by Theorem 1.9, is a rotation if N meets L and a translation otherwise.

Also, by a suitable choice of lines, the product of a translation and a rotation can also be expressed as the product of two reflections. □

We are about to show that all isometries are the products of at most three reflections. For simplicity, we assume the results of the following lemma and corollary. Basically, the lemma states that every euclidean isometry is determined by its effect on a triangle.

**Lemma 1.12.** *Any isometry $f$ of $\mathbb{R}^2$ is determined by the images f(A), f(B), f(C) of three points A, B, C not in a line.*

**Corollary 1.13.** *If $L$ is the line of points equidistant from points $P$ and $Q$, then reflection in $L$ exchanges $P$ and $Q$.*

**Theorem 1.14** (Three Reflections Theorem)**.** *Any isometry $f$ of $\mathbb{R}^2$ is the product of one, two, or three reflections.*

*Proof.* Consider the points $A, B, C$ not in a line and their $f$-images $f(A), f(B), f(C)$.

Case(i) If two points of $A, B, C$ coincide with their $f$-images, say $f(A) = A$ and $f(B) = B$, then the reflection in the line $L$ through $A, B$ sends $C$ to $f(C)$ by corollary 1.13.

Case(ii) If only one point of $A, B, C$ coincides with its $f$-image, say $f(A) = A$, first perform reflection $\overline{g}$ in the line $M$ of points equidistant to $B$ and $f(B)$. Then, $\overline{g}$ sends $A, B$ to $f(A), f(B)$ respectively. If $\overline{g}(C) = f(C)$, then we are done. If not, then by case(i), we perform another reflection $\overline{h}$ in the line $L$ through $f(A), f(B)$, which sends $\overline{g}C$ to $f(C)$.

Case(iii) None of the points of $A, B, C$ coincide with their $f$-images. We perform at most three reflections $\overline{g}, \overline{h}, \overline{i}$, in lines respectively equidistant from $A$ and $f(A)$, $\overline{g}(B)$ and $f(B)$, $\overline{h}(C)$ and $f(C)$. By similar arguments as in the previous two cases, we are ensured that the product of one, two, or three of these reflection sends $A$ to $f(A)$, $B$ to $f(B)$, $C$ to $f(C)$, thus proving our theorem. □

It still remains to be seen how we will classify the euclidean isometries into translations, rotations, and glide reflections. This will become possible by the realization that we can separate the isometries into two groups, one group containing the products of even-numbered reflections, and a second group containing the products of odd-numbered reflections. We naturally recognize the group of even-numbered reflections as the orientation-preserving isometries, and the other group as the orientation-reversing isometries.

**Corollary 1.15.** *The isometries of $\mathbb{R}^2$ form a group Iso($\mathbb{R}^2$), and the products of even numbers of reflections form a subgroup $Iso^+(\mathbb{R}^2)$ of index 2.*

*Proof.* It is clear that associativity holds for products of reflections, hence, also for products of isometries. There is also an identity isometry. The theorem establishes that every isometry also has an inverse. Since reflections are self-inverses, the inverse of a isometry $\overline{r}_{L_1}...\overline{r}_{L_n}$ is $\overline{r}_{L_n}...\overline{r}_{L_1}$. Thus, the isometries form a group, Iso($\mathbb{R}^2$).

It also follows that the products $\overline{r}_{L_1}...\overline{r}_{L_{2n}}$ of even numbers of reflections form a subgroup, Iso$^+(\mathbb{R}^2)$ since products and inverses of such isometries are of the the same form.

To show that Iso$^+(\mathbb{R}^2)$ is of index 2, we show that the coset $Iso^+(\mathbb{R}^2) \cdot \overline{r} =$ {products of odd numbers of reflections} is not Iso$^+(\mathbb{R}^2)$. This is equivalent to showing that $\overline{r} \notin Iso^+(\mathbb{R}^2)$. By corollary 1.11, Iso$^+(\mathbb{R}^2)$ contains only rotations and translations. We deduce that reflections cannot belong in this set since the fixed-point set of non-trivial rotations is one fixed point, and the fixed-point set of translations is zero, whereas the fixed-point set of reflections is a line of points. □

We have already shown that isometries which are the products of even-numbered reflections are translations and rotations. It still remains to show that the products of three reflections and one reflection are another type of isometry: the glide reflection.

**Definition 1.16.** Glide Reflections

A glide reflection is an orientation-reversing isometry, described as the product of one or three reflections (the product of a translation and a reflection). From this definition, reflections can also be described as glide reflections, in particular those with trivial translations.

Now we show that the remaining euclidean isometries are glide reflections.

**Theorem 1.17.** *A product $\overline{r}_N\overline{r}_M\overline{r}_L$ of reflections in lines $L, M, N$ is a glide reflection.*

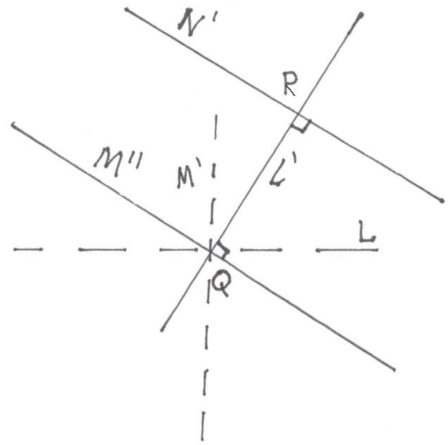


FIGURE 2

*Proof.* Case (i). L,M, N have a common point or are parallel.

First suppose that L, M, N have a common point $P$. Then, by corollary 1.10, $\overline{r}_M\overline{r}_L$ is a rotation about $P$, and $\overline{r}_{M'}\overline{r}'_L$ for any other lines $L', M'$ through $P$ separated by the same signed angle. We can choose $M' = N$ which gives

$$\overline{r}_N\overline{r}_M\overline{r}_L = \overline{r}_N\overline{r}_N\overline{r}'_L = \overline{r}_{L'},$$

which shows that $\overline{r}_N\overline{r}_M\overline{r}_L$ is a reflection, and hence also a glide reflection. Now, suppose that $L, M, N$ are parallel to one another. Then by theorem 1.9, we can take any two lines $L', M'$ having the same signed distance as lines $L, M$, and $\overline{r}_N\overline{r}_M = \overline{r}'_N\overline{r}'_M$. In particular, we can choose two lines such that $M' = N$, and by the same calculation, $\overline{r}_N\overline{r}_M\overline{r}_L$ is a glide reflection.

Case(ii). Without loss of generality, assume that only $M, N$ intersect at $P$. Then, $\overline{r}_N\overline{r}_M$ is a rotation about $P$ and $\overline{r}_N\overline{r}_M = \overline{r}_{N'}\overline{r}_{M'}$ for any lines $M', N'$ through $P$ separated by the same signed angle. In particular, we can choose a line $M'$ such that $M' \perp L$ at some point $Q$ (Figure 2), so

$$\overline{r}_N\overline{r}_M\overline{r}_L = \overline{r}_N\overline{r}_N\overline{r}'_L = \overline{r}_{L'}, \text{ where } M', L \text{ are perpendicular.}$$

Now $\overline{r}_{M'}\overline{L}$ a rotation about $Q$ and $\overline{r}_{M'}\overline{L} = \overline{r}_{M''}\overline{L'}$ for perpendicular lines $M'', L'$ through $Q$. Specifically, we choose $L'$ perpendicular to $N'$, at some point $R$. Then,

$$\overline{r}_N\overline{r}_M\overline{r}_L = \overline{r}_{N'}\overline{r}_{M'}\overline{r}_L = \overline{r}_{N'}\overline{r}_{M''}\overline{r}_{L'}$$

and since $L'$ is the common perpendicular of $M''$ and $N'$, $\overline{r}_{N'}\overline{r}_{M''}$ is a translation in the direction of $L'$, which makes $\overline{r}_{N'}\overline{r}_{M''}\overline{r}_{L'}$ a glide reflection. □

We have completed a classification of the euclidean isometries by showing that each isometry of $\mathbb{R}^2$ is either a rotation, translation, or glide reflection.

## 2. EUCLIDEAN SURFACES

If $\Gamma$ is a group of isometries, then $\mathbb{R}^2/\Gamma$ is a set of equivalence classes of the form $\{gP | g \in \Gamma\}$ for $P \in \mathbb{R}^2/\Gamma$. Generally, we refer to $\mathbb{R}^2/\Gamma$ as the *quotient space* and each equivalence class as a *point* of the quotient space. In our paper, the equivalence

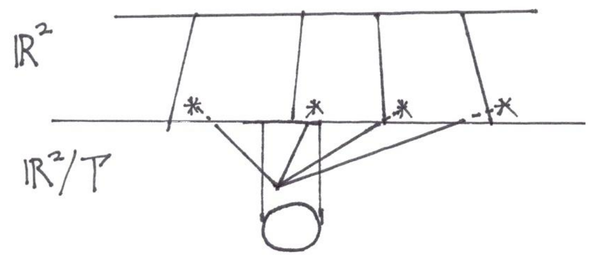


FIGURE 3. The asterisk (*) marks points of the same $\Gamma$-orbit

class of a point $P \in \mathbb{R}^2/\Gamma$ is referred to as the $\Gamma$-orbit of $P$ (abbreviated as $\Gamma P$). We define distance between two points $\Gamma P, \Gamma Q \in \mathbb{R}^2/\Gamma$ by

$$d_{\mathbb{R}^2/\Gamma}(\Gamma P, \Gamma Q) = min\{d(P', Q') | P' \in \Gamma P, Q' \in \Gamma Q\},$$

where $d$ denotes euclidean distance. We say a surface $S$ is a *euclidean surface* if for each $A \in S$, there is a $\epsilon > 0$ such that the $\epsilon$ neighborhood of A,

$$D_\epsilon(A) = \{B \in S | d_S(A, B) < \epsilon\}$$

is isometric to a euclidean disc. To be exact, there is a one-to-one correspondence $B \leftrightarrow B'$ between the points $B \in D_\epsilon(A)$ and the points $B' \in D_\epsilon(A')$ such that

$$d_S(B_1, B_2) = d(B_1', B_2')$$

Furthermore, we say a euclidean surface $S$ is *connected* if each $A, B \in S$ are joined by a series of line segments (a polygonal path) lying within euclidean discs of $S$, and *complete* if each line segment on $S$ can be continued indefinitely.

We would like to show eventually that every complete, connected euclidean surface $S = \mathbb{R}^2/\Gamma$ for some group $\Gamma$. With this in mind, we first provide an example of such a surface generated by a group of integer translations: the cylinder.

**Example 2.1** (If $\Gamma$ is the group of integer horizontal translations, then $C = \mathbb{R}^2/\Gamma$ is a complete and connected euclidean surface)**.** The $\Gamma$-orbit of a point $(x, y) \in C$ is a set of the form $\{(x + n, y) | n \in \mathbb{Z}\}$. From this definition, we note that distance on $C$ is well-defined, since for any $P \in C$, there is a nearest $Q' \in \Gamma Q$. And if $d(P, Q) < 1/2$, we have

$$d_C(\Gamma P, \Gamma Q) = d(P, Q)$$

which shows that $C$ is a euclidean surface.

Since $C$ is constructed by "rolling up" the euclidean plane (Figure 3), it is clear that $C$ is also connected and complete. We can find a line that joins any two points on $C$ by taking the pre-image of those points in $\mathbb{R}^2$, joining them with a line segment, and then taking the image of that line segment, which will then be a line segment in $C$. Likewise, to extend a line segment in $C$, we first extend the line segment in $\mathbb{R}^2$, and then map it to $C$ under the $\Gamma$-orbit.

An important consideration to be made is that not every group $\Gamma$ of isometries will generate a quotient space which is a euclidean surface. For instance, if $\Gamma$ is the group generated by rotation $r_{\pi/4}$, then for any $\epsilon > 0$, the $\epsilon$-neighborhood of $O$ contains four members of the same $\Gamma$-orbit, thus a "circle" with radius $\delta$ centered at $O \in \mathbb{R}^2/\Gamma$ will have a circumference of $\delta\pi/2$, instead of the euclidean $2\pi\delta$.

In other cases, $\Gamma$ may generate a quotient space where the distance between two distinct points on $\mathbb{R}^2/\Gamma$ is undefined. For instance if $\alpha$ is a translation by an irrational real number, and the group $\Gamma$ is generated by 1(the unit translation) and $\alpha$, then $\mathbb{R}^2/\Gamma$ is not a euclidean surface.

*Proof.* We begin by considering the set $\{n\alpha \mid n \in \mathbb{N}\}$. All of the members of this set must be distinct modulo 1, else

$$m\alpha - n\alpha = \text{an integer } p,$$

and

$$\alpha = \frac{p}{m-n},$$

which contradicts the irrationality of $\alpha$. Hence, the set $A = \{n\alpha - [n\alpha] \mid n \in \mathbb{N}\} \subset \Gamma$ is an infinite subset in $[0, 1]$.

The infinitude of $A$ implies that, for any $\epsilon > 0$, there are $\beta, \gamma \in A$ such that $|\beta - \gamma| < \epsilon$. Since $\Gamma$ is a group, $\beta, \gamma \in \Gamma$ implies $\beta - \gamma \in \Gamma$ and also $p(\beta - \gamma) \in \Gamma$ for any integer $p$. The latter includes translations within $\epsilon$ of any real length. This means that for any point $P \in \mathbb{R}^2$, the $\Gamma$-orbit of $P$ includes points within every neighborhood of any point. $\square$

Thus, to ensure that our group $\Gamma$ will always generate a euclidean surface, we require $\Gamma$ to be discontinuous.

**Definition 2.2.** A isometry group $\Gamma$ of the euclidean plane is called *discontinuous* if no $P \in \mathbb{R}^2$ has a $\Gamma$-orbit with a limit point. By limit point, we mean a point whose neighborhood includes infinitely many orbit points of ($\Gamma$P).

From this definition, we see that the example given above is of a non-discontinuous group, as every point $P \in \mathbb{R}^2$ is a limit point.

The following lemma and theorem allow us to realize the four types of quotient spaces which satisfy the properties of a complete, connected euclidean space. Namely, they are the cylinder, twisted cylinder, torus, and Klein bottle.

**Lemma 2.3.** *If $\Gamma$ is a group of isometries of $\mathbb{R}^2$, then $\Gamma$ is discontinuous and fixed point free if and only if each $P \in \mathbb{R}^2$ has a neighborhood $D_P$ in which each point belongs to a different $\Gamma$-orbit.*

*Proof.* Suppose $\Gamma$ is discontinuous and fixed point free, and consider any $P \in \mathbb{R}^2$. Since $\Gamma$ is discontinuous, there is a $\delta > 0$ such that all points in the $\Gamma$-orbit of $P$ are at a distance $\geq \delta$ from $P$. Then, since $\Gamma$ is fixed point free, $gP$ is at a distance $\geq \delta$ from $P$ for each $g \neq 1$ in $\Gamma$. Thus, we are guaranteed that the whole neighborhood of $D_P$ with radius $\delta/3$ is shifted $\delta/3$ to a position disjoint from from $D_P$ by $g$. Thus, $D_P$ cannot contain two points in the same $\Gamma$-orbit.

Conversely, suppose each $P \in \mathbb{R}^2$ has a neighborhood $D_P$ in which each point belongs to a different $\Gamma$-orbit. Then $\Gamma$ must be discontinuous, otherwise some $P \in \mathbb{R}^2$ would be a limit point, and have members of the same $\Gamma$-orbit in all its neighborhoods.

Also, $\Gamma$ must be fixed point free point. If not, consider a fixed point $Q$ of some $g \neq 1$ in $\Gamma$. Since $g$ is not the identity, it cannot fix three points in a line (which would make $g$ the identity), then $g$ moves some point $R$ arbitrarily close to $Q$ to another another point $gR$, which is equally close to $Q$ (since $g$ is an isometry).

Thus, any neighborhood $D_Q$ includes distinct points $R, gR$ in the same $\Gamma$-orbit, contrary to the hypothesis. □

**Theorem 2.4.** *A discontinuous, fixed point free group* $\Gamma$ *of isometries of* $\mathbb{R}^2$ *is generated by one or two elements.*

Following Lemma 2.3, we need only consider translations or proper glide reflections as possible generators of $\Gamma$, since both reflections and rotations have fixed points. We prove this theorem for translations only, though the idea is similar if we include glide reflections.

*Proof.* Suppose that $\Gamma$ contains translations only. Choose a point $P \in \mathbb{R}^2$. Since $\Gamma$ is discontinuous, there is a minimum distance $\delta > 0$ between $P$ and any other point in the $\Gamma$-orbit of $P$. We choose our first generator $t_1 \in \Gamma$ as one of these nearest members of $\Gamma P$ to $P$.

We claim that the powers $...t_1^{-1}, 1, t_1, t_1^2, ...$ of $t_1$ include all translations in $\Gamma$ with the same direction as $t_1$. If $t \in \Gamma$ is another translation in the same direction as $t_1$, and if $t_1^m P$ is the closest possible point to $tP$, then $t^{-1}t_1^m \in \Gamma$ would be a translation in the same direction as $t_1$, but shorter, contrary to the choice of $t_1$.

Thus, if the the set $\{...t_1^{-1}, 1, t_1, t_1^2, ...\}$ does not exhaust $\Gamma$, then the remaining translations have a different direction. In this case, we choose another translation of minimal length and call it $t_2$. Since $t_1$ and $t_2$ are in different directions, they generate a *lattice* of points in $\mathbb{R}^2$–the vertices of a tessellation of $\mathbb{R}^2$ by equal parallelograms (see Figure). We show that this lattice is the whole $\Gamma$-orbit of $P$.

Suppose that the powers of $t_1 t_2 = \{t_1^m t_2^n | m, n \in \mathbb{Z}\}$ do not exhaust $\Gamma$, then there is a translation $t \in \Gamma$ which is not a lattice translation. If $t_1^m t_2^n P$ is the point closest to $tP$, then $t^{-1}t_1^m t_2^n$ is a translation shorter than $t_1$ or $t_2$, which contradicts our choices of $t_1$ and $t_2$.

This is equivalent to showing that any point $Q$ within a parallelogram is separated from at least one vertex by a distance less than the length of the longest side. This is clear from the figure (see figure). Since $Q$ must lie in one-half of the parallelogram, $Q$ must lie inside a circle with the long side of the parallelogram as the radius. □

**Corollary 2.5.** $S = \mathbb{R}^2/\Gamma$ *is a cylinder, twisted cylinder, torus, or Klein bottle.*

From the previous theorems, we arrive at these surfaces depending on the generators of $\Gamma$.
1. (Cylinder) - Generated by a single translation.
2.(Twisted Cylinder) - Generated by a single glide reflection.
3.(Torus) - Generated by two translations.
4. (Klein Bottle) - Generated by a glide reflection and translation (or two glide reflections).

We have found all the possible complete, connected, euclidean surfaces which can be generated by different isometry groups, but it remains to be shown that these are the only surfaces possible. Thus, we first show that euclidean plane is a cover for the any such surface, and then prove that there is a discontinuous, fixed-point free group of isometries which generate any of these surfaces.

To show that the euclidean plane $\mathbb{R}^2$ covers a complete, connected euclidean surface $S$, we use a device known as the pencil map. Since the *pencil* of a point is defined to be the family of rays which pass through a point, each point $P \in \mathbb{R}^2$ is uniquely determined by the ray from $O$ (origin) through $P$ and the length $|OP|$ of $OP$. Given a point $O^S \in S$ and an isometry $p : D_\epsilon(O) \to D_\epsilon(O^S)$ (by $S$ being a euclidean surface), we define the *pencil map* $p : \mathbb{R}^2 \to S$ as the extension of each of these line segments in $D_\epsilon(O^S)$. For instance, given a point $P \in \mathbb{R}^2$, $p(P)$ is found by extending the $p$-image of $OP \cap D_\epsilon(O)$ to distance $|OP|$. Since $S$ is complete, allowing us to extend each line segment indefinitely, the pencil map is defined for all points in the euclidean plane. We now show that pencil map $p$ is a *covering* of $\mathbb{R}^2$ onto $S$, in the sense that each $P \in \mathbb{R}^2$ is locally isometric.

**Theorem 2.6** (The Pencil Map)**.** *The pencil map $p$ has the properties*
*(i) each $P \in \mathbb{R}^2$ has a neighborhood on which $p$ is an isometry, and*
*(ii) $p$ is onto $S$.*

*Proof.* (i) Suppose on the contrary that $P \in \mathbb{R}^2$ has no neighborhood on which $p$ is an isometry. Then, this is a point where $p$ is not locally isometric. Such points form a closed set because if some point $Q$ has a neighborhood on which $p$ is an isometry, then so have all points in this neighborhood. Thus, the points on line segment $OP$ at which $p$ is not locally isometric form a nonempty closed set, and hence have a least member. Without loss of generality we can assume that this least member (the one nearest to $O$) is $P$.

Since $p(P)$ lies in the euclidean surface $S$, it has an $\epsilon$-neighborhood $D_\epsilon(p(P))$ isometric to a disc of $\mathbb{R}^2$, which we can choose to be $D_\epsilon(P)$. We rotate $D_\epsilon(P)$ to make the isometry agree with $p$ on $OP \cap D_\epsilon(P)$. Finally, we can reflect in $OP$ to make the isometry agree with $p$ on any subdisc of $D_\epsilon(P)$ centered on $OP$, where $p$ is an isometry. By hypothesis, $p$ is an isometry on a sufficiently small neighborhood $D_\delta(Q)$ of any $Q \in D_\epsilon(P)$ between $O$ and $P$ (see figure).

Now, there are two maps, an isometry $f : D_\epsilon(P) \to D_\epsilon(p(P))$ and the pencil map $p : D_\epsilon(P) \to D_\epsilon(p(P))$, which agree on $D_\delta(Q)$. But $f$ and $p$ also agree on the extensions of all rays from the origin which pass through $D_\delta(Q)$ because $f$ preserves length and straightness. The union of these line segments includes a disc neighborhood of $P$, which is a contradiction of the hypothesis.

(ii) To prove that $p$ is onto $S$ we observe that $p$ maps any closed line segment $L$ of $\mathbb{R}^2$ onto a line segment of $S$. By compactness and the local isometry of $p$, $L$ can be divided into subsegments, each lying in a disc of $\mathbb{R}^2$ which is mapped isometrically by $p$. Then $p(L)$ is a union of line segments of $S$, each meeting its predecessor at angle $\pi$; hence $p(L)$ is a line segment of $S$.

Now if $p$ is not onto $S$, consider the nonempty set $S - p(\mathbb{R}^2)$. Since $p$ is a local isometry, $p(\mathbb{R}^2)$ includes an $\epsilon$-neighborhood of each $p(P)$, and hence is open, so $S - p(\mathbb{R}^2)$ is closed. If $P^S \in S - p(\mathbb{R}^2)$, connectedness of $S$ gives a polygonal path from $O^S$ to $P^S$, and then there is a first point $Q^S$ of this path in $S - p(\mathbb{R}^2)$. If $R^S$ is the last vertex of this path in $p(\mathbb{R}^2)$, let $R^S = p(R)$ and consider any line segment $L$ out of $R$ which is mapped by $p$ onto an initial segment of $R^S Q^S$. By extending $L$ sufficiently we can make its $p$-image include all of $R^S Q^S \cap p(\mathbb{R}^2)$ and hence, by continuity, it will include $Q^S$ also, which is a contradiction. □

Though we have constructed a covering $p : \mathbb{R}^2 \to S$, we still need to find a group $\Gamma$ for which $S = \mathbb{R}^2/\Gamma$. This group is the covering isometry group. We say a isometry $g : \mathbb{R}^2 \to \mathbb{R}^2$ is a *covering isometry* for a covering $f : \mathbb{R}^2 \to S$, if $fgP = fP$ for all $P \in \mathbb{R}^2$. We now show that the covering isometries form a group $\Gamma$.

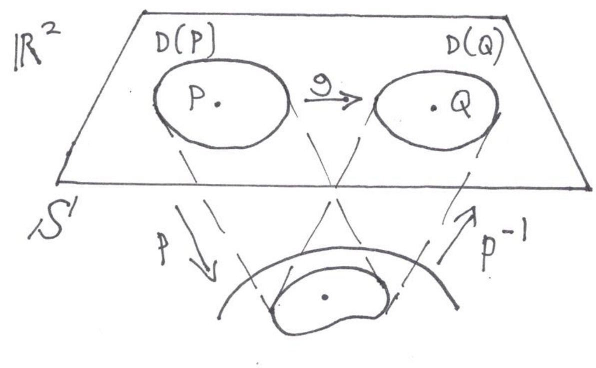


FIGURE 4

**Exercise 2.7.** Suppose that $g_1, g_2$ are covering isometries for a covering $f$. We first show that $g_1, g_2$ is a covering isometry.

Since $g_2P \in \mathbb{R}^2$, then $fg_1(g_2P) = fg_2P$, and hence $fg_2P = fP$. This shows that $fg_1g_2P = fP$, which makes $g_1g_2$ a covering isometry.

To show that any covering isometry $g$ has an inverse, we begin by writing any $P \in \mathbb{R}^2$ as $g^{-1}Q$, using the fact that $g$ is invertible. Then,

$fgg^{-1}Q = fg^{-1}Q$, i.e., $fQ = fg^{-1}Q$ for all $Q \in \mathbb{R}^2$, which says that $g^{-1}$ is a covering isometry as well.

Now, to prove that $p$ is a $\Gamma$-orbit map under the covering isometry group.

**Theorem 2.8.** *If $pP = pQ$, then $Q = gP$ for some cover $p$ and covering isometry $g$ (i.e., $P, Q$ are in the same $\Gamma$-orbit).*

*Proof.* By the local isometry property of $p$, there are disc neighborhoods $D(P)$ and $D(Q)$ mapped isometrically by $p$ onto a disc $p(D(P)) = p(D(Q))$ of $S$. Thus, $D(P) \xrightarrow{p} pD(P) \xrightarrow{p^{-1}} D(Q)$ is an isometry $g : D(P) \to D(Q)$. Since $D(P)$ contains three points not in a line, $g$ is one of the euclidean isometries classified in section 1. Now, we show that $g$ is a covering isometry for $p$, that $pR = pgR$ for all $R \in \mathbb{R}^2$.

Suppose for a contradiction that $pR \neq pgR$ for some $R \in \mathbb{R}^2$. Then, consider the set $\{R|pR = pgR\}$, which we claim is open by the fact that $p$ and $pg$ agee on the open disc $D(P)$. Then, the set $\{R|pR \neq pgR\}$ is closed.

Since the $\{R|pR \neq pgR\}$ is closed, there must be a least element (the one closest to P), which we shall call $R'$. Given that $R'$ is the nearest point to $P$ within the set, consider a sequence $R'_1, R'_2, ...$ of points between $P$ and $R'$ which converge to $R'$. By hypothesis,

$$pR'_i = pgR'_i$$

and by continuity of $p$ and $g$,

$$p(\lim_{i\to\infty} R_i) = pg(\lim_{i\to\infty} R_i);$$

which shows that $pR = pgR$, and hence is a contradiction. Thus, $pR = pgR$ for all $R \in \mathbb{R}^2$, and $g$ is a covering isometry. □

**Corollary 2.9** (Killing-Hopf theorem)**.** *Each complete, connected euclidean surface is of the form* $\mathbb{R}/\Gamma$*, and hence is either a cylinder, twisted cylinder, torus, or Klein bottle (if not* $\mathbb{R}^2$ *itself).*

**Acknowledgments.** I would like to thank my mentor, Daniel Studemund, for introducing me to the topics of this paper, and for his help throughout the paper-writing process. I would also like to thank Professor May for making the REU an exciting and interesting experience for students of all mathematical backgrounds.

## References

[1] John Stillwell. Geometry of Surfaces. 1992 Springer-Verlag New York, Inc.