A Rigorous Introduction to Brownian Motion

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Abstract

In this paper we develop the basic properties of Brownian motion then go on to answer a few questions regarding its zero set and its local maxima.

Contents

1 The Basics

The concept of a Brownian motion was discovered when Einstein observed particles oscillating in liquid. Since fluid dynamics are so chaotic and rapid at the molecular level, this process can be modeled best by assuming the particles move randomly and independently of their past motion. We can also think of Brownian motion as the limit of a random walk as its time and space increments shrink to 0. In addition to its physical importance, Brownian motion is a central concept in stochastic calculus which can be used in finance and economics to model stock prices and interest rates.

1.1 Brownian Motion Defined

Since we are trying to capture physical intuition, we define a Brownian motion by the properties we want it to have and worry about proving the existence of and explicitly constructing such a process later.

Definition 1. *An* $\mathbb{R}^d$*-valued stochastic process* $\{B(t) : t \geq 0\}$ *is called a d-dimensional* ***Brownian motion*** *starting at* $x \in \mathbb{R}^d$ *if it has the following four properties:*

- *Start at x: B(0) = x*
- *Independent increments: for all* $0 \leq t_1 \leq t_2 \leq \ldots \leq t_n$*, the increments* $B(t_n) - B(t_{n-1}), \ldots, B(t_2) - B(t_1)$ *are independent random variables*
- *Normality: for all* $t \geq 0$ *and* $h > 0$ *the increment* $B(t+h) - B(t)$ *is distributed* $N(0, h)$
- *Continuity: almost surely,* $t \mapsto B(t)$ *is continuous*

The first property anchors the stochastic process in space. The second captures the continually random nature of a particle that is being constantly buffeted by fluid molecules. The third is required because the expected displacement of a particle should be proportional to the time it has been traveling and should be symmetrically distributed about the starting point. Physical motion is continuous which explains the fourth requirement.

While a Brownian motion is frequently denoted $\{B(t) | t \geq 0\}$ to stress the fact that it is actually an uncountable family of random variables, we will use B_t as shorthand, understanding that t varies over the non-negative reals. The bulk of the first and third sections apply to general Brownian motions and in the fourth we specialize to the linear case.

The construction of Brownian motion is tedious and beyond the scope of this paper. But we should remember that it is the characteristics of Brownian motion, rather than its construction, which define it. Indeed, there are even different constructions. The details of the construcion will not be used in this paper.

1.2 Nondifferentiability of Brownian motion

The most striking quality of Brownian motion is probably its nowhere differentiability.

Theorem 1. *Almost surely, Brownian motion is nowhere differentiable*

The proof consists primarily of a long computation which we do not present. We will prove later that in any small interval to the right of some time s, B_t attains values greater than and less than B_s. So for all $\epsilon > 0$ and $s \geq 0$, we can choose some $h \in (s, s+\epsilon)$ such that $\frac{B_{s+h} - B_s}{h}$ is either positive or negative. This supports the idea that the upper and lower derivatives of B_t at every point are $+\infty$ and $-\infty$, respectively, although a good deal of computational work goes into proving that the upper and lower limits diverge as $\epsilon \to 0$. Nonetheless, knowing that they do diverge does give us insight into how rapidly and erratically Brownian motion jumps around.

The theorem can also be understood directly from the definition of Brownian motion. If B_t were differentiable at some point s, we would know where it was going in some small time interval in the future, but the independent increment property of B_t should make us skeptical of such a prediction.

The next proposition is a manifestation of the combination of Brownian motion's nondifferntiability and its continuity.

Proposition 1. *Almost surely, B_t is not monotonic on any interval.*

Proof. Fix $0 \leq a < b$. Let $P(a,b)$ be the probability that B_t is monotonic on (a,b). Then, by independence of increments,

$$P(a,b) \leq \frac{1}{2} \cdot P(a, \frac{a+b}{2}) \cdot P(\frac{a+b}{2}, b) \leq \frac{1}{2} \cdot P(a, \frac{a+b}{2})$$

since, even if B_t is monotonic on $(a, \frac{a+b}{2})$ and $(\frac{a+b}{2}, b)$, the probability that it is monotonic in the same direction on both intervals is $\frac{1}{2}$. Iterating the divisions of the interval into halves n times we get

$$P(a,b) \leq (\frac{1}{2})^n \cdot P(a, a + \frac{b-a}{2^n})$$

Taking the limit as $n \to \infty$, we see $P(a,b) \leq (\frac{1}{2})^n \cdot P(a, a + \frac{b-a}{2^n}) \leq \frac{1}{2}^n \to 0$, showing $P(a,b) = 0$, so any fixed interval is almost surely not monotonic.

By taking the countable union over all intervals with rational endpoints we can see that B_t is almost surely not monotonic on any interval with rational endpoints. By the density of $\mathbb{Q} \subset \mathbb{R}$, there is an interval with rational endpoints contained within every interval. So every interval contains a subinterval which is almost surely not monotonic, thus every interval is almost surely not monotonic. $\square$

We will use this result later when discussing the maxima of B_t.

1.3 Scaling Properties of Brownian Motion

We often study transformations of functions which leave certain properties invariant, and it is natural to ask what transformations of B_t have the same distribution.

Example 1. *$-B_t$ is a Brownian motion. Continuity and independence are clearly maintained by negative multiplication and, since the normal distribution is symmetric about zero, all the increments have the proper means and variances.*

We now move on to more interesting and useful transformations.

Proposition 2. ***Rescaling**: If B_t is a standard Brownian motion, then so is the process $X_t = aB_{\frac{t}{a^2}}$, for all $a > 0$.*

Proof. Continuity and independence of increments still hold. For all $t > s \geq 0$, the normal random variable $X(t) - X(s) = a(B(\frac{t}{a^2}) - B(\frac{s}{a^2}))$ is distributed $aN(0, \frac{t-s}{a^2}) \stackrel{d}{=} N(0, t-s)$, so $X(t) - X(s) \sim N(0, t-s)$ as desired. □

This proposition tells us that B_t is a Brownian motions on all time scales as long as we compensate for the change in variance of the increments by taking a scalar multiple of the process. More surprisingly, we can invert the domain of B_t and still have a Brownian motion.

Proposition 3. ***Time-inversion**: Let B_t be a standard Brownian motion. Then the process*

$$X_t = \begin{cases} 0 & : t = 0 \\ tB_{\frac{1}{t}} & : t \neq 0 \end{cases}$$

is also a standard Brownian motion

Proof. For Brownian motions,

$$\text{Cov}(B_t, B_{t+s}) = \text{Cov}(B_t, B_{t+s} - B_t) + \text{Cov}(B_t, B_t) = t$$

for all $t, s \geq 0$. For our process X_t,

$$\begin{aligned} \text{Cov}(X_t, X_{t+s}) &= \text{Cov}(tB_{\frac{1}{t}}, (t+s)B_{\frac{1}{t+s}}) \\ &= t(t+s)\text{Cov}(B_{\frac{1}{t}}, B_{\frac{1}{t+s}}) = t(t+s)\frac{1}{t+s} = t \end{aligned}$$

So $\text{Cov}(X_t, X_{t+s} - X_t) = \text{Cov}(X_t, X_{t+s}) - \text{Var}(X_t) = t - t = 0$. Because the random variables X_{t+s} and X_t are normal, $\text{Cov}(X_t, X_{t+s} - X_t) = 0$ implies that $X_{t+s} - X_t$ and X_t are independent. And $\text{Var}(X_{t+s} - X_t) = \text{Var}(X_{t+s}) + \text{Var}(X_t) - 2\text{Cov}(X_{t+s}, X_t) = (t+s) + t - 2t = s$, so our increments are independent and have the right variances.

Continuity is clear for $t > 0$. We know that X_t has the distribution of a Brownian motion on $\mathbb{Q}$, so

$$0 = \lim_{n \to \infty} X(\frac{1}{n}) = \lim_{t \to 0} X(t)$$

and we conclude that X_t is continuous at $t = 0$, so X_t satisfies the properties of a standard Brownian motion. $\square$

We end with section with an example which demonstrates the computational usefulness of these alternative expressions for Brownian motion.

Example 2. *Let B_t be a standard Brownian motion and $X_t = tB_{\frac{1}{t}}$. X_t is a standard Brownian motion, so*

$$\lim_{t\to\infty} \frac{X_t}{t} = \lim_{t\to\infty} B_{\frac{1}{t}} = B_0 = 0$$

2 The Relevant Measure Theory

We assume the reader is familiar with the elements of basic probability theory such as expectation, covariance, normal random variables, etc. But we do add rigor to these notions by developing the underlying measure theory, which will be necessary for our discussion of the Markov properties.

Definition 2. *A σ-**algebra** Σ on a set S is a subset of 2^S, where 2^S is the power set of S, satisfying:*

- $\{\emptyset\} \subset \Sigma$
- *for all* $A \in \Sigma$, $A^c \in \Sigma$
- *for all sequences* $A_0, A_1, \ldots \in \Sigma$, $\bigcup_{i=0}^{\infty} A_i \in \Sigma$

By de Morgan's laws we can see that σ-algebras are closed under countable intersections as well. The σ-algebra will be our object of measurement, so now we need to develop our method of measurement.

Definition 3. *A **measure** is a countably additive map $\mu : \Sigma \mapsto [0, \infty]$, where $\Sigma \subset 2^S$ is our σ-algebra on some set S. A countably additive map is one such that for any sequence $A_1, A_2, \ldots \in \Sigma$ of disjoint events, $\mu(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$. A probability measure is a measure μ such that $\mu(S) = 1$.*

Our definition implies that $\mu(\{\emptyset\}) = 0$ because $\mu(\{\emptyset\}) = \mu(\{\emptyset\} \cup \{\emptyset\}) = \mu(\{\emptyset\}) + \mu(\{\emptyset\})$. Our next definition collects these notions.

Definition 4. *The **probability triple** is the triple $(\Omega, \mathcal{F}, P)$ where $\mathcal{F}$ is a σ-algebra on the set Ω and $P : \Sigma \mapsto [0, 1]$ is a probability measure. We call Ω the sample space and $\mathcal{F}$ the collection of (P-measurable) events.*

This triple provides the background for the study probability. In the foreground are random variables.

Definition 5. *A **random variable** is an $\mathcal{F}$-measurable map $X : \Omega \mapsto \mathbb{R}$, meaning that the preimage $X^{-1}(B) \subset \mathcal{F}$ for all $B \in \mathcal{B}(\mathbb{R})$. The law of X is $P(X^{-1}) : \mathcal{B}(\mathbb{R}) \mapsto [0, 1]$.*

The random variable X is a correspondence between events and sets in $\mathbb{R}$, which formalizes the notion that X takes on certain values when certain events occur. Of course, this correspondence is not that interesting in itself; what interests us is the probability of X lying in sets in $\mathbb{R}$, which is given by the law of X. For the law of X to be well defined we need $X^{-1}(\mathcal{B}(\mathbb{R}) \subset \mathcal{F}$, since $\mathcal{F}$ is the domain of P, which is why we require X to be $\mathcal{F}$-measurable.

A stochastic process is a family of random variables that evolves over time, and up to this point we have viewed these random variables from time 0. But we can also look at the process at some time s at which the set $\{X_t | 0 \leq t \leq s\}$ is known, and the probability of events occuring past s will depend on this information.

Definition 6. *A **filtration** on a probability space $(\Omega, \mathcal{F}, P)$ is a family $\{\mathcal{F}_t | t \geq 0\}$ of σ-algebras such that $\mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F}$ for all $s < t$. A stochastic process $\{X_t | t \geq 0\}$ is adapted to the filtration if X_t is $\mathcal{F}_t$ measurable for all $t \geq 0$.*

An adapted filtration captures the intuition of our information which evolves along with our process: our information grows as time goes on.

3 Markov Properties of Brownian motion

The Markov properties tell us at what times s a Brownian motion $\{B_{t+s} | t \geq 0\}$–referred to as B_{t+s} in the future–has the same distribution as a Brownian motion started at B_s or, alternatively, when the process $B_{t+s} - B_s$ is a standard Brownian motion. We will refer to this phenomenon as Brownian motion starting anew at time s. Our independence of increments requirement might seem to make this property trivial, and, for deterministic times, it does.

Theorem 2. ***(Markov Property)*** *Let B_t be a Brownian motion and fix $s \geq 0$. $B_{t+s} - B_s$ is a standard Brownian motion independent of $\{B_t | 0 \leq t \leq s\}$.*

Proof. It is clear that B_{t+s} is a Brownian motion. Subtracting a constant only changes the starting point, and, in particular, subtracting B_s makes the process a standard Brownian motion. Independence of B_t before time s follows from the independence of increments of Brownian motion. □

A far more interesting and important class of times is random times, meaning times defined by some randomly occurring event. Brownian motion does not necessarily start afresh at such times. We provide an example, but first state a definition.

Definition 7. *A continuous function f is said to attain a* ***maximum*** *on an interval I at $s \in I$ if*

$$f(s) \geq f(t) \text{ for all } t \in I$$

We say f attains a ***local maximum*** *at s if there exists a non-degenerate interval I containing s on which $f(s)$ is a maximum. We say the (local) maximum is strict if the above inequality can be replaced by a strict inequality.*

Example 3. *Let s be a time that B_t attains a strict local maximum and define $X_t = B_{t+s} - B_s$. Then there exists some δ such that for all $r \in (s-\delta, s+\delta)$, $B_r < B_s$. So $P(X_{\frac{\delta}{2}} - X_0 < 0) = P(B_{s+\frac{\delta}{2}} - B_s > 0) = 0$. So the increment $X_{\frac{\delta}{2}} - X_0 < 0$ is certainly not normal, thus X_t is not a Brownian motion and B_t does not start anew at s.*

This example shows we need to be careful when considering random times. Nonetheless, Brownian motion does start anew at some random times.

Definition 8. *A random time $T \in [0, \infty]$ defined on a probability space with filtration $\mathcal{F}_t$ is a* ***stopping time*** *if $\{T \leq s\} \in \mathcal{F}_s$ for every $s > 0$.*

Given our heuristic understanding of $\mathcal{F}_t$ as the information up to time t, a random time is a stopping time if we can determine whether it has occurred before s based only on knowing the information up to s. This explains why B_t does not start anew at the time s where B_t attains a local maximum because we do not know that B_s is a local maximum until time $s+\delta$. Stopping times get their name because if we stop exactly at time s we can determine whether s is our stopping time; we do not need to go forward in time to see that s is in fact the time we want. We provide an example of a stopping time.

Definition 9. *For a Brownian motion in $\mathbb{R}^d$ and $a \in \mathbb{R}^d$, define $T_a = \inf\{t \geq 0 | B(t) = a\}$ as the first time B_t hits a.*

T_a is a stopping time because we can determine if $s = T_a$ by observing whether $B_s = a$ and whether $B_r = a$ for any $0 \leq r < s$, both of which are known at time s. For the same reason, the second, third, or nth time B_t hits a are also stopping times, but the last time B_t hits a is not a stopping time because we would need to see infinitely far into the future to know the process never returns to a again.

T_a is really only useful for linear Brownian motions, as it turns out that Brownian motions almost surely never hit any specific $a \in \mathbb{R} \setminus \{0\}$ for $d \geq 2$, while in the 1-dimensional case T_a is almost surely finite for all a.

We now state the strong Markov property of Brownian motion and justify our emphasis on stopping times.

Theorem 3. *(**Strong Markov Property**) For every almost surely finite stopping time T, the process $B_{T+t} - B_T$ is a standard Brownian motion independent of $\mathcal{F}_T$.*

Equivalently, conditional on $\mathcal{F}_T$, B_{T+t} is a Brownian motion started at B_T, so, for Brownian motions, we can say that the only useful information contained in $\mathcal{F}_T$ is the value of B_t at T: Brownian motion starts anew at T.

The strong Markov property allows us to prove the reflection principle.

Theorem 4. *(**Reflection Principle**)Let T be a stopping time. The process given by reflecting a Brownian motion about time T is a Brownian motion. More precisely, the process defined by*

$$B_t^* = \begin{cases} 2B(T) - B(t) & : t \geq T \\ B(t) & : 0 \leq t \leq T \end{cases}$$

is a Brownian motion.

Proof. By the strong Markov property, $B(t+T) - B(T)$ is a standard Brownian motion independent of $\{B(t)|0 \leq t \leq T\}$. So $B(T) - B(t+T)$ must be as well. If we attach the process $\{B(t)|0 \leq t \leq T\}$ in front of $\{B(t+T) - B(T)|t \geq 0\}$ we will get a new Brownian motion: the properties of Brownian motion all hold independently within the concatenated processes, independence of increments holds between them because the processes we concatenate are independent by the strong Markov property and continuity holds because the process are equal at T, the point where we glue them together. In the same way, we attach $\{B(t)|0 \leq t \leq T\}$ to the front of $B(T) - B(t+T)$, which will give us a process identical to the former concatenation, which is just B_t. Since this latter concatenation is B_t^*, we conclude B_t^* is a standard Brownian motion. □

The reflection principle is extremely useful. We will first use it to compute the density of the maximal process.

Proposition 4. *Let B_t be a linear Brownian motion and M_t be its maximal process, defined as $M_t = \max_{0 \leq s \leq t} B_s$. Then $P(M_t > a) = 2P(B_t > a)$ for all $a > 0$.*

Proof.

$$\begin{aligned} P(M_t > a) &= P(B_t > a \text{ or } M_t > a \text{ while } B_t \leq a) \\ &= P(B_t > a) + P(M_t > a \text{ and } B_t \leq a) \end{aligned}$$

since the two events are disjoint. $M_t > a$ implies $T_a < t$, so by reflecting about T_a, we see that $M_t > a$ and $B_t \leq a$ if and only if $B_t^* > a$. So $P(M_t > a) = P(B_t > a) + P(B_t^* > a) = 2P(B_t > a)$ □

4 Further Properties of Brownian motion

Throughout this section we specialize to linear Brownian motions.

4.1 The Zeroes of Brownian motion

We define the Zero set, Z, as the times B_t hits 0, or $Z = \{t \geq 0 | B_t = 0\}$. We first show that Z is small in the sense of area.

Theorem 5. *With probability one, the Lebesgue measure of Z is 0.*

Proof. Let $|Z|$ be the Lebesgue measure of Z. We compute the expectation:

$$\begin{aligned} E|Z| = E\int_0^\infty \chi_{\{0\}}(W_t)\,dt &= \int_0^\infty (1 \cdot P(W_t = 0) + 0 \cdot P(W_t \neq 0))\,dt \\ = \int_0^\infty P(W_t = 0)\,dt = 0 \end{aligned}$$

We know $|Z|$ is non-negative since it is a measure. We now show that non-negative random variables with expectation 0 are almost surely 0.

Suppose $X \geq 0$ is a random variable with $EX = 0$ and fix $a > 0$. Then

$$0 = EX = \int_\Omega X \cdot dP \geq \int_{\{X \geq a\}} X \cdot dP \geq a \cdot P(X \geq a) \geq 0$$

So $P(X \geq a) = 0$ for all $a > 0$. Letting $a \to 0+$, we see $P(X > 0) = 0$, and X is almost surely 0. We conclude that E|Z| = 0 ⇒ |Z| = 0 almost surely. □

We should have expected this result because we know that B_t is almost surely nonzero for all nonzero t since the increment $B_t - B_0 = B_t$ is a normal random variable. Nonetheless, B_t hits 0 infinitely many times in any interval to the right of the origin.

Proposition 5. *Almost surely, B_t has infinitely many zeros in every time interval $(0, \epsilon)$, where $\epsilon > 0$.*

Proof. We induct on the number of zeros in $(0, \epsilon)$.

First, we show that there must be a zero in this interval. Let M_t^+ and M_t^- be the maximal and minimal processes, respectively, and fix $\epsilon > 0$. We know that $P(M_\epsilon^+ > a) = 2 \cdot P(B_\epsilon > a)$. By taking the limit as $a \to 0+$, we see $P(M_\epsilon^+ > 0) = 2 \cdot P(B_\epsilon > 0) = 1$ since B_t is symmetric. By symmetry, $P(M^-(\epsilon) < 0) = 1$. Since B_t is almost surely continuous, we employ the intermediate value theorem and conclude that $B_t = 0$ for some $t \in (0, \epsilon)$.

Now take some finite set S of zeros of B_t on the interval $(0, \epsilon)$. Let $T = \min(t | t \in S)$ be the earliest member of S. Since ϵ was arbitrary when we established that B_t had a zero in $(0, \epsilon)$, by the same argument we can see that B_t almost surely has a zero in $(0, T)$. By the minimality of T this zero must not be in S, thus there is no finite set containing all the times B_t hits zero, so Z is almost surely infinite. □

This allows us to fully characterize Z.

Theorem 6. *Almost surely, Z is a perfect set, that is, Z is closed and has no isolated points.*

Proof. Since B_t is continuous, $Z = B_t^{-1}(\{0\})$ must be closed because it is the inverse image of the closed set $\{0\}$.

Consider the time $\tau_q = \inf\{t \geq q | B_t = 0\}$, where q is a rational number. This is clearly a stopping time, and it is almost surely finite because B_t almost surely crosses 0 for arbitrarily large t. Moreover, the infimum is a minimum because Z is almost surely closed. We apply the strong Markov property at τ_q and get that $B_{t+\tau_q} - B_{\tau_q} = B_{t+\tau_q}$ is a standard Brownian motion. Because we already know that a Brownian motion crosses 0 in every small interval to the right of the origin, τ_q is not isolated from the right in Z.

Now suppose we have some $z \in Z$ that is not in $\{\tau_q | q \in \mathbb{Q}\}$. Take some sequence, q_n, of rational numbers that converges to z. For each q_n, there must exist some $t_n \in Z$ such that $q_n \leq t_n < z$ since $z \neq \tau_{q_n}$. Because $q_n \to$ z, $t_n \to z$, so z is not isolated from the left in Z. □

It is a fact from analysis that perfect sets are uncountable. This shows that even though Z has measure 0, it still is very big in a sense.

4.2 Maxima of Brownian motion

Because Brownian motions change directions so frequently and dramatically it is not surprising that they frequently attains local maxima.

Theorem 7. *Almost surely, the set of times where $B(t)$ attains a local maximum is dense in $[0, \infty)$*

Proof. We begin by showing that B_t almost surely has a local maximum on every fixed interval. A time B_t attains local maximum is a point preceded by an increasing interval and followed by a decreasing interval. So B_t has no local max on (a, b) if and only if it is monotonic on (a, b) or monotonically decreasing until some point $c \in (a, b)$ then monotonically increasing. The former event has probability 0. And for all $c \in (a, b)$, $(a, \frac{a+b}{2}) \subset (a, c)$ or $(\frac{a+b}{2}, b) \subset (c, b)$, so B_t being monotonic on (a, c) and (c, b) implies that B_t is monotonic on either $(a, \frac{a+b}{2})$ or $(\frac{a+b}{2}, b)$, and both these events occur with probability 0.

By taking a countable union, we see that all intervals with rational endpoints almost surely have a time B_t attains a local maximum. By the density of $\mathbb{Q} \in \mathbb{R}$, there exists such a rational interval within every interval. So every real interval contains a rational interval which contains a time B_t attains a local max, thus contains a time B_t attains a local max. □

Continuing a theme, having shown that the set of times Brownian motion attains a maximum is big in the sense of density in $[0, \infty)$, we now show that it is small in the sense of cardinality. First we must prove two lemmas.

Lemma 1. *On any two fixed intervals, B_t almost surely does not attain the same maximum.*

Proof. Fix two disjoint closed intervals, let M_t be the maximal process and let m be the maximum value attained by B_t on the first of the two disjoint intervals. Name the second interval $[a, b]$. Consider the time $\tau = \inf\{t \geq a | B_t = m\}$. Since m is known strictly before the time a, τ is a stopping time. If $\tau \notin [a, b]$, then B_t never attains m on $[a, b]$ and certainly does not have m as a maximum on this interval, so suppose $\tau \in [a, b]$. Since $B_b \neq m$ almost surely, we can suppose $\tau \in [a, b)$. By the strong Markov property, $X_t = B_{t+\tau} - B_\tau = B_{t+\tau} - m$ is a Brownian motion. Fix $0 < \epsilon < b - \tau$ so that $(\tau, \epsilon + \tau) \subset [a, b)$. We know there exists some $s \in (0, \epsilon)$ such that $X_s > 0$ since X_t is a standard Brownian motion. Thus there exists some $s' = s + \tau \in (\tau, \epsilon + \tau) \subset (a, b]$ such that $B_{s'} = m + X_{s'-\tau} = m + X_s > m$, showing that the maximum of B_t on $[a, b]$ is strictly greater than m. □

It is important to note that we have not proved that almost surely no two disjoint intervals have the same maximum. In fact, this is not true; almost surely, there exist disjoint intervals with the same maximum. Let $T_{a,n}$ be the nth time the process hit a. Since B_t is strictly above or below a on the intervals $(T_{a,n-1}, T_{a,n})$ for all n, and it has equal probability of being above or below, there almost surely exists some n such that the maximum of B_t on $[T_{a,n-1}, T_{a,n}]$ is a, which is also the maximum of B_t on $[0, T_{a,1}]$.

Lemma 2. *Almost surely, every local maximum of Brownian motion is a strict local maximum.*

Proof. Suppose B_t attains a local maximum at time s. Then there exists some δ such that B_s is a maximum on the interval $(s-\delta, s+\delta)$. If B_s is not a strict maximum, then there exists some $r \in (s-\delta, s+\delta)$ such that $B_r = B_s$. We can find disjoint closed intervals with rational endpoints around r and s contained in $(s-\delta, s+\delta)$. But, with probability 1, B_t does not attain the same maximum on any two rational intervals, so such an r almost surely does not exist. Thus, almost surely, every time B_t attains a local maximum, it attains a strict local maximum. □

With these two lemma we can move on to a major result.

Theorem 8. *Almost surely, the set of times B_t attains a local maximum is countable.*

Proof. Let S be the set of times B_t attains a local maximum. Define $f : \mathbb{Q} \times \mathbb{Q} \to S$ by $f(p,q) = \inf\{t \geq p | B_t = \max_{p \leq s \leq q}(B_s)\}$. This will contain S, and thus show that S is almost surely countable, if B_t almost surely attains a strict maximum on some rational interval at every time it attains a local maximum. By the above lemma, we can almost surely find neighborhoods N_s around each $s \in S$ on which B_s is a strict maximum. And, by the density of $\mathbb{Q} \subset \mathbb{R}$, we can find rational intervals within each N_s containing each s on which B_s is a strict maximum. □

This classification of local maxima is very important: now, if we prove that some property almost surely holds for an arbitrary local maximum, we can take a countable union and show that the property almost surely holds for all local maxima. We demonstrate this technique.

Corollary 1. *The local maxima of B_t are almost surely distinct.*

Proof. Since there is a countable number of times B_t attains a local maximum, there is a countable number of disjoint intervals on which these local maxima are maxima, and thus there is countable number of pairs of these intervals. We know that, almost surely, B_t does not attain the same maximum on any pair of intervals. Thus we take a countable union over all pairs of disjoint intervals on which a local maximum of B_t is a max and prove the corollary. □

We move on to providing a few computations. The reflection principle will be our key ingredient.

Proposition 6. *T_a has the distribution given by*

$$P(T_a \in dt) = \frac{a}{\sqrt{2\pi t^3}} \exp(\frac{-a^2}{2t})dt$$

Proof.

$$P(T_a \le t) = P(M_t \ge a) = 2P(B_t \ge a) = 2\int_a^\infty \frac{1}{\sqrt{2\pi t}} \exp(\frac{-x^2}{2t})dx$$
$$= \int_0^t \frac{a}{\sqrt{2\pi s^3}} \exp(\frac{-a^2}{2s})ds$$

substituting $x = a\sqrt{\frac{t}{s}}$. Differentiating with respect to t gives the result. □

We will use the reflection principle again in the next proposition.

Proposition 7. *The processes M_t and $M_t - B_t$ have the joint distribution*

$$P(M_t \in da, M_t - B_t \in db) = \frac{2(a+b)}{\sqrt{2\pi t^3}} \exp(\frac{-(a+b)^2}{2t})dadb$$

Proof.

$$P(M_t \le a, B_t \le b) = P(T_a \le t, B_t \le b) = P(T_a \le t, 2a - B_t^* \le b)$$
$$= P(2a - B_t^* \le b) = P(B_t^* \ge 2a - b) = \int_{2a-b}^\infty \frac{1}{\sqrt{2\pi t}} \exp(\frac{-x^2}{2t})dx$$
$$= \int_a^\infty \frac{2}{\sqrt{2\pi t}} \exp(\frac{-(2y-b)^2}{2t})dy$$

where B_t^* is the Brownian motion B_t reflected about T_a and we substituted $x = 2y - b$. Taking the derivative with respect to b we get

$$P(M_t \le a, B_t \in db) = \int_a^\infty -\frac{2(2y-b)}{\sqrt{2\pi t^3}} \exp(\frac{-(2y-b)^2}{2t})dydb$$

Now differentiating with respect to a we get

$$P(M_t \in a, B_t \in db) = \frac{2(2a-b)}{\sqrt{2\pi t^3}} \exp(\frac{-(2a-b)^2}{2t})dadb$$

So

$$P(M_t \in a, M_t - B_t \in db) = P(M_t \in da, B_t \in a - db) = \frac{2(a+b)}{\sqrt{2\pi t^3}} \exp(\frac{-(a+b)^2}{2t})dadb$$

□

Corollary 2. *We can now show that $M_t - B_t$ has the same distribution as $|B_t|$. Integrating $P(M_t \in a, M_t - B_t \in db)$ over a gives*

$$P(M_t - B_t \in da) = \frac{-2}{\sqrt{2\pi t}} \exp(\frac{-(a+b)^2}{2t}) da|_0^\infty = \frac{2}{\sqrt{2\pi t}} \exp(\frac{-a^2}{2t}) da$$

This proposition leads us to another calculation.

Proposition 8. *Let $\theta_t = \max\{s \le t \,|\, B(s) = M(s)\}$. The joint distribution of θ_t and M_t is given by*

$$P(M_t \in da, \theta_t \in ds) = \frac{a}{\pi\sqrt{(t-s)s^3}} \exp(-\frac{a^2}{2s}) da ds$$

and thus the cumulative distribution function of θ is

$$P\{\theta_t \le s\} = \frac{2}{\pi} \arcsin\sqrt{\frac{s}{t}}$$

Proof. Let X_t be the standard Brownian motion given by $X_t = B_{t+s} - B_s$ and let N_t be the maximal process of X_t. Then

$$\begin{aligned} N_{t-s} &= \max\{X_u | 0 \le u \le t-s\} = \max\{B_{u+s} - B_s | 0 \le u \le t-s\} \\ &= \max\{B_u | s \le u \le t\} - B_s \end{aligned}$$

We know $\theta_t \le s$ if and only if $M_s = M_t$ which happens if and only if $M_s \ge \max\{B_u | s \le u \le t\} = N_{t-s} + B_s$, and we can almost surely replace the inequality by a strict inequality. So

$$\begin{aligned} &P(M_t \in da,\ \theta_t \le s) = P(M_s \in da,\ M_s - B_s > N_{t-s}) \\ &= \int_{c\in[0,\infty)} P(M_s \in da,\ M_s - B_s > c \mid N_{t-s} \in dc)\ P(N_{t-s} \in dc) \\ &= \int_{c\in[0,\infty)} P(M_s \in da,\ M_s - B_s > c)\ P(N_{t-s} \in dc) \\ &= \int_{c\in[0,\infty)} \int_{b>c} P(M_s \in da,\ M_s - W_s \in db)\ P(N_{t-s} \in dc) \\ &= \int_0^\infty \int_c^\infty \frac{2(a+b)}{\sqrt{2\pi s^3}} \exp(\frac{-(a+b)^2}{2s}) \frac{2}{\sqrt{2\pi(t-s)}} \exp(\frac{-c^2}{2(t-s)}) db dc da \\ &= \frac{2da}{\pi\sqrt{s(t-s)}} \int_0^\infty \exp(\frac{-(a+c)^2}{2s}) \exp(\frac{-c^2}{2(t-s)}) dc \\ &= \frac{2da}{\pi\sqrt{s(t-s)}} \int_0^\infty \exp(-\frac{t(c + \frac{t-s}{t}a)^2}{2s(t-s)}) \exp(-\frac{a^2}{2t}) dc \\ &= \frac{2da}{\pi\sqrt{t}} \exp(-\frac{a^2}{2t}) \int_{a\sqrt{\frac{t-s}{st}}}^\infty \exp(-\frac{u^2}{2}) du \end{aligned}$$

where we substituted $u = \frac{c+\frac{t-s}{t}a}{\sqrt{\frac{s(t-s)}{t}}} = c\sqrt{\frac{t}{s(t-s)}} + a\sqrt{\frac{t-s}{st}}$. Differentiating with respect to s, we get

$$P(M_t \in da, \theta_t \in ds) = \frac{2da}{\pi\sqrt{t}}\exp(-\frac{a^2}{2t})(\exp(-\frac{(a\sqrt{\frac{t-s}{st}})^2}{2})ds)\frac{d}{ds}(a\sqrt{\frac{t-s}{st}})$$
$$= \frac{2da}{\pi\sqrt{t}}\exp(-\frac{a^2}{2t})(-\exp(-\frac{a^2(t-s)}{2st})ds)\frac{-a\sqrt{t}}{2\sqrt{(t-s)s^3}}$$
$$= \frac{a}{\pi\sqrt{(t-s)s^3}}\exp(-\frac{a^2}{2s})dads$$

To get to the cumulative distribution of θ_t, we integrate:

$$P(\theta_t \leq s) = \int_{r\in[0,s]}\int_{a\in[0,\infty)} P(M_t \in da, \theta_t \in dr)$$
$$= \int_0^s\int_0^\infty \frac{a}{\pi\sqrt{(t-r)r^3}}\exp(-\frac{a^2}{2r})dadr = \int_0^s \frac{dr}{\pi\sqrt{(t-r)r}}$$
$$= \int_0^{\sqrt{\frac{s}{t}}} \frac{2du}{\pi\sqrt{1-u^2}} = \frac{2}{\pi}\arcsin\sqrt{\frac{s}{t}}$$

where we substituted $u = \sqrt{\frac{r}{t}}$. □

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