THE REPRESENTATIONS OF THE SYMMETRIC GROUP

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Abstract. Young tableau is a combinatorial object which provides a convenient way to describe the group representations of the symmetric group, S_n. In this paper, we prove several facts about the symmetric group, group representations, and Young tableaux. We then present the construction of Specht modules which are irreducible representations of S_n.

Contents

1. The Symmetric Group. S_n

Definitions 1.1. The symmetric group, S_Ω, is a group of all bijections from Ω to itself under function composition. The elements $\pi \in S_\Omega$ are called **permutations**. In particular, for $\Omega = \{1, 2, 3, \dots, n\}$, S_Ω is **the symmetric group of degree** n, denoted by S_n.

Example 1.2. $\sigma \in S_7$ given by $\begin{array}{c|ccccccc} i & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \sigma(i) & 2 & 5 & 6 & 4 & 7 & 3 & 1 \end{array}$ is a permutation.

Definition 1.3. A **cycle** is a string of integers which represents the element of S_n that cyclically permutes these integers. The cycle $(a_1\ a_2\ a_3\ \dots\ a_m)$ is the permutation which sends a_i to a_{i+1} for $1 \leq i \leq m-1$ and sends a_m to a_1.

Proposition 1.4. *Every permutation in S_n can be written as a product of disjoint cycles.*

Proof. Consider $\pi \in S_n$. Given $i \in \{1, 2.3, \dots, n\}$, the elements of the sequence $i, \pi(i), \pi^2(i), \pi^3(i), \dots$ cannot all be distinct. Taking the first power p such that $\pi^p(i) = i$, we have the cycle $(i\ \pi(i)\ \pi^2(i)\ \dots\ \pi^{p-1}(i))$. Iterate this process with an element that is not in any of the previously generated cycles until each element of $\{1, 2, 3, ..., n\}$ belongs to exactly one of the cycles generated. Then, π is the product of the generated cycles. □

Definition 1.5. If $\pi \in S_n$ is the product of disjoint cycles of lengths $n_1, n_2, \dots, n_r$ such that $n_1 \leq n_2 \leq \dots \leq n_r$, then the integers $n_1, n_2, \dots, n_r$ are called the **cycle type** of π.

For instance, σ in *Example 1.2.* can be expressed as $\sigma = (4)(3\,6)(1\,2\,5\,7)$ and its cycle type is $1, 2, 4$. A 1-cycle of a permutation, such as (4) of σ, is called a **fixed point** and usually omitted from the cycle notation. Another way to represent the cycle type is as a partition:

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Definition 1.6. A **partition** of n is a sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ where the λ_i are weakly decreasing and $\sum_{i=1}^{l} \lambda_i = n$. If $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ is a partition of n, we write $\lambda \vdash n$.

σ corresponds to the partition $\lambda = (4, 2, 1)$.

Definitions 1.7. In any group G, elements g and h are **conjugates** if $g = khk^{-1}$ for some $k \in G$. The set of all elements conjugate to a given g is called the **conjugacy class** of g and is denoted by K_g.

Proposition 1.8. *Conjugacy is an equivalence relation. Thus, the distinct conjugacy classes partition G.*

Proof. Let $a \sim b$ if a and b are conjugates. Since $a = \epsilon a \epsilon^{-1}$ where ϵ is the identity element of G, $a \sim a$ for all $a \in G$, and conjugacy is reflexive. Suppose $a \sim b$. Then, $a = kbk^{-1} \Leftrightarrow b = (k^{-1})a(k^{-1})^{-1}$. Hence, $b \sim a$, and conjugacy is symmetric. If $a \sim b$ and $b \sim c$, $a = kbk^{-1} = k(lcl^{-1})k^{-1} = (kl)c(kl)^{-1}$ for some $k, l \in G$, and $a \sim c$. Thus, conjugacy is transitive. □

Proposition 1.9. *In S_n, two permutations are in the same conjugacy class if and only if they have the same cycle type. Thus, there is a natural one-to-one correspondence between partitions of n and conjugacy classes of S_n.*

Proof. Consider $\pi = (a_1\ a_2\ \ldots\ a_l)\cdots(a_m\ a_{m+1}\ \ldots\ a_n) \in S_n$. For $\sigma \in S_n$,

$$\sigma\pi\sigma^{-1} = (\sigma(a_1)\ \sigma(a_2)\ \ldots\ \sigma(a_l))\cdots(\sigma(a_m)\ \sigma(a_{m+1})\ \ldots\ \sigma(a_n)).$$

Hence, conjugation does not change the cycle type. □

Definition 1.10. A 2-cycle is called a **transposition**.

Proposition 1.11. *Every element of S_n can be written as a product of transpositions*

Proof. For $(a_1\ a_2\ \ldots\ a_m) \in S_n$,

$$(a_1\ a_2\ \ldots\ a_m) = (a_1\ a_m)(a_1\ a_{m-1})\cdots(a_1\ a_2)$$

Since every cycle can be written as a product of transpositions, by *Proposition 1.4.*, every permutation can be expressed as a product of transpositions. □

Definition 1.12. If $\pi = \tau_1\tau_2 ... \tau_k$, where the τ_i are transpositions, then the **sign** of π is $sgn(\pi) = (-1)^k$.

Proposition 1.13. *The map $sgn : S_n \to \{\pm 1\}$ is a well-defined homomorphism. In other words, $sgn(\pi\sigma) = sgn(\pi)sgn(\sigma)$.*

The proof of *Proposition 1.13* may be found in [1].

2. Group Representations

Definitions 2.1. Mat_d, **the full complex matrix algebra of degree** d, is the set of all $d \times d$ matrices with entries in $\mathbb{C}$, and GL_d, **the complex general linear group of degree** d, is the group of all $X = (x_{i,j})_{d\times d} \in Mat_d$ that are invertible with respect to multiplication.

Definition 2.2. A **matrix representation** of a group G is a group homomorphism $X : G \to GL_d$.

Definition 2.3. For V a vector space, GL(V), the **general linear group** of V is the set of all invertible linear transformations of V to itself.

In this study, all vector spaces will be over $\mathbb{C}$ and of finite dimension.
Since $GL(V)$ and GL_d are isomorphic as groups if $dim\ V = d$, we can think of representations as group homomorphisms into the general linear group of a vector space.

Definitions 2.4. Let V be a vector space and G be a group. Then V is a **G-module** if there is a group homomorphism $\rho : G \to GL(V)$. Equivalently, V is a **G-module** if there is an action of G on V denoted by gv for all $g \in G$ and $v \in V$ which satisfy:

(1) $gv \in V$
(2) $g(cv + dw) = c(gv) + d(gw)$
(3) $(gh)v = g(hv)$
(4) $\epsilon v = v$

for all $g, h \in G$; $v, w \in V$; and $c, d \in \mathbb{C}$

Proof. (The Equivalence of Definitions) By letting $gv = \rho(g)(v)$, (1) means $\rho(g)$ is a transformation from V to itself; (2) represents that the transformation is linear; (3) says ρ is a group homomorphism; and (4) in combination with (3) means $\rho(g)$ and $\rho(g^{-1})$ are inverse maps of each other and, thus, invertible. □

When there is no confusion arises about the associated group, the prefix **G-** will be dropped from terms, such as shortening **G-module** to **module**.

Definition 2.5. Let V be a G-module. A **submodule** of V is a subspace W that is closed under the action of G, i.e., $w \in W \Rightarrow gw \in W$ for all $g \in G$. We write $W \leq V$ if W is a submodule of V.

Definition 2.6. A nonzero G-module V is **reducible** if it contains a nontrivial submodule W. Otherwise, V is said to be **irreducible**.

Definitions 2.7. Let V be a vector space with subspaces U and W. Then V is the **direct sum** of U and W, written $V = U \oplus W$, if every $v \in V$ can be written uniquely as a sum $v = u + w$, $u \in U$, $w \in W$. If V is a G-module and U, W are G-submodules, then we say that U and W are **complements** of each other.

Definition 2.8. An **inner product** on a vector space V is a map $< \cdot, \cdot >: V \times V \to \mathbb{C}$ that satisfies:

(1) $< x, y > = \overline{< y, x >}$
(2) $< ax, y > = a < x, y >$
(3) $< x + y, z > = < x, z > + < y, z >$
(4) $< x, x > \geq 0$ with equality only for $x = 0$

for $x, y, z \in V$ and $a \in \mathbb{C}$

Definition 2.9. For $< \cdot, \cdot >$ an inner product on a vector space V and a subspace W, the **orthogonal complement** of W is $W^\perp = \{v \in V : < v, w > = 0 \text{ for all } w \in W\}$

Note that $V = W \oplus W^\perp$.

Definition 2.10. An inner product $< \cdot, \cdot >$ on a vector space V is **invariant** under the action of G if $< gv, gw > = < v, w >$ for all $g \in G$ and $v, w \in V$.

Proposition 2.11. *Let V be a G-module, W a submodule, and $< \cdot, \cdot >$ an inner product on V. If $< \cdot, \cdot >$ is invariant under the action of G, then $W^{\perp}$ is also a G-submodule.*

Proof. Suppose $g \in G$ and $u \in W^{\perp}$. Then, for any $w \in W$,

$$< gu, w >=< g^{-1}gu, g^{-1}w >=< u, g^{-1}w >= 0$$

Hence, $gu \in W^{\perp}$, and $W^{\perp}$ is a G-submodule. □

Theorem 2.12. ***(Maschke's Theorem)*** *Let G be a finite group and let V be a nonzero G-module. Then, $V = W^{(1)} \oplus W^{(2)} \oplus ... \oplus W^{(k)}$ where each $W^{(i)}$ is an irreducible G-submodule of V.*

Proof. Induction on $d = dim\ V$

- Base Case: if $d = 1$, V itself is irreducible. Hence, $V = W^{(1)}$.

- Inductive Case: For $d > 1$, assume true for $d' < d$.
  If V is irreducible, $V = W^{(1)}$.
  Suppose V is reducible. Then, V has a nontrivial G-submodule, W.
  Let $B = \{v_1, ..., v_d\}$ be a basis for V. Consider the unique inner product on V that satisfies

  $$< v_i, v_j >= \delta_{i,j} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

  for basis elements in B.
  For any $v, w \in V$, let

  $$< v, w >' = \sum_{g \in G} < gv, gw >$$

  (1)

  $$< v, w >' = \sum_{g \in G} < gv, gw >= \sum_{g \in G} \overline{< gw, gv >} = \overline{< w, v >'}$$

  (2)

  $$< av, w >' = \sum_{g \in G} < g(av), gw >= \sum_{g \in G} a < gv, gw >= a < v, w >'$$

  (3)

  $$\begin{aligned} < v + w, z > &= \sum_{g \in G} < g(v + w), gz > \\ &= \sum_{g \in G} < gv, gz > + < gw, gz > \\ &= < v, z >' + < w, z >' \end{aligned}$$

  (4)

  $$< v, v >' = \sum_{g \in G} < gv, gv > \geq 0 \text{ and } < 0, 0 >' = \sum_{g \in G} < g0, g0 >= 0$$

Hence, $< \cdot, \cdot >'$ is an inner product on V.
Moreover, since, for $h \in G$,

$$\begin{aligned} < hv, hw >' &= \sum_{g \in G} < ghv, ghw > \\ &= \sum_{k \in G} < kv, kw > \\ &= < v, w >', \end{aligned}$$

$< \cdot, \cdot >'$ is invariant under the action of G.
Let $W^{\perp} = \{v \in V :< v, w >' = 0 \text{ for all } w \in W\}$ Then, $V = W \oplus W^{\perp}$, and $W^{\perp}$ is a G-submodule by *Proposition 2.11*. Since W and $W^{\perp}$ can be written as direct sums of irreducibles by the inductive hypothesis, V can be expressed as a direct sum of irreducibles.

□

Definition 2.13. Let V and W be G-modules. Then a G-homomorphism is a linear transformation $\theta : V \to W$ such that

$$\theta(gv) = g\theta(v)$$

for all $g \in G$ and $v \in V$.

Definition 2.14. Let V and W be G-modules. A G**-isomorphism** is a G-homomorphism $\theta : V \to W$ that is bijective. In this case, we say that V and W are G**-isomorphic**, or G**-equivalent**, denoted by $V \cong W$. Otherwise, we say that V and W are G**-inequivalent**.

Proposition 2.15. *Let $\theta : V \to W$ be a G-homomorphism. Then,*

(1) $ker\ \theta$ is a G-submodule of V
(2) $im\ \theta$ is a G-submodule of W

Proof. (1) Since $\theta(0) = 0$, $0 \in ker\ \theta$ and $ker\ \theta \neq \emptyset$, and if $v_1, v_2 \in ker\ \theta$ and $c \in \mathbb{C}$, $\theta(v_1 + cv_2) = \theta(v_1) + c\theta(v_2) = 0 + c0 = 0$ and $v_1 + cv_2 \in ker\ \theta$. Hence, $ker\ \theta$ is a subspace of V. Suppose $v \in ker\ \theta$. Then, for any $g \in G$

$$\begin{aligned} \theta(gv) &= g\theta(v) \\ &= g0 \\ &= 0 \end{aligned}$$

Thus, $gv \in ker\ \theta$ and $ker\ \theta$ is a G-submodule of V.

(2) $0 \in im\ \theta$ and $im\ \theta \neq \emptyset$, and if $w_1, w_2 \in W$ and $c \in \mathbb{C}$, there exist $v_1, v_2 \in V$ such that $\theta(v_1) = w_1$ and $\theta(v_2) = w_2$ and $\theta(v_1 + cv_2) = \theta(v_1) + c\theta(v_2) = w_1 + cw_2$. Thus, $w_1 + cw_2 \in im\ \theta$ and $im\ \theta$ is a subspace of W. Suppose $w \in im\ \theta$. Then, there exists $v \in V$ such that $\theta(v) = w$. For any $g \in G$, $gv \in V$ and

$$\theta(gv) = g\theta(v) = gw$$

Hence, $gw \in im\ \theta$ and $im\ \theta$ is a G-submodule of W.

□

Theorem 2.16. ***(Schur's Lemma)*** *Let V and W be irreducible G-modules. If $\theta : V \to W$ is a G-homomorphism, then either*

(1) θ is a G-isomorphism, or

(2) θ is the zero map

Proof. Since V is irreducible and $ker\ \theta$ is a submodule by *Proposition 2.15.*, $ker\ \theta = \{0\}$ or $ker\ \theta = V$. Similarly, $im\ \theta = \{0\}$ or $im\ \theta = W$. If $ker\ \theta = \{0\}$ and $im\ \theta = W$, θ is a G-isomorphism, and if $ker\ \theta = V$ and $im\ \theta = \{0\}$, θ is the zero map. □

Corollary 2.17. *Let V be a irreducible G-module. If $\theta : V \to V$ is a G-homomorphism, $\theta = cI$ for some $c \in \mathbb{C}$, multiplication by a scalar.*

Proof. Since $\mathbb{C}$ is algebraically closed, θ has an eigenvalue $c \in \mathbb{C}$. Then, $\theta - cI$ has a nonzero kernel. By *Theorem 2.16.*, $\theta - cI$ is the zero map. Hence, $\theta = cI$. □

Definition 2.18. Given a G-module V, the corresponding **endomorphism algebra** is

$$End\ V = \{\theta : V \to V : \theta \text{ is a } G\text{-homomorphism}\}$$

Definition 2.19. The **center** of an algebra A is

$$Z_A = \{a \in A : ab = ba \text{ for all } b \in A\}$$

Let $E_{i,j}$ be the matrix of zeros with exactly 1 one in position (i, j).

Proposition 2.20. *The center of Mat_d is*

$$Z_{Mat_d} = \{cI_d : c \in \mathbb{C}\}$$

Proof. Suppose that $C \in Z_{Mat_d}$. Consider

$$CE_{i,i} = E_{i,i}C$$

$CE_{i,i}$($E_{i,i}C$, respectively) is all zeros except for the ith column(row, respectively) which is the same as that of C. Hence, all off-diagonal elements must be 0.
For $i \neq j$,

$$C(E_{i,j} + E_{j,i}) = (E_{i,j} + E_{j,i})C$$

$$\begin{pmatrix} 0 & \dots & c_{1,j} & \dots & c_{1,i} & \dots \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots \\ 0 & \dots & c_{i,j} & \dots & c_{i,i} & \dots \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots \\ 0 & \dots & c_{j,j} & \dots & c_{j,i} & \dots \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} 0 & \dots & 0 & \dots & 0 & \dots \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots \\ c_{j,1} & \dots & c_{j,i} & \dots & c_{j,j} & \dots \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots \\ c_{i,1} & \dots & c_{i,i} & \dots & c_{i,j} & \dots \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots \end{pmatrix}$$

Then, $c_{i,i} = c_{j,j}$. Hence, all the diagonal elements must be equal, and $C = cI_d$ for some $c \in \mathbb{C}$. □

Note that, for $A, X \in Mat_d$ and $B, Y \in Mat_f$,

$$(A \oplus B)(X \oplus Y) = AB \oplus XY$$

Theorem 2.21. *Let V be a G-module such that*

$$V \cong m_1V^{(1)} \oplus m_2V^{(2)} \oplus \cdots m_kV^{(k)}$$

where the $V^{(i)}$ are pairwise inequivalent irreducibles and $dim\ V^{(i)} = d_i$. Then,

(1) $dim\ V = m_1d_1 + m_2d_2 + \cdots m_kd_k$
(2) $End\ V \cong \bigoplus_{i=1}^{k} Mat_{m_i}$
(3) $dim\ Z_{End\ V} = k$.

Proof.

(1) Clear.
(2) By *Theorem 2.16.* and *Corollary 2.17.*, $\theta \in End\ V$ maps each $V^{(i)}$ into m_i copies of $V^{(i)}$ as multiplications by scalars. Hence,
$$End\ V \cong Mat_{m_1} \oplus Mat_{m_2} \oplus \cdots \oplus Mat_{m_k}$$
(3) Consider $C \in Z_{End\ V}$. Then,
$$CT = TC \text{ for all } T \in End\ V \cong \bigoplus_{i=1}^{k} Mat_{m_i}$$
where $T = \bigoplus_{i=1}^{k} M_{m_i}$ and $C = \bigoplus_{i=1}^{k} C_{m_i}$.
$$\begin{aligned} CT &= (\bigoplus_{i=1}^{k} C_{m_i})(\bigoplus_{i=1}^{k} M_{m_i}) \\ &= \bigoplus_{i=1}^{k} C_{m_i} M_{m_i} \end{aligned}$$
Similarly, $TC = \bigoplus_{i=1}^{k} M_{m_i} C_{m_i}$. Hence,
$$C_{m_i} M_{m_i} = M_{m_i} C_{m_i} \text{ for all } M_{m_i} \in Mat_{m_i}$$
By *Proposition 2.20.*, $C_{m_i} = c_i I_{m_i}$ for some $c_i \in \mathbb{C}$. Thus,
$$C = \bigoplus_{i=1}^{k} c_i I_{m_i}$$
and $dim\ Z_{End\ V} = k$.

□

Proposition 2.22. *Let V and W be G-modules with V irreducible. Then, $dim\ Hom(V, W)$ is the multiplicity of V in W.*

Proof. Let m be the multiplicity of V in W. By *Theorem 2.16.* and *Corollary 2.17.*, $\theta \in Hom(V, W)$ maps V into m copies of V in W as multiplications by scalars. Hence,
$$dim\ Hom(V, W) = m$$

□

Definition 2.23. For a group $G = \{g_1, g_2, \ldots, g_n\}$, the corresponding **group algebra** of G is a G-module
$$\mathbb{C}[G] = \{c_1 g_1 + c_2 g_2 + \cdots + c_n g_n : c_i \in \mathbb{C} \text{ for all } i\}$$

Proposition 2.24. *Let G be a finite group and suppose $\mathbb{C}[G] = \bigoplus_{i=1}^{k} m_i V^{(i)}$ where the $V^{(i)}$ form a complete list of pairwise inequivalent irreducible G-modules. Then,*
$$\text{number of } V^{(i)} = k = \text{ number of conjugacy classes of } G$$

Proof. For $v \in \mathbb{C}[G]$, let the map $\phi_v : \mathbb{C}[G] \to \mathbb{C}[G]$ be right multiplication by v. In other words,

$$\phi_v(w) = wv \text{ for all } w \in \mathbb{C}[G]$$

Since $\phi_v(gw) = (gw)v = g(wv) = g\phi_v(w)$, $\phi_v \in End\, \mathbb{C}[G]$.
Claim: $\mathbb{C}[G] \cong End\, \mathbb{C}[G]$
Consider $\psi : \mathbb{C}[G] \to End\, \mathbb{C}[G]$ such that $\psi(v) = \phi_v$.

$$\psi(v)\psi(w) = \phi_v\phi_w = \phi_{wv} = \psi(wv)$$

If $\psi(v) = \phi_v$ is the zero map, then

$$0 = \phi_v(\epsilon) = \epsilon v = v.$$

Hence, ψ is injective.
Suppose $\theta \in End\, \mathbb{C}[G]$ and let $v = \theta(\epsilon) \in \mathbb{C}[G]$. For any $g \in G$,

$$\theta(g) = \theta(g\epsilon) = g\theta(\epsilon) = gv = \phi_v(g)$$

Since θ and ϕ_v agree on a basis G, $\theta = \phi_v$ and ψ is surjective. Thus, ψ is an anti-isomorphism, and $\mathbb{C}[G] \cong End\, \mathbb{C}[G]$.
By (3) *of Theorem 2.21.*, $k = dim\, Z_{End\, \mathbb{C}[G]} = dim\, Z_{\mathbb{C}[G]}$.
Consider $z = c_1g_1 + c_2g_2 + \cdots + c_ng_n \in Z_{\mathbb{C}[G]}$.
For all $h \in G$, $zh = hz \Leftrightarrow z = hzh^{-1} \Leftrightarrow$

$$c_1g_1 + c_2g_2 + \cdots + c_ng_n = c_1hg_1h^{-1} + c_2hg_2h^{-1} + \cdots + c_nhg_nh^{-1}$$

Since hg_ih^{-1} runs over the conjugacy class of g_i, all elements of each conjugacy class have the same coefficient. If G has l conjugacy classes $K_1, \ldots, K_l$, let

$$z_i = \sum_{g \in K_i} g \text{ for } i = 1, \ldots, l.$$

Then, any $z \in Z_{\mathbb{C}[G]}$ can be written as

$$z = \sum_{i=1}^{l} d_i z_i.$$

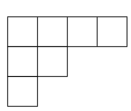
Hence,

$$\text{number of conjugacy classes} = dim\, Z_{\mathbb{C}[G]} = k = \text{number of } V^{(i)}.$$

□

3. Young Tableaux

Definition 3.1. Suppose $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l) \vdash n$. The **Young diagram**, or **shape**, of λ is a collection of boxes arranged in l left-justified rows with row i containing λ_i boxes for $1 \le i \le l$.

Example 3.2.  is the Young diagram of $\lambda = (4, 2, 1)$.

Definition 3.3. Suppose $\lambda \vdash n$. **Young tableau of shape** λ is an array t obtained by filling the boxes of the Young diagram of λ with the numbers $1, 2, \ldots, n$ bijectively.

Let $t_{i,j}$ stand for the entry of t in the position (i, j) and $sh\, t$ denote the shape of t.

Example 3.4. $t = \begin{array}{|c|c|c|c|} \hline 2 & 5 & 6 & 4 \\ \hline 7 & 3 \\ \cline{1-2} 1 \\ \cline{1-1} \end{array}$ is a Young tableau of $\lambda = (4, 2, 1)$, and $t_{1,3} = 6$.

$\pi \in S_n$ acts on a tableau $t = (t_{i,j})$ of $\lambda \vdash n$ as follows:

$$\pi t = (\pi t_{i,j}) \text{ where } \pi t_{i,j} = \pi(t_{i,j})$$

Definitions 3.5. Two λ-tableaux t_1 and t_2 are **row equivalent**, $t_1 \sim t_2$, if corresponding rows of the two tableaux contain the same elements. A **tabloid of shape** λ, or λ**-tabloid**, is then $\{t\} = \{t_1 : t_1 \sim t\}$ where $sh\, t = \lambda$.

If $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l) \vdash n$, then the number of tableaux in a λ-tabloid is

$$\lambda_1!\lambda_2!...\lambda_l! \stackrel{\text{def}}{=} \lambda!.$$

Hence, the number of λ-tabloids is $n!/\lambda!$.

Example 3.6. For $s = \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array}$, $\{s\} = \left\{ \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array}, \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 4 & 3 \\ \hline \end{array}, \begin{array}{|c|c|} \hline 2 & 1 \\ \hline 3 & 4 \\ \hline \end{array}, \begin{array}{|c|c|} \hline 2 & 1 \\ \hline 4 & 3 \\ \hline \end{array} \right\} \stackrel{\text{def}}{=} \begin{array}{cc} \hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array}$

Definition 3.7. Suppose $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_m)$ are partitions of n. Then λ **dominates** μ, written $\lambda \trianglerighteq \mu$, if $\lambda_1 + \lambda_2 + \ldots + \lambda_i \geq \mu_1 + \mu_2 + \ldots + \mu_i$ for all $i \geq 1$. If $i > l$ ($i > m$, respectively), then we take λ_i (μ_i, respectively) to be zero.

Lemma 3.8. ***(Dominance Lemma for Partitions)*** *Let t^λ and s^μ be tableaux of shapes λ and μ, respectively. If for each index i, the elements of row i in s^μ are all in different columns of t^λ, then $\lambda \trianglerighteq \mu$.*

Proof. Since the elements of row 1 in s^μ are all in different columns of t^λ, we can sort the entries in each column of t^λ so that the elements of row 1 in s^μ all occur in the first row of $t^\lambda_{(1)}$. Then, since the elements of row 2 in s^μ are also all in different columns of t^λ and, thus, $t^\lambda_{(1)}$, we can re-sort the entries in each column of $t^\lambda_{(1)}$ so that the elements of rows 1 and 2 in s^μ all occur in the first two rows of $t^\lambda_{(2)}$. Inductively, the elements of rows $1, 2, \ldots, i$ in s^μ all occur in the first i rows of $t^\lambda_{(i)}$. Thus,

$$\begin{aligned} \lambda_1 + \lambda_2 + \ldots + \lambda_i &= \text{number of elements in the first } i \text{ rows of } t^\lambda_{(i)} \\ &\geq \text{number of elements in the first } i \text{ rows of } s^\mu \\ &= \mu_1 + \mu_2 + \ldots + \mu_i \end{aligned}$$

□

Definition 3.9. Suppose $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_m)$ are partitions of n. Then $\lambda > \mu$ in **lexicographic order** if, for some index i,

$$\lambda_j = \mu_j \text{ for } j < i \text{ and } \lambda_i > \mu_i$$

Proposition 3.10. *If $\lambda, \mu \vdash n$ with $\lambda \trianglerighteq \mu$, then $\lambda \geq \mu$.*

Proof. Suppose $\lambda \neq \mu$. Let i be the first index where they differ. Then, $\sum_{j=1}^{i-1} \lambda_j = \sum_{j=1}^{i-1} \mu_j$ and $\sum_{j=1}^{i} \lambda_j > \sum_{j=1}^{i} \mu_j$. Hence, $\lambda_i > \mu_i$. □

4. Representations of the Symmetric Group

Definition 4.1. Suppose $\lambda \vdash n$. Let $M^\lambda = \mathbb{C}\{\{t_1\}, \ldots, \{t_k\}\}$, where $\{t_1\}, \ldots, \{t_k\}$ is a complete list of λ-tabloids. Then M^λ is called the **permutation module corresponding to** λ.

M^λ is indeed an S_n-module by letting $\pi\{t\} = \{\pi t\}$ for $\pi \in S_n$ and t a λ-tableau. In addition, $dim\ M^\lambda = n!/\lambda!$, the number of λ-tabloids.

Definition 4.2. Any G-module M is **cyclic** if there is a $v \in M$ such that $M = \mathbb{C}Gv$ where $Gv = \{gv : g \in G\}$. In this case, we say that M is **generated by** v.

Proposition 4.3. *If $\lambda \vdash n$, then M^λ is cyclic, generated by any given λ-tabloid.*

Definition 4.4. Suppose that the tableau t has rows $R_1, R_2, \ldots, R_l$ and columns $C_1, C_2, \ldots, C_k$. Then,

$$R_t = S_{R_1} \times S_{R_2} \times \ldots \times S_{R_l}$$

and

$$C_t = S_{C_1} \times S_{C_2} \times \ldots \times S_{C_k}$$

are the **row-stabilizer** and **column-stabilizer** of t, respectively.

Example 4.5. For t in *Example 3.4.*, $R_t = S_{\{2,4,5,6\}} \times S_{\{3,7\}} \times S_{\{1\}}$ and $C_t = S_{\{1,2,7\}} \times S_{\{3,5\}} \times S_{\{6\}} \times S_{\{4\}}$.

Given a subset $H \subseteq S_n$, let $H^+ = \sum_{\pi \in H} \pi$ and $H^- = \sum_{\pi \in H} sgn(\pi)\pi$ be elements of $\mathbb{C}[S_n]$. If $H = \{\pi\}$, then we denote H^- by π^-.
For a tableau t, let $\kappa_t = C_t^- = \sum_{\pi \in C_t} sgn(\pi)\pi$. Note that if t has columns $C_1, C_2, ..., C_k$, then $\kappa_t = \kappa_{C_1}\kappa_{C_2}...\kappa_{C_k}$.

Definition 4.6. If t is a tableau, then the associated **polytabloid** is $e_t = \kappa_t\{t\}$.

Example 4.7. For s in *Example 3.6.*,

$$\begin{aligned} \kappa_s &= \kappa_{C_1}\kappa_{C_2} \\ &= (\epsilon - (1\,3))(\epsilon - (2\,4)) \end{aligned}$$

Thus,

$$e_t = \begin{array}{cc} \hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array} - \begin{array}{cc} \hline 3 & 2 \\ \hline 1 & 4 \\ \hline \end{array} - \begin{array}{cc} \hline 1 & 4 \\ \hline 3 & 2 \\ \hline \end{array} + \begin{array}{cc} \hline 3 & 4 \\ \hline 1 & 2 \\ \hline \end{array}$$

Lemma 4.8. *Let t be a tableau and π be a permutation. Then,*

(1) $R_{\pi t} = \pi R_t \pi^{-1}$
(2) $C_{\pi t} = \pi C_t \pi^{-1}$
(3) $\kappa_{\pi t} = \pi \kappa_t \pi^{-1}$
(4) $e_{\pi t} = \pi e_t$

Proof.

(1)

$$\begin{aligned} \sigma \in R_{\pi t} &\Leftrightarrow \sigma\{\pi t\} = \{\pi t\} \\ &\Leftrightarrow \pi^{-1}\sigma\pi\{t\} = \{t\} \\ &\Leftrightarrow \pi^{-1}\sigma\pi \in R_t \\ &\Leftrightarrow \sigma \in \pi R_t \pi^{-1} \end{aligned}$$

(2) and (3) can be shown analogously to (1).

(4)

$$e_{\pi t} = \kappa_{\pi t}\{\pi t\} = \pi\kappa_t\pi^{-1}\{\pi t\} = \pi\kappa_t\{t\} = \pi e_t$$

□

Definition 4.9. For a partition $\lambda \vdash n$, the corresponding **Specht module,** S^λ**,** is the submodule of M^λ spanned by the polytabloids e_t, where $sh\,t = \lambda$.

Proposition 4.10. *The S^λ are cyclic modules generated by any given polytabloid.*

Given any two λ-tabloids t_i, t_j in the basis of M^λ, let their inner product be

$$< \{t_i\}, \{t_j\} >= \delta_{\{t_i\},\{t_j\}} = \begin{cases} 1 & \text{if } \{t_i\} = \{t_j\} \\ 0 & \text{otherwise} \end{cases}$$

and extend by linearity in the first variable and conjugate linearity in the second to obtain an inner product on M^λ.

Lemma 4.11. ***(Sign Lemma)*** *Let $H \le S_n$ be a subgroup.*

(1) If $\pi \in H$, then

$$\pi H^- = H^-\pi = sgn(\pi)H^-$$

(2) For any $u, v \in M^\lambda$,

$$< H^-u, v >=< u, H^-v >$$

(3) If the transposition $(b\,c) \in H$, then we can factor

$$H^- = k(\epsilon - (b\,c))$$

where $k \in \mathbb{C}[S_n]$.

(4) If t is a tableau with b, c in the same row of t and $(b\,c) \in H$, then

$$H^-\{t\} = 0$$

Proof.

(1)

$$\begin{aligned} \pi H^- &= \pi \sum_{\sigma\in H} sgn(\sigma)\sigma \\ &= \sum_{\sigma\in H} sgn(\sigma)\pi\sigma \\ &= \sum_{\tau\in H} sgn(\pi^{-1}\tau)\tau && \text{(by letting } \tau = \pi\sigma) \\ &= \sum_{\tau\in H} sgn(\pi^{-1})sgn(\tau)\tau \\ &= sgn(\pi^{-1})\sum_{\tau\in H} sgn(\tau)\tau \\ &= sgn(\pi)H^- \end{aligned}$$

$H^-\pi = sgn(\pi)H^-$ can be proven analogously.

(2)

$$\begin{aligned} < H^- u, v > &= \sum_{\pi \in H} < sgn(\pi)\pi u, v > \\ &= \sum_{\pi \in H} < u, sgn(\pi^{-1})\pi^{-1} v > \\ &= \sum_{\tau \in H} < u, sgn(\tau)\tau v > \qquad \text{(by letting } \tau = \pi^{-1}) \\ &= < u, H^- v > \end{aligned}$$

(3) Consider the subgroup $K = \{\epsilon, (b\,c)\} \le H$. Let $\{k_i : i \in I\}$ be a transversal such that $H = \sqcup_{i\in I} k_i K$. Then, $H^- = (\sum_{i\in I} k_i)(\epsilon - (b\,c))$.

(4) $(b\,c)\{t\} = \{t\}$. Hence,

$$H^-\{t\} = k(\epsilon - (b\,c))\{t\} = k(\{t\} - \{t\}) = 0$$

□

Corollary 4.12. *Let t be a λ-tableau and s be a μ-tableau, where $\lambda, \mu \vdash n$. If $\kappa_t\{s\} \neq 0$, then $\lambda \trianglerighteq \mu$. Moreover, if $\lambda = \mu$, then $\kappa_t\{s\} = \pm e_t$*

Proof. Suppose b and c are two elements in the same row of s. If they are in the same column of t, then $(b\,c) \in C_t$ and $\kappa_t\{s\} = 0$ by (4) of *Sign Lemma*. Hence, the elements in each row of s are all in different columns in t, and $\lambda \trianglerighteq \mu$ by *Dominance Lemma*.
If $\lambda = \mu$, then $\{s\} = \pi\{t\}$ for some $\pi \in C_t$. Then, by (4) of *Sign Lemma*,

$$\kappa_t\{s\} = \kappa_t \pi\{t\} = sgn(\pi)\kappa_t\{t\} = \pm e_t$$

□

Corollary 4.13. *If $u \in M^\mu$ and $sh\, t = \mu$, then $\kappa_t u$ is a multiple of e_t.*

Proof. Let $u = \sum_{i\in I} c_i\{s_i\}$ where $c_i \in \mathbb{C}$ and s_i are μ-tableaux. By *Corollary 4.12.*, $\kappa_t u = \sum_{i\in J} \pm c_i e_t = (\sum_{i\in J} \pm c_i)e_t$ for some $J \subseteq I$. □

Theorem 4.14. ***(Submodule Theorem)*** *Let U be a submodule of M^μ. Then,*

$$U \supseteq S^\mu \quad \text{or} \quad U \subseteq S^{\mu\perp}$$

Thus, S^μ is irreducible.

Proof. For $u \in U$ and a μ-tableau t, $\kappa_t u = ce_t$ for some $c \in \mathbb{C}$ by *Corollary 4.13.*.
Suppose that there exists a u and t such that $c \neq 0$. Then, since U is a submodule, $ce_t = \kappa_t u \in U$. Hence, $e_t \in U$ and $S^\mu \subseteq U$ since S^μ is cyclic.
Otherwise, $\kappa_t u = 0$ for all $u \in U$ and all μ-tableau t. Then, by (2) of *Sign Lemma*,

$$< u, e_t > = < u, \kappa_t\{t\} > = < \kappa_t u, \{t\} > = < 0, \{t\} > = 0.$$

Since e_t span S^μ, $u \in S^{\mu\perp}$ and $U \subseteq S^{\mu\perp}$.
$S^\mu \cap S^{\mu\perp} = 0$. Hence, S^μ is irreducible. □

Proposition 4.15. *If $\theta \in Hom(S^\lambda, M^\mu)$ is nonzero, then $\lambda \trianglerighteq \mu$. Moreover, if $\lambda = \mu$, then θ is multiplication by a scalar.*

Proof. Since $\theta \neq 0$, there exists a basis element $e_t \in S^\lambda$ such that $\theta(e_t) \neq 0$. Because $M^\lambda = S^\lambda \oplus S^{\lambda\perp}$, we can extend θ to an element of $Hom(M^\lambda, M^\mu)$ by letting $\theta(S^{\lambda\perp}) = \{0\}$. Then,

$$0 \neq \theta(e_t) = \theta(\kappa_t\{t\}) = \kappa_t\theta(\{t\}) = \kappa_t(\sum_i c_i\{s_i\})$$

where $c_i \in \mathbb{C}$ and s_i are μ-tableaux. Hence, by *Corollary 4.12.*, $\lambda \trianglerighteq \mu$.
If $\lambda = \mu$, $\theta(e_t) = ce_t$ for some $c \in \mathbb{C}$ by *Corollary 4.12.*. For any permutation π,

$$\theta(e_{\pi t}) = \theta(\pi e_t) = \pi\theta(e_t) = \pi(ce_t) = ce_{\pi t}$$

Thus, θ is multiplication by c. □

Theorem 4.16. *The S^λ for $\lambda \vdash n$ form a complete list of irreducible S_n-modules.*

Proof. Since the number of irreducible modules equals the number of conjugacy classes of S_n by *Proposition 2.24.*, it suffices to show that they are pairwise inequivalent. Suppose $S^\lambda \cong S^\mu$. Then, there exists a nonzero $\theta \in Hom(S^\lambda, M^\mu)$ since $S^\lambda \subseteq M^\mu$. Thus, by *Proposition 4.15.*, $\lambda \trianglerighteq \mu$. Analogously, $\lambda \trianglelefteq \mu$. Hence, $\lambda = \mu$. □

Corollary 4.17. *The permutation modules decompose as*

$$M^\mu = \bigoplus_{\lambda \trianglerighteq \mu} m_{\lambda\mu} S^\lambda$$

where the diagonal multiplicity $m_{\mu\mu} = 1$.

Proof. If S^λ appears in M^μ with nonzero multiplicity, then there exists a nonzero $\theta \in Hom(S^\lambda, M^\mu)$ and $\lambda \trianglerighteq \mu$ by *Proposition 4.15.*. If $\lambda = \mu$, then

$$m_{\mu\mu} = dim\ Hom(S^\mu, M^\mu) = 1$$

by *Propositions 2.22. and 4.15.*. □

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