GAUSS MAP

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Abstract. This paper is intended as a introduction to Gaussian Curvature and the Gauss Map. Before studying this, a review of Jacobian Determinants and Surfaces in $\mathbb{R}^3$ will be reviewed so Gaussian Curvature can be adequately studied.

Contents

1. Jacobian, Geometric Interpretations and Examples

1.1. **Motivation for Understanding.** We are motivated to understand the Jacobian of a given function because it will allow us to understand certain geometric properties of f, namely:

(1) Area
(2) Orientation, and
(3) Invertibility.

Definition 1.1. Given $f : \mathbb{R}^2 \to \mathbb{R}^2$ written $f(x, y) = (f_1(x, y), f_2(x, y))$ the **Jacobian** of f at $(x, y) \in \mathbb{R}^2$ is:

$$Df_{(x,y)} = \begin{pmatrix} \frac{\delta f_1}{\delta x} & \frac{\delta f_1}{\delta y} \\ \frac{\delta f_2}{\delta x} & \frac{\delta f_2}{\delta y} \end{pmatrix}.$$

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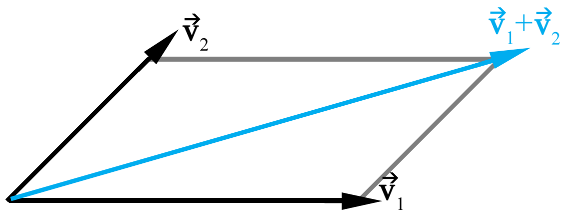
1.2. **Area.** The Jacobian of a matrix allows us to understand the area. When the determinant of the Jacobian is not equal to zero, the area is not annihilated but may be enlarged or shrunk.

Definition 1.2. Given a $2x2$ matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the **determinant** of A is the area of the parallelogram spanned by $\overrightarrow{v_1} = (a, b)$ and $\overrightarrow{v_2} = (c, d)$. The formula for a $2x2$ matrix is

$$det(A) = a \cdot d - b \cdot c.$$

A visual aid for this can be seen in Figure 1.1.

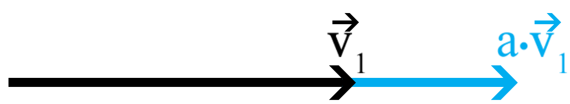
Figure 1.1



The span of two independent vectors form a parallelogram.

However, this is only how it looks when the vectors are linearly independent. Instead, we can suppose that $\overrightarrow{v_1}$ and $\overrightarrow{v_2}$ are linearly dependent. That is $\overrightarrow{v_1} = a \cdot \overrightarrow{v_2}$ for some $a \in \mathbb{R}$ then $det(A) = 0$. A visual example for understanding why the span of a set of dependent vectors will always equal to zero can be seen in Figure 1.2.

Figure 1.2



The span of two dependent vectors form a line (with zero area).

Example 1.3. Suppose that $f(x, y) = (cos(\theta)x - sin(\theta)y, sin(\theta)x + cos(\theta)y)$ for $\theta \in \mathbb{R}$ which is a rotation about the origin of angle θ. Then

$$det(Df_{(x,y)}) = \begin{bmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cost(\theta) \end{bmatrix} = 1.$$

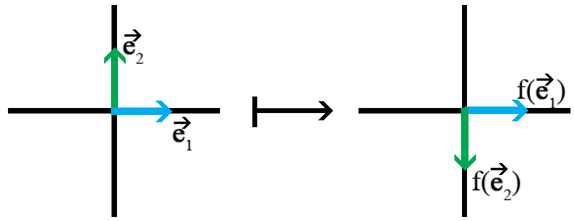
This implies that f preserves area. For a visual example of this, we can let $\theta = \frac{\pi}{2}$. Then $f(x, y) = (-y, x)$. So,

$$f(\overrightarrow{e_1}) = f(1, 0) = (0, 1) = \overrightarrow{e_2}$$

$$f(\overrightarrow{e_2}) = f(-1, 0) = -(1, 0) = -\overrightarrow{e_1}.$$

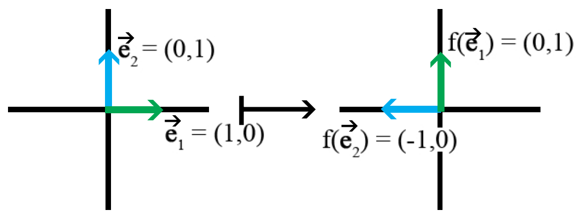
As we can see, this simply rotates the vectors which would not change the area spanned by the vectors. See Figure 1.3.

Figure 1.4



$f(x, y) = (x, -y)$ reverses orientation.

Figure 1.3



Functions that rotate a graph, such as $f(x, y) = (-y, x)$ preserve area.

1.3. **Orientation.**

Definition 1.4. A function $f : \mathbb{R}^2 \to \mathbb{R}^2$ preserves orientation if $det(Df_{(x,y)}) > 0$ for all $(x, y) \in \mathbb{R}^2$. And, f reverses orientation if $det(Df_{(x,y)}) < 0$ for all $(x, y) \in \mathbb{R}^2$. Looking again at Figure 1.3, we see that the function $f(x, y) = (-y, x)$ preserves orientation which should be expected because the determinant of the Jacobian is greater than zero.

Example 1.5. Consider $f(x, y) = (x, -y)$. Then, $det(Df_{x,y}) = -1$. Looking at Figure 1.4, we can use the right hand rule to see that f does reverse orientation.

1.4. **Invertability.** To understand whether a function is in invertible, we shall call upon the help of the Inverse Function Theorem.

Theorem 1.6. *(The Inverse Function Theorem) Suppose that $f : \mathbb{R}^2 \to \mathbb{R}^2$ is a smooth function such that $Df_{(x,y)}$ is non-singular at $(x_0, y_0) \in \mathbb{R}^2$, then there exists a neighborhood U of (x_0, y_0) and $f(U) = V$ of $f(x_0, y_0)$ such that the restriction*

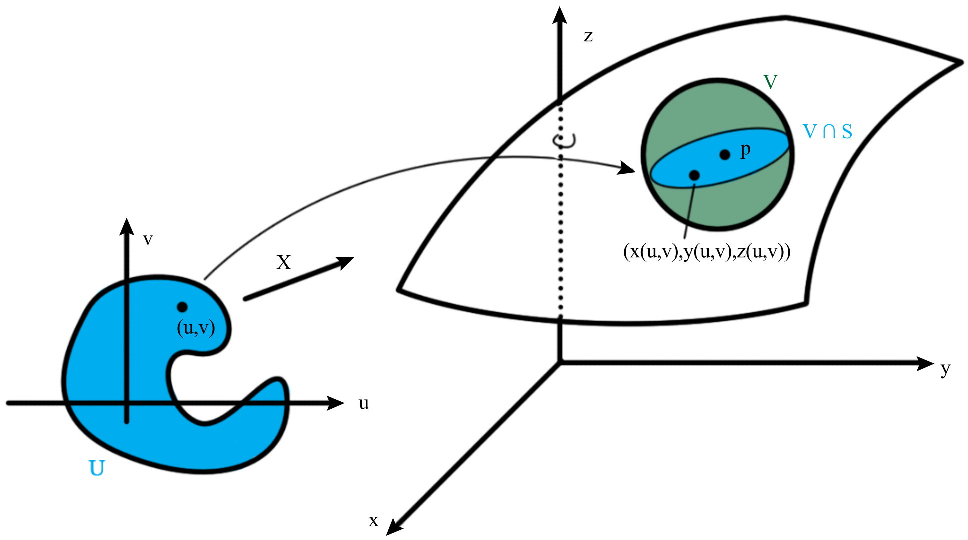
$$f : U \to V$$

is a diffeomorphism. That is, there exists a smooth function $f^{-1} : V \to U$ such that $f \circ f^{-1} = 1 : U \to U$ and $f^{-1} \circ f = 1 : V \to V$.

2. Regular Surfaces in S^3

Definition 2.1. A subset $S \subset \mathbb{R}^3$ is called a regular surface if, for each point $p \in S$, there exists a neighborhood $V \in \mathbb{R}^3$ and a map $\overrightarrow{x} : U \to V \cap S$. where U is an open set in $\mathbb{R}^2$ and U is onto $V \cap S \subset \mathbb{R}^3$ such that (see Figure 2.1):

Figure 2.1



Explanation of Regular Surface

(1) $\overrightarrow{x}$ is a differentiable. This means if

$$\overrightarrow{x}(u,v) = (x(u,v), y(u,v), z(u,v)), \qquad (u,v) \in U$$

the functions $x(u,v)$, $y(u,v)$, and $z(u,v)$ have continuous partial derivatives for all orders in U.

(2) $\overrightarrow{x}$ is a homeomorphism. We already know $\overrightarrow{x}$ is continuous by condition one, thus $\overrightarrow{x}$ has an inverse $\overrightarrow{x}^{-1} : V \cap S \to U$ which is also continuous

(3) For each $q \in U$, the differential $d\overrightarrow{x}_q : \mathbb{R}^2 \to \mathbb{R}^3$ is one-to-one.

2.1. **Stereographic Projection.**

Example 2.2. The Unit Sphere I shall show that the unit sphere is a regular surface in two different ways. The first will involve using six, simpler parameterizations while the second requires only two, however more slightly more complicated, parameterizations. We must show each parameterizations is a regular surface satisfying the three conditions. Of course, the unit sphere is:

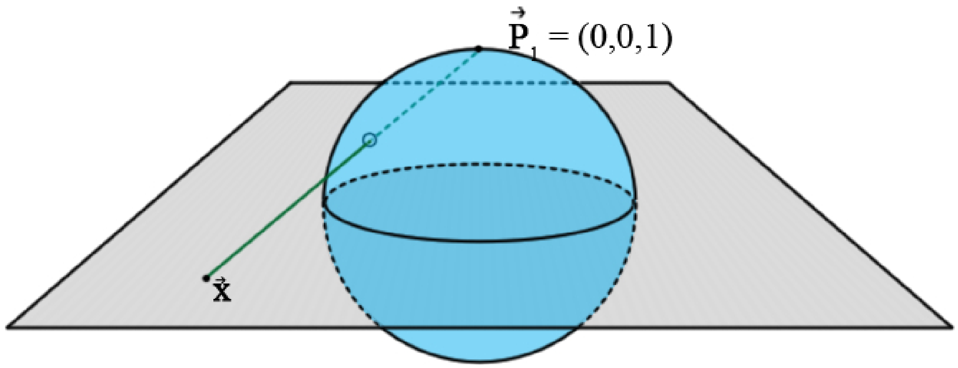
$$S^2 = \{(x,y,z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 = 1\}.$$

Version 1: We will first define the open part the unit sphere above the xy plane. This is the map $x_1 : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ given by:

$$x_1(x,y) : (x, y, +\sqrt{1-(x^2+y^2)}).$$

Version 2: Stereographic Projection Let:
$\pi(x) : \mathbb{R}^2 \to S^2 \setminus \{p\}$

Figure 1.2



$\mathtt{L} = \{t\overrightarrow{x} + (1-t)\overrightarrow{p} | t \in [o,1]\}$.

If $\overrightarrow{x} = (x_1, x_2, 0)$, then $t\overrightarrow{x} + (1-t)\overrightarrow{p} = (tx_1, tx_2, (1-t))$.

Note this is on the sphere when $t^2x_1^2+t^2x_2^2+(1-t)^2 = 1$. Recall, $||\overrightarrow{x}||^2 = x_1^2+x_2^2$, $t^2||\overrightarrow{x}||^2 + (1-t)^2 = 1$.

So,
$t^2 \cdot ||\overrightarrow{x}||^2 + t^2 - 2t + 1 = 1$ and
$t^2(||\overrightarrow{(}x)||^2 + 1) - 2t = 0$.

Thus, $t = \frac{2}{1+||\overrightarrow{x}||^2}$

$$\pi(\overrightarrow{x}) = \frac{2}{1+||\overrightarrow{x}||^2} \cdot \overrightarrow{x} + (1 - \frac{2}{1+||\overrightarrow{x}||^2})\overrightarrow{p}$$

So,

$$= \frac{2}{1+||\overrightarrow{x}||^2} \cdot \overrightarrow{x} + (\frac{||\overrightarrow{x}||^2 - 1}{1+||\overrightarrow{x}||^2})\overrightarrow{p}$$

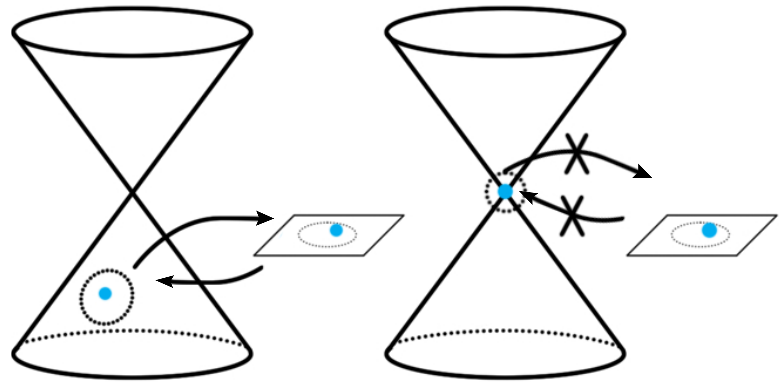
$$= (\frac{2x+1}{1+||\overrightarrow{x}||^2}, \frac{2x_2}{1+||\overrightarrow{x_0}||}, \frac{||\overrightarrow{x}||^2 - 1}{||\overrightarrow{x}||^2 + 1}).$$

2.2. **Regular Value Theorem.**

Theorem 2.3. *: Regular Value Theorem ($\mathbb{R}^3$ to $\mathbb{R}$)*
Let $F : \mathbb{R}^3 \to \mathbb{R}$ be a smooth function. Then, for $a \in \mathbb{R}$, the set $F^{-1}(a) = \{\overrightarrow{p} \in \mathbb{R} | F(\overrightarrow{p}) = a\}$ is a regular surface if $D_pF \neq \overrightarrow{0}$ for all $\overrightarrow{p} \in F^{-1}(a)$.

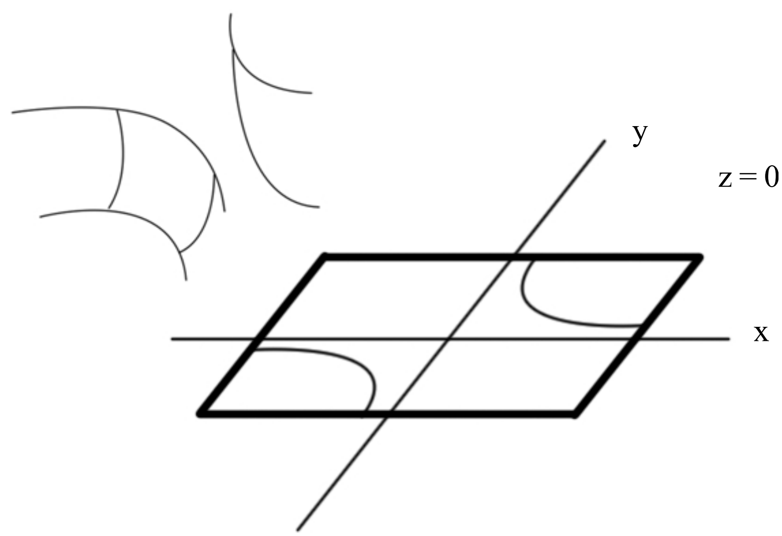
For instance, the set $F^{-1}(0)$ is not a surface for the function $F(x,y,z) = x^2 + y^2 - z^2$ because $D_pF = (2x, 2y, -2z)$ which is equal to $\overrightarrow{0}$ when x, y, and z are all identically equal to zero. See Figure 2.2. However, the surface defined by $F(x,y,z) = xy - z^2$ is a surface for $F^{-1}(1)$ for $D_pF = (y, x, -2z)$ which is only equal to zero when x,y, and z are all identically equal to zero. But $F(0,0,0) \neq -1$. See Figure 2.3. To prove the Regular Value Theorem, the Implicit Function Theorem will be used.

Figure 2.2



$F(x, y, z) = x^2 + y^2 - z^2$ is not a regular surface for $F^{-1}(0)$.

Figure 2.3



$F(x, y, z) = xy - z^2$ is a regular surface for $F^{-1}(1)$.

Theorem 2.4. *: Implicit Function Theorem ($\mathbb{R}^2$ to $\mathbb{R}^3$)*
If $F : \mathbb{R}^3 \to \mathbb{R}$ satisfies $\frac{DF}{dz}$ nonzero at some point (x, y, z) with $F(x, y, z) = a$, then there exists a neighborhood of (x, y) in $\mathbb{R}^2$ and a smooth function g from the

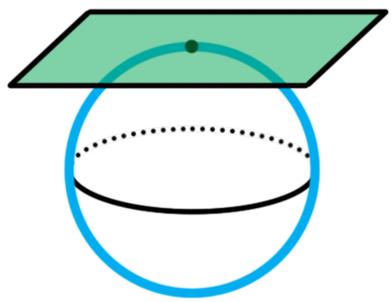
neighborhood to $\mathbb{R}^3$ *satisfying* $g(x,y) = (x,y,z)$ *and, in a neighborhood of* (x,y,z), $F(x',y',z') = a$ *if, and only if,* $g(x',y') = z'$.

Proof. Consider a function $F : \mathbb{R}^3 \to \mathbb{R}$ such that F is smooth and a point $a \in \mathbb{R}$ that is a regular value. That is, for each $\overrightarrow{p} \in F^{-1}(a) = \{(x,y,z)|F(x,y,z) = a\}$ $D_{\overrightarrow{p}}F \neq \overrightarrow{0}$. Without loss of generality, let $\frac{\delta f}{\delta z}(p) \neq 0$. Write $\overrightarrow{p} = (p_1, p_2, p_3)$. Near $\overrightarrow{p} \in V$, by the Implicit Function Theorem, there exists a neighborhood $U \in \mathbb{R}^2$ where $(p_1,p_2) \in U$ and $\phi : U \to \mathbb{R}^3$ with $\phi(p_1,p_2) = \overrightarrow{p}$ and for $(x,y,z) \in V$, $F(x,y,z) = a$ if and only if $\phi(x,y) = z$. This means, locally, $F^{-1}(a)$ is the graph of a smooth function. □

3. Gauss Map

3.1. **Tangent Space.** : Consider the surface $S \in \mathbb{R}^3$ and a point $p \in S$ (see figure 3.1).

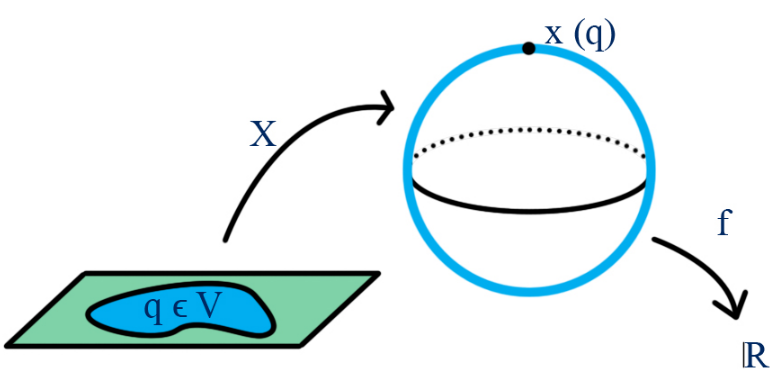
Figure 3.1



Surface $S \in \mathbb{R}^3$ and point $p \in S$.

Let U be a neighborhood containing p and consider a coordinate chart $\overrightarrow{X} : V \to U$ where V is a space in $\mathbb{R}^2$ containing the point q such that $\overrightarrow{x}(q) = p$ (see Figure 3.2).

Figure 3.2



Say $F : S \to \mathbb{R}$ is a smooth function. To differentiate F at p, differentiate $F \circ x$ at $q \in \overrightarrow{v}$ where $\overrightarrow{x}(q) = p$.

Because $\overrightarrow{x}(q) = p$, the tangent space at p is the 2-Dimensional vector space spanned by $\frac{\delta \overrightarrow{x}}{\delta u} \equiv \overrightarrow{X}_u(q)$ and $\frac{\delta \overrightarrow{x}}{\delta v} \equiv \overrightarrow{X}_v(q)$ where (u, v) are the coordinates of v.

Given $\overrightarrow{v} \in T_pM$, the directional derivative of F at p is defined by

$$\begin{aligned} D_p f(\overrightarrow{v}) &= D_q(f \circ \overrightarrow{x})(\tilde{v}) \quad \textit{where } d_{\overrightarrow{x}q}(\tilde{v}) = \overrightarrow{v} \\ &= D_p f \circ D_q \overrightarrow{v} \\ &= D_p f \circ \overrightarrow{v}. \end{aligned}$$

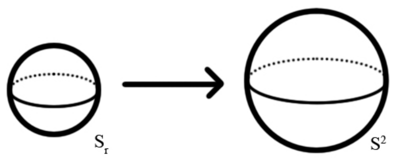
3.2. **Gauss Map.** Examples

$S_r = \{\overrightarrow{x} \in \mathbb{R}^3 | ||\overrightarrow{x}|| = r\} =$ sphere radius r $\subseteq \mathbb{R}^3$

The Gauss Map $N : S_r \to S$ is given by:

$$N(\overrightarrow{x}) = \frac{x}{||\overrightarrow{x}||} = \frac{\overrightarrow{x}}{r}.$$

Figure 3.3

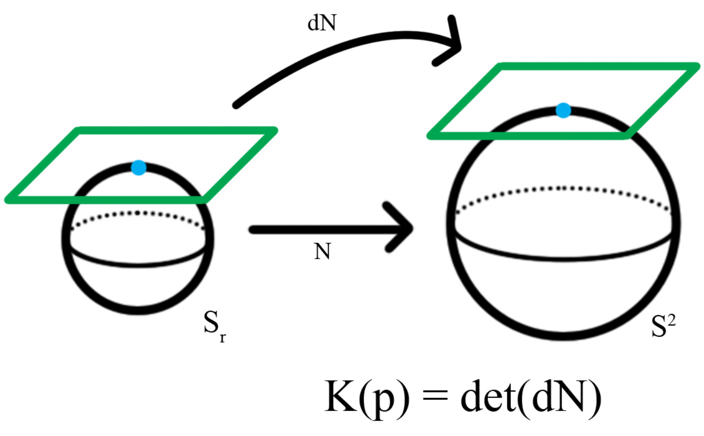


$\tilde{N} : \mathbb{R}^3 \to \mathbb{R}^3$

Consider $\tilde{N}(x, y, z) = (\frac{\overrightarrow{x}}{r}, \frac{\overrightarrow{y}}{r}, \frac{\overrightarrow{z}}{r})$. This means that:

$$\begin{aligned} d\tilde{N} &= \begin{bmatrix} \frac{1}{r} & 0 & 0 \\ 0 & \frac{1}{r} & 0 \\ 0 & 0 & \frac{1}{r} \end{bmatrix} \\ &= \frac{1}{r} \cdot I \end{aligned}$$

Figure 3.4



Note, $N = \tilde{N}|_{Sr}$. Given $d\tilde{N} = \frac{1}{r} I_3$ at p, what is dN?

Given $\overrightarrow{v} \in T_pS_r$, take $\sigma : (-\epsilon, \epsilon) \to S_r$ smooth satisfying $\sigma(0) = \overrightarrow{p}$ and $\sigma'(0) = \overrightarrow{v}$.

By definition,

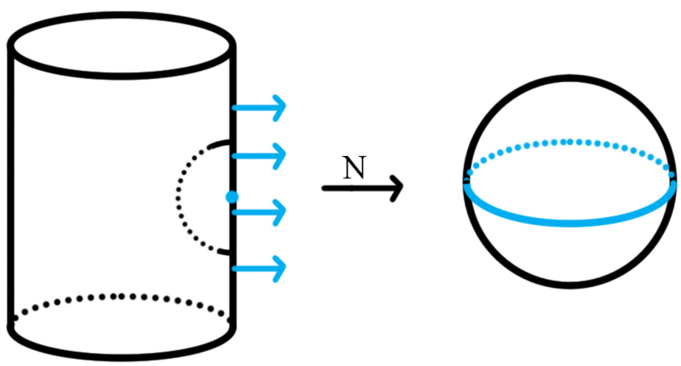
$$\begin{aligned} d_pN\overrightarrow{v} &= \frac{d}{dt}(N \circ \sigma)|_{t=0} \\ &= \frac{d}{dt}(\tilde{N} \circ \sigma)|_{t=0} \\ &= d\tilde{N} \cdot \sigma'(0) \end{aligned}$$

and

$$\begin{aligned} d_pN(\overrightarrow{v} &= d_p\tilde{N} \cdot \sigma'(0) \\ &= \frac{1}{r} \cdot I(\overrightarrow{(}v) \\ &= \frac{1}{r} \cdot \overrightarrow{v}. \end{aligned}$$

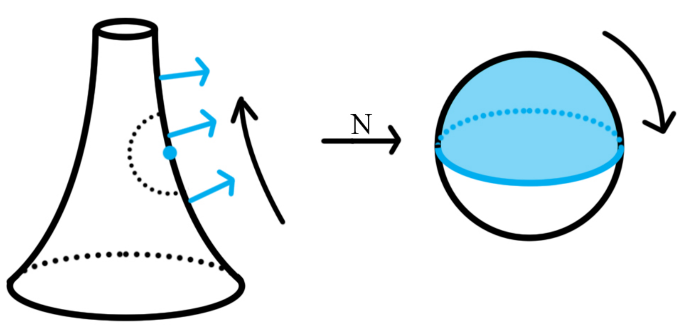
This means, for all $\overrightarrow{v} \in T_pS_r$, $d_pN(\overrightarrow{v}) = \frac{1}{r} \cdot \overrightarrow{v}$. Hence, $d_pN = \frac{1}{r}I_2$. Hence, for each $p \in S_r$, the Gaussian curvature $K(p) = det(d_pN) = det(\frac{1}{r} \cdot I_2) = \frac{1}{r^2}$.

Figure 3.5



Given $p \in \mathcal{C}$, $K(p) = det(d_pN) = 0$.

Figure 3.6



N reverses orientation but remains a local diffeomorphism with $det(d_pN)0$.

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References

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