SOME EXAMPLES OF SUBDIVISION OF SMALL CATEGORIES

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Abstract. The purpose of this paper is to build up the basic conceptual framework and underlying motivations that will allow us to understand categorical subdivision, and then work through some specific examples.

Contents

1. Subdivision of Simplicial Sets

The idea of subdividing a category arises from the notion of a barycentric subdivision of a simplicial complex. A barycentric subdivision decomposes an n-simplex into a simplicial complex of $n!$ n-simplices whose geometric realization is homotopy equivalent to the original simplicial complex. It is with this idea in mind that we will define the subdivision of a simplicial set.

First, we have the notion of a geometric simplicial complex, a space built from simplices. We can move towards an abstract definition of a simplicial complex.

Definition 1.1 (Abstract Simplicial Complex)**.** A *simplicial complex* is a set X and a collection of its finite subsets S (called the *simplices* of X) with the property that for any simplex K of X, every subset of K is also a simplex of X.

This gives us the face and degeneracy maps from class [2], whereby we can include lower dimensional simplices in higher dimensional ones as faces and we can project from a higher dimensional simplex onto one of its faces. The key thing to take away from this abstraction from the geometric simplicial complexes from topology is the importance of the face and degeneracy maps, in particular their composition properties. Especially when working with ordered simplicial complexes (which we can do by imposing a partial ordering on X so that the simplices are the totally ordered subsets), these face and degeneracy maps become maps of finite totally ordered sets. This motivates the following pair of definitions.

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Definition 1.2 ($\mathbf{\Delta}$)**.** Let $[n]$ represent the finite poset $\{0 < 1 < \ldots < n\}$. Then let $\mathbf{\Delta}$ denote the category that has as its objects the finite posets $[n]$ for all nonnegative integers n and as its arrows the monotonic functions between these posets.

Definition 1.3 (Simplicial Sets)**.** A *simplicial set* is a contravariant functor $K : \mathbf{\Delta^{op}} \to \mathbf{Set}$. The category of simplicial sets is the category of all such contravariant functors, $[\mathbf{\Delta^{op}}, \mathbf{Set}]$, and it is denoted $\mathbf{sSet}$.

While abstract, we note that each object of $\mathbf{\Delta}$ picks out a set which serves as the set of simplices of that dimension and we inherit the face and degeneracy maps with the composition properties we love from the monotonic maps in $\mathbf{\Delta}$ and by contravariancy.

In the way we have subdivision of simplicial complexes, we can define subdivision on simplicial sets, a functor $Sd : \mathbf{sSets} \to \mathbf{sSets}$, which is explained in detail in May's notes [2]. With this functor, we can define subdivision on small categories by composing and precomposing it with functors to and from $\mathbf{Cat}$, respectively. These are the fundamental category and nerve functors.

2. The Nerve of him....

Here, we will construct the nerve functor $N : \mathbf{Cat} \to \mathbf{sSet}$. The nerve functor gives us a way to turn small categories into simplicial sets in a fairly intuitive way. First, we note that every poset, and in particular the posets $[n]$, can be seen as categories with at most one arrow between any pair of objects and monotonic maps between posets can then be seen as functors. Thus, we have an inclusion $\Delta : \mathbf{\Delta} \to \mathbf{Cat}$. For a given small category $\mathcal{C}$, define $N\mathcal{C}_n$ to be the set of all covariant functors $[n] \to \mathcal{C}$. For an arrow $\mu : [m] \to [n]$, define $\mu^* := - \circ \mu : N\mathcal{C}_n \to N\mathcal{C}_m$. This collection of sets $N\mathcal{C}_n$ defines a simplicial set, where the objects $[n]$ of $\mathbf{\Delta}$ are mapped to the corresponding set $N\mathcal{C}_n$ and arrows μ in $\mathbf{\Delta}$ are mapped to μ^*.

Now suppose we have a functor $F : \mathcal{C} \to \mathcal{D}$. We get a natural transformation $F^* : N\mathcal{C} \to N\mathcal{D}$ given by $F^*_n = F \circ - : N\mathcal{C}_n \to N\mathcal{D}_n$. Thus, we have a functor $N : \mathbf{Cat} \to \mathbf{sSet}$ given on objects by $\mathcal{C} \mapsto N\mathcal{C}$ and on arrows by $F \mapsto F^*$.

Definition 2.1 (Nerve Functor)**.** The covariant functor $N : \mathbf{Cat} \to \mathbf{sSet}$ is called the *nerve functor*.

3. Fundamental Category

Now we describe the fundamental category functor, $\tau_1 : \mathbf{sSets} \to \mathbf{Cat}$, allowing us to associate a category to a simplicial set in a canonical way, and it will turn out that this functor is left adjoint to the nerve functor.

Starting with a simplicial set K, take as the objects of $\tau_1 K$ the elements of K_0, the 0-simplices of K. For every 1-simplex y, have an arrow $y : d_1 y \to d_0 y$. Tack on all formal composites that make sense (i.e. form yz if $d_0 z = d_1 y$). Finally, we impose the following two relations:

$$s_0 x = 1_x \text{ for } x \in K_0 \text{ and } d_1 z = d_0 z \circ d_2 z \text{ for } z \in K_2$$

This gives us a well-defined category [2].

Definition 3.1 (Fundamental Category Functor)**.** The functor $\tau_1 : \mathbf{sSet} \to \mathbf{Cat}$ is called the *fundamental category functor*.

Now that we have all these pieces in place, we are in a position to define categorical subdivision.

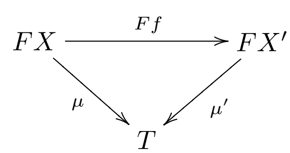
Definition 3.2 (Subdivision)**.** We define the subdivision functor $Sd : \mathbf{Cat} \to \mathbf{Cat}$ as $Sd := \tau_1 \circ Sd \circ N$, where the Sd in the composition is the subdivision functor on simplicial sets.

While this definition makes conceptual sense, it is not computationally friendly. To see concrete examples of categorical subdivision, we need a more constructive look at subdivision.

4. A Whole New Section

So in order to understand the construction of the subdivision functor, we first need to introduce the concept of a left fiber.

Definition 4.1 ([1])**.** [Left Fiber] Let $F : \mathcal{C} \to D$ be a functor and let $T \in \text{ob } \mathcal{D}$. The *left fiber* F/T of F over T is the category of pairs (X, μ), where $X \in \text{ob } \mathcal{C}$ and $\mu : FX \to T$ with arrows $f : (X, \mu) \to (X', \mu')$ corresponding to arrows $f : X \to X'$ in $\mathcal{C}$ s.t. $\mu' \circ Ff = \mu$. Diagrammatically,



Let's consider the objects of $\mathbf{\Delta}$, the posets $[n]$, as categories. Then we have an inclusion functor $\Delta : \mathbf{\Delta} \to \mathbf{Cat}$. If we consider an arbitrary small category $\mathcal{C}$ in $\mathbf{Cat}$, we can look at the left fiber of Δ over $\mathcal{C}$: $\Delta/\mathcal{C}$. The objects of this category are the all the arrows $[p] \to \mathcal{C}$ for all $[p]$, which are exactly all the the simplices of $N\mathcal{C}$.

In the following discussion, we will use $[n]$ to represent the image of $[n]$ under Δ, but for any arrows $f : [m] \to [n]$, $f_* : [m] \to [n]$ will represent the image of f in $\mathbf{Cat}$. If we're already using subscripts for something else, then we'll use a superscript $*$: f^*.

Now that we have this category, consider a q simplex $X \in N\mathcal{C}_q$ and a surjective (monotonic) map $s : [q+1] \to [q]$ in $\mathbf{\Delta}$. If $d, d' : [q+1] \to [q]$ are the two right inverses of s, then we say that $d_*, d'_* : ([q], X) \to ([q+1], Xs)$ are *elementary equivalent*, denoted $d_* \approx d'_*$. Clearly, $\approx$ is symmetric and reflexive. Let $\sim$ be the minimal equivalence relation on the arrows of $\Delta/\mathcal{C}$ generated by $\approx$. The we can define $[\Delta/\mathcal{C}] := (\Delta/\mathcal{C})/\sim$. Now that we have this, we can define the subdivision of a small category $\mathcal{C}$:

Definition 4.2 (Subdvision)**.** For a small category $\mathcal{C}$, the *subdivision* of $\mathcal{C}$, $Sd(\mathcal{C})$, is the full subcategory of $[\Delta/\mathcal{C}]$ taken on the nondegenerate simplices of $\mathcal{C}$, where a nondegenrate simplex is one in which none of the q composed arrows is an identity arrow for some object in $\mathcal{C}$.

This functor is equal to the functor given earlier in terms of simplicial sets, i.e. $Sd = \tau_1 \circ sd \circ N$ [1].

5. Concrete Examples

Here in this section, we will compute a few concrete examples of subdivisions. We will start with some trivial examples and work up to some interesting ones, including one infinite example. We'll see multiple times instances of how two consecutive subdivisions yields a poset.

It's worth noting at this point that when we compute these examples, since our result will only depend on nondegenerate simplices, we will omit them in the early stages of the computation (despite the fact that they should be there in the categories we're looking at at that stage) because they'll be irrelevant in the final result.

Further, when we are looking for arrows in $\Delta/\mathcal{C}$, we do not need to consider the σ_i maps. To see this, suppose we have an arrow in $\Delta/\mathcal{C}$ corresponding to some $\sigma_i : [q+1] \to [q]$. Then this tells us that the $(q+1)$-simplex we are considering factors through as a q-simplex, and so one of its composable arrows is an identity. Thus, the $(q+1)$-simplex is degenerate, and doesn't appear in $Sd(\mathcal{C})$.

Finally, we will only consider the arrows δ_i. The reason for this is that any arrows in $\mathbf{\Delta}$ can be written as a composition of the σ_i and δ_i, and so the arrows in $\Delta/\mathcal{C}$ corresponding to these compositions are just the composition of the arrows for the associated δ_i and σ_i (remembering of course that the σ_i will become irrelevant). Having this, let's consider our first concrete example.

Consider the category $\mathbf{1}$ consisting of a single object $*$ and its identity arrow. First, let's make a list of all the nondegenerate simplices of $\mathbf{1}$. We only have one nondegenerate 0-simplex, $* : [0] \to \mathbf{1}$, and no nondegenerate simplices of higher dimension. Since there are no other simplices, then there are no nonidentity arrows in $\Delta/\mathcal{C}$, and since there are no nontrivial composable pairs, then we have no elementary equivalent pairs. Thus, $Sd(\mathbf{1}) = \mathbf{1}$.

Now consider the category $\mathbf{1}'$, consisting of a single object $*$ which has one nonidentity arrow $a : * \to *$. In this example, we'll assume that $f \circ f = 1_*$. Now, we have one 0-simplex, $* : [0] \to \mathbf{1}'$, and one nondegenerate 1-simplex, $a : [1] \to \mathbf{1}'$. Again, there are no other nondegenerate simplices. Thus, we are considering a category with two objects: $*$ and a. Our potential arrows are $\delta_0, \delta_1 : [0] \to [1]$, which in $\Delta/\mathcal{C}$ will be of the form $\delta_0^*, \delta_1^* : X \to a$ for some object (simplex) X. We are looking for simplices $X : [0] \to \mathbf{1}'$ such that $X = a \circ \delta_0$. $a \circ \delta_0$ picks out the codomain (by deleting 0) of a, which is 0, so $X = 0$. Analogously, we have an arrow $\delta_1^* : 0 \to a$ since δ_1 picks out 0 (by deleting 1). δ_0 and δ_1 cannot be right inverses for the same map, so they are not elementary equivalent. Therefore, we get that

$$Sd(\mathbf{1}') = \; 0 \underset{\delta_1}{\overset{\delta_0}{\rightrightarrows}} a$$

We also note the following useful heuristic: If we view $\mathbf{1}'$ as $0 \xrightarrow{a} 0$, then it's subdivision is $0 \xrightarrow{\delta_0} a \xleftarrow{\delta_1} 0$.

Our next example is the category $\mathbf{2}$: $0 \xrightarrow{a} 1$. There are three nondegenerate simplices: $0 : [0] \to \mathbf{2}$, $1 : [0] \to \mathbf{2}$, and $a : [1] \to \mathbf{2}$. Since there is no nonidentity arrow $[0] \to [0]$, we can assume that all nonidentity arrows in $Sd(\mathbf{2})$ will be either of the form $0 \to a$ or $1 \to a$. Following the lead of the previous example, we see that these two arrows are $\delta_0 : 1 \to a$ and $\delta_1 : 0 \to a$. Again, these are not elementary equivalents, so we have that $Sd(\mathbf{2}) = 0 \xrightarrow{\delta_1} a \xleftarrow{\delta_1} 1$.

Our next example is the category $\mathcal{C}$ that is just a corner of arrows: $0 \xrightarrow{a} 2 \xleftarrow{b} 1$. The heuristic tells us that our subdivision should look like this:

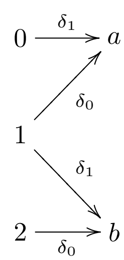
$$0 \longrightarrow a \longleftarrow 2 \longrightarrow b \longleftarrow 1$$

In $\mathcal{C}$, we have three 0-simplices and two 1-simplieces:

$$\begin{array}{ll} 0 : [0] \to \mathcal{C} & a : [1] \to \mathcal{C} \\ 1 : [0] \to \mathcal{C} & b : [1] \to \mathcal{C} \\ 2 : [0] \to \mathcal{C} & \end{array}$$

There are no nonidentity maps $[0] \to [0]$. Any map $[1] \to [1]$ is either a constant map (giving us a degeneracy) or the identity (by monotonicity), so we only need to consider arrows $[0] \to [1]$. There are no arrows $0 \to b$ or $2 \to a$ since the images of 0 and b are disjoint and the images of 2 and a are disjoint. Thus, we have four possible arrows in $Sd(\mathcal{C})$, all of which turn out to actually be arrows. Thus, we have that

$Sd(\mathcal{C}) =$

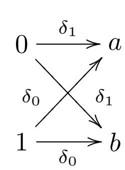


Or, if we linearize it, we get

$$0 \xrightarrow{\delta_1} a \xleftarrow{\delta_0} 2 \xrightarrow{\delta_1} b \xleftarrow{\delta_0} 1,$$

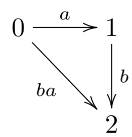
which is exactly what we expected. Note that $Sd(\mathcal{C}) = Sd^2(\mathbf{2})$, which is a poset.

Now let's consider the category $\mathcal{P}$ consisting if two objects 0 and 1 and two parallel nonisomorphism arrows $a, b : 0 \to 1$. Diagrammatically, $\mathcal{P} = 0 \overset{a}{\underset{b}{\rightrightarrows}} 1$. This category has two 0-simplices, $0 : [0] \to \mathcal{P}$ and $1 : [0] \to \mathcal{P}$, and two 1-simplices, $a : [1] \to \mathcal{P}$ and $b : [1] \to \mathcal{P}$. There are no other nondegenerate simplices. This gives us four relevant arrows in $[\Delta/\mathcal{P}]$: $\delta_0 : 1 \to a$, $\delta_1 : 0 \to a$, $\delta_0 : 1 \to b$, and $\delta_1 : 0 \to b$. Since none of these arrows are elementary equivalent pairs, we have that $Sd(\mathcal{P}) =$



It's worth noting that $Sd(\mathcal{P}) = Sd^2(\mathbf{1}')$, and that it's a poset. If we look at the geometric realization of $\mathbf{1}'$, $|N(\mathbf{1}')|$, we observe that it's homeomorphic to S^1 and that $Sd^2(\mathbf{1}')$ is the 4-point model that is weakly homotopy equivalent to S^1.

Now let's examine an example with a nontrivial composition of arrows. Consider the category $\mathcal{T}$:



Let's make a list of all the nondegenerate simplices:

0-simplices	1-simplices	2-simplices
$0 : [0] \to \mathcal{T}$	$a : [1] \to \mathcal{T}$	$b \circ a : [2] \to \mathcal{T}$
$1 : [0] \to \mathcal{T}$	$b : [1] \to \mathcal{T}$	
$2 : [0] \to \mathcal{T}$	$ba : [1] \to \mathcal{T}$	

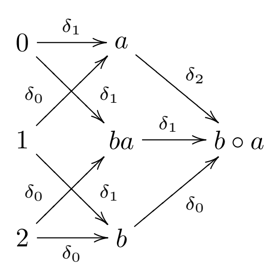
Here, we distinguish between the 1-simplex corresponding to the composite ba and between the 2-simplex consisting of the two composable arrows $0 \xrightarrow{a} 1 \xrightarrow{b} 2$ which we denote $b \circ a$. So our subdivided category will have these seven objects. Now let's looks at arrows in our new category. We have six arrows going from the 0-simplices to the 1-simplices which are the maps that compose with the degeneracy maps δ_i to give us maps that pick out the domain and codomain of each arrow as in each of the previous examples. This gives us the six arrows

$\delta_0 : 1 \to a$	$\delta_1 : 1 \to b$
$\delta_1 : 0 \to a$	$\delta_0 : 2 \to ba$
$\delta_0 : 2 \to b$	$\delta_1 : 0 \to ba$

We also get three more arrows that pick out the "faces" of the commutative triangle:

$$\delta_0 : b \to b \circ a \quad \delta_1 : ba \to b \circ a \quad \delta_2 : a \to b \circ a$$

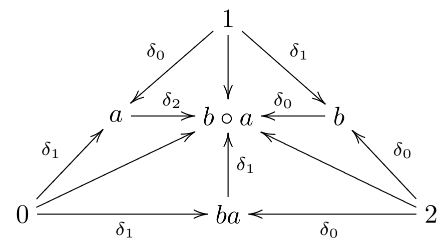
Thus, we get the following diagram for $\Delta/\mathcal{T}$ (restricted to nondegenerate simplices):



We leave out the composite arrows and we don't claim that this diagram commutes since we have not shown, for example, that $\delta_2\delta_1 = \delta_1\delta_0$ for the two composite arrows $0 \to b \circ a$. What we would like to show is that these three pairs of composites are elementary equivalent, that is:

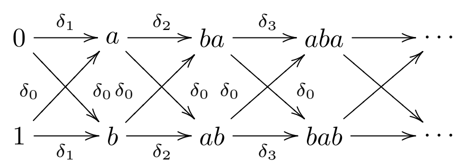
$$\delta_2\delta_1 \approx \delta_1\delta_0 \quad \delta_2\delta_1 \approx \delta_0\delta_1 \quad \delta_0\delta_0 \approx \delta_1\delta_0$$

There is only a single map $s : [2] \to [0]$. By composition with any of the six maps listed above, we get $1_{[0]}$ since there are no other maps $[0] \to [0]$, which proves elementary equivalence. Thus, the diagram above commutes in $Sd(\mathcal{T})$. We can rearrange this diagram and get the following diagram which commutes because the unlabeled composites are elementary equivalent and are identified in $[\Delta/\mathcal{T}]$:



This should remind us very strongly of the result of barycentric subdivision of a 2-simplex in algebraic topology.

For our last finite example, consider the groupoid $\mathcal{G} : 0 \underset{b}{\overset{a}{\rightleftarrows}} 1$, where $ba = 1_0$ and $ab = 1_1$. We observe that $\mathcal{G}$ has two nondegenerate simplices of each dimension: the 0-simplices 0 and 1, the 1-simplices a and b, and for each $[q]$ for $q > 2$ the two compositions of length q $aba\ldots$ and $bab\ldots$. As for the arrows, there are two ways to include a string of alternating as and bs of length q into one of length $q+1$: add the appropriate letter to the beginning (δ_0) or to the end (δ_{q+1}). It will turn out [1] that all composites will work out to be appropriately elementary equivalent, so that we get $Sd(\mathcal{G}) =$



And our final example is our only infinite example. Consider the category $\mathcal{A}$

$$\cdots \longrightarrow \bullet \longleftarrow \bullet \longrightarrow \bullet \longleftarrow \bullet \longrightarrow \bullet \longleftarrow \cdots$$

We know how each corner of arrows $\bullet \longrightarrow \bullet \longleftarrow \bullet$ subdivides by a previous example. Since we have no nontrivial compositions of arrows and since each corner subdivides into two corners, we get that $Sd(\mathcal{A}) = \mathcal{A}$.

Acknowledgements

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References

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[2] J. P. May. Subdivisions of Small Categories. http://math.uchicago.edu/ may/FINITE/Subdivide.pdf